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Harvesting Herbivore with Plant in Refuge on Plant-Herbivore System with Allee Effect

Research Article*

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Abstract: In this paper, we study plant-herbivore interactions where the herbivore is exposed to the risk of disease and harvesting while the plant has the ability to use a refuge. The feasibility and stability conditions of the equilibrium points of the system and the numerical simulations were verified to analyze the theoretical results and to investigate the properties of the system.

Keywords: Allee effect, susceptible herbivore - infective herbivore, constant refuge-random refuge, stability analysis.

1. Introduction

Plant-Herbivore models are of great interest to researchers in mathematics because they deal with environmental issues. The first differential equation model of predator-prey type, called Lotka-Volterra equation, was found by Alfred Lotka and Vito Volterra in 1920 [5, 10]. These equations expressed the relationship between two or more species. Plant-herbivore interactions are influenced by many factors. One of these factors is the hiding behaviour of plant and it reduces the chance of extinction due to predation [2, 4, 11-16]. Population harvesting has a strong effect on the plant-herbivore system with disease and it can be used to prevent infectious disease, so that the stability will be maintained [6, 8]. Allee effect describes a positive interaction among individuals at low population sizes and these interactions may be critical for survival [1, 3, 7, 9, 17].

In this paper, we study plant-herbivore interactions where the herbivore is exposed to the risk of disease and harvesting while the plant has the ability to use a refuge with Allee effect.

2. Mathematical Model

We consider two models the ability of plant to use constant refuge and the ability to use random refuge.

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2.1. The Constant Refuge Model with Allee Effect

The Mathematical model is written as

$$\begin{aligned}\frac{dx}{dt} &= rx\,(x-\theta)\,(1-x) - ax(1-m)(y+z)\\ \frac{dy}{dt} &= b\,(1-m)\,xy + \text{yz} - \mu y\\ \frac{dz}{dt} &= b\,(1-m)\,xz - \text{yz} - \sigma z\end{aligned} \tag{1}$$

The parameters were described in Table 1 We assume that the less effective herbivore shall be easier to harvest ($\mu > \sigma$), we

Parameters	Description
x	Density of Plant
y	Density of Infected Herbivore
z	Density of Susceptible Herbivore
r	Maximum birth-rate of plant, which can be scaled to be 1 by altering the time scale
θ	Allee threshold; $0 < \theta < 1$
m	The ability of the plant to use constant refuge; $0 < m \leq 1$
μ	Harvesting rate of infected herbivore; $\mu>\sigma$
σ	Harvesting rate of susceptible herbivore
λ	Contact rate between the infected and susceptible herbivore
a	Attack rate of herbivore
b	The growth rate of herbivore; a>b

Table 1: Variables and Parameters used Model (1)

also assume that infected herbivore does not become susceptible again and finally the disease does not affect the ability of the infected herbivore attacking plant.

Dynamics of Submodels : In order to understand the full dynamics of system (1), we should have a complete picture of the dynamics of the following two submodels:

Sub-model I: The plant-herbivore system in the absence of the disease in (1) is represented as

$$\begin{aligned}\frac{dx}{dt} &= x\,(x-\theta)\,(1-x) - ax(1-m)z\\ \frac{dz}{dt} &= bx(1-m)z - \sigma z\end{aligned} \tag{2}$$

For convenience, we introduce a disease-free demographic reproduction number for herbivore $R_0^z = \frac{b}{\sigma}$ and it is based upon the assumptions that the plant is at unit density and the disease is absent. The value of $R_0^z < 1$ indicates that the herbivore cannot invade while the value of $R_0^z > 1$ indicates that the herbivore may invade.

Equilibrium points : The system (2) has four equilibrium points which are

1. $E_1^z = (0,0)$
2. $E_2^z = (\theta,0)$
3. $E_3^z = (1,0)$
4. $E_4^z = (x^*,z^*)$, where $x^* = \frac{1}{R_0^z(1-m)}$ and $z^* = \frac{1}{a(1-m)}\left(\frac{1}{R_0^z(1-m)} - \theta\right)\left(1 - \frac{1}{R_0^z(1-m)}\right)$

Dynamical behaviour : In this section, we study the local behaviour of the system (2) about each equilibrium points. The Jacobian matrix is given by

$$J^z(x,z) = \begin{bmatrix} (x-\theta)(1-x) + x(1-2x+\theta) - a(1-m)z & -ax(1-m) \\ b(1-m)z & bx(1-m)-\sigma \end{bmatrix} \tag{3}$$

Proposition 2.1. *The equilibrium point $E_1^z = (0,0)$ is locally asymptotically stable.*

Proof. The Jacobian matrix evaluated at E_1^z is given by

$$J^z(0,0) = \begin{bmatrix} -\theta & 0 \\ 0 & -\sigma \end{bmatrix}$$

The equilibrium point $E_1^z = (0,0)$ is always locally asymptotically stable since its eigenvalues are $\lambda_1 = -\theta(<0)$, $\lambda_2 = -\sigma(<0)$. □

Proposition 2.2. *The equilibrium point $E_2^z = (\theta,0)$ is a*

(a). Saddle if $R_0^z < \frac{1}{\theta(1-m)}$

(b). Source if $R_0^z > \frac{1}{\theta(1-m)}$

Proof. The Jacobian matrix evaluated at E_2^z is given by

$$J^z(\theta,0) = \begin{bmatrix} \theta(1-\theta) & -a\theta(1-m) \\ 0 & b\theta(1-m)-\sigma \end{bmatrix}$$

The eigenvalues of $J^z(\theta,0)$ are

$$\lambda_1 = \theta(1-\theta) > 0 \;\; \& \;\; \lambda_2 = b\theta(1-m) - \sigma = \begin{cases} < 0 \;\; if \;\; R_0^z < \frac{1}{\theta(1-m)} \\ > 0 \;\; if \;\; R_0^z > \frac{1}{\theta(1-m)} \end{cases}$$

Therefore, $E_2^z = (\theta,0)$ is a saddle if $R_0^z < \frac{1}{\theta(1-m)}$ and is a source if $R_0^z > \frac{1}{\theta(1-m)}$. □

Proposition 2.3. *The equilibrium point $E_3^z = (1,0)$ is a*

(a). Stable if $R_0^z < \frac{1}{1-m}$

(b). Saddle if $R_0^z > \frac{1}{1-m}$

Proof. The Jacobian matrix evaluated at E_3^z is given by

$$J^z(1,0) = \begin{bmatrix} \theta-1 & -a(1-m) \\ 0 & b(1-m)-\sigma \end{bmatrix}$$

The eigenvalues of $J^z(1,0)$ are

$$\lambda_1 = (\theta-1) < 0; \;\; \lambda_2 = b(1-m) - \sigma = \begin{cases} < 0 \;\; if \;\; R_0^z < \frac{1}{1-m} \\ > 0 \;\; if \;\; R_0^z > \frac{1}{1-m} \end{cases}$$

Therefore, $E_3^z = (1,0)$ is locally asymptotically stable if $R_0^z < \frac{1}{1-m}$ and is a saddle if $R_0^z > \frac{1}{1-m}$. □

Proposition 2.4. *The equilibrium point E_4^z exists and is locally asymptotically stable if* $\frac{1}{1-m} < R_0^z < \min\left\{\frac{1}{\theta(1-m)}, \frac{2}{(1-m)(1+\theta)}\right\}$

Proof. The unique interior equilibrium $E_4^z = (x^*, z^*) = \left(\frac{1}{R_0^z(1-m)}, \frac{1}{a(1-m)}\left(\frac{1}{R_0^z(1-m)} - \theta\right)\left(1 - \frac{1}{R_0^z(1-m)}\right)\right)$ exists only if $\frac{1}{1-m} < R_0^z < \frac{1}{\theta(1-m)}$. The Jacobian matrix evaluated at E_4^z is given by

$$J^z(x^*, z^*) = \begin{bmatrix} \frac{1}{R_0^z(1-m)}(1 - \frac{2}{R_0^z(1-m)} + \theta) & \frac{-a}{R_0^z} \\ \frac{b}{a}\left(\frac{1}{R_0^z(1-m)} - \theta\right)\left(1 - \frac{1}{R_0^z(1-m)}\right) & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A & -B \\ C & 0 \end{bmatrix}$$

Whose characteristic equation is $\lambda^2 - A\lambda + BC = 0$ where $BC > 0$ and $\begin{cases} A < 0 \; if \; R_0^z < \frac{2}{(1-m)(1+\theta)} \\ A > 0 \; if \; R_0^z > \frac{2}{(1-m)(1+\theta)} \end{cases}$ This indicates that the eigenvalues of $J^z(x^*, z^*)$ are $\lambda_1 = \frac{A - \sqrt{A^2 - 4BC}}{2}$ and $\lambda_2 = \frac{A + \sqrt{A^2 - 4BC}}{2}$ when $A^2 > 4BC$ (or) $\lambda_1 = \frac{A - i\sqrt{4BC - A^2}}{2}$ and $\lambda_2 = \frac{A + i\sqrt{4BC - A^2}}{2}$ when $A^2 < 4BC$. Therefore, $E_4^z = (x^*, z^*)$ exists and is locally asymptotically stable if $\frac{1}{1-m} < R_0^z < \min\left\{\frac{1}{\theta(1-m)}, \frac{2}{(1-m)(1+\theta)}\right\} = \frac{2}{(1-m)(1+\theta)}$. □

Sub-model II: The plant-herbivore system in the absence of predation in (2) is represented as

$$\frac{dx}{dt} = x(x - \theta)(1 - x) - ax(1-m)y$$
$$\frac{dy}{dt} = b(1-m)xy - \mu y \tag{4}$$

We introduce the basic reproductive number $R_0^y = \frac{\lambda}{\mu}$. The value of $R_0^y < 1$ indicates that the infection cannot invade while the value of $R_0^y > 1$ indicates that the disease can invade.

Equilibrium points : The system (4) has four equilibrium points which are

1. $E_5^y = (0, 0)$
2. $E_6^y = (\theta, 0)$
3. $E_7^y = (1, 0)$
4. $E_8^y = (x^*, y^*)$ where $x^* = \frac{1}{R_0^y(1-m)}$ and $y^* = \frac{1}{a(1-m)}\left(\frac{1}{R_0^z(1-m)} - \theta\right)\left(1 - \frac{1}{R_0^z(1-m)}\right)$.

Dynamical behaviour : In this section, we study the local behaviour of the system (4) about each equilibrium points. The Jacobian matrix is given by

$$J^y(x, y) = \begin{bmatrix} (x - \theta)(1 - x) + x(1 - 2x + \theta) - a(1-m)y & -ax(1-m) \\ b(1-m)y & b(1-m)x - \mu \end{bmatrix} \tag{5}$$

Proposition 2.5. *The equilibrium point $E_5^y = (0, 0)$ is locally asymptotically stable.*

Proof. The Jacobian matrix evaluated at E_5^y is given by

$$J^y(0, 0) = \begin{bmatrix} -\theta & 0 \\ 0 & -\mu \end{bmatrix}$$

The equilibrium point $E_5^y = (0, 0)$ is locally asymptotically stable since its eigenvalues are $\lambda_1 = -\theta\,(< 0)$, $\lambda_2 = -\mu(< 0)$. □

Proposition 2.6. *The equilibrium point $E_6^y = (\theta, 0)$ is a*

(a). Saddle if $R_0^y < \frac{1}{\theta(1-m)}$.

(b). Source if $R_0^y > \frac{1}{\theta(1-m)}$.

Proof. The Jacobian matrix evaluated at E_6^y is given by

$$J^y(\theta, 0) = \begin{bmatrix} \theta(1-\theta) & -a\theta(1-m) \\ 0 & \mathrm{b}\theta(1-m) - \mu \end{bmatrix}$$

The eigenvalues of $J^y(\theta, 0)$ are $\lambda_1 = \theta(1-\theta) > 0$ and $\lambda_2 = b\theta(1-m) - \mu = \begin{cases} < 0 \;\; if \; R_0^y < \frac{1}{\theta(1-m)} \\ > 0 \;\; if \; R_0^y > \frac{1}{\theta(1-m)} \end{cases}$. Therefore, $E_6^y = (\theta, 0)$ is a saddle if $R_0^y < \frac{1}{\theta(1-m)}$ and is a source if $R_0^y > \frac{1}{\theta(1-m)}$. □

Proposition 2.7. *The equilibrium point $E_7^y = (1, 0)$ is a*

(a). Stable if $R_0^y < \frac{1}{1-m}$

(b). Saddle if $R_0^y > \frac{1}{1-m}$.

Proof. The Jacobian matrix evaluated at E_7^y is given by

$$J^y(1, 0) = \begin{bmatrix} \theta - 1 & -a(1-m) \\ 0 & \mathrm{b(1-m)} - \mu \end{bmatrix}$$

The eigenvalues of $J^y(1, 0)$ are $\lambda_1 = (\theta - 1) < 0$ and $\lambda_2 = b(1-m) - \mu = \begin{cases} < 0 \;\; if \; R_0^y < \frac{1}{1-m} \\ > 0 \;\; if \; R_0^y > \frac{1}{1-m} \end{cases}$. Therefore, $E_7^y = (1, 0)$ is locally asymptotically stable if $R_0^y < \frac{1}{1-m}$ and is a saddle if $R_0^y > \frac{1}{1-m}$. □

Proposition 2.8. *The equilibrium point E_8^y exists and is locally asymptotically stable if* $\frac{1}{1-m} < R_0^y < \min\left\{\frac{1}{\theta(1-m)}, \frac{2}{(1-m)(1+\theta)}\right\}$.

Proof. The unique interior equilibrium $E_8^y = (x^*, y^*) = \left(\frac{1}{R_0^y(1-m)}, \frac{1}{a(1-m)}\left(\frac{1}{R_0^y(1-m)} - \theta\right)\left(1 - \frac{1}{R_0^y(1-m)}\right)\right)$ exists only if $\frac{1}{1-m} < R_0^y < \frac{1}{\theta(1-m)}$. The Jacobian matrix evaluated at E_8^y is given by

$$\begin{aligned} J^y(x^*, y^*) &= \begin{bmatrix} \frac{1}{R_0^y(1-m)}(1 - \frac{2}{R_0^y(1-m)} + \theta) & -\frac{a}{R_0^y} \\ \frac{b}{a}\left(\frac{1}{R_0^y(1-m)} - \theta\right)\left(1 - \frac{1}{R_0^y(1-m)}\right) & 0 \end{bmatrix} \\ &= \begin{bmatrix} A & -B \\ C & 0 \end{bmatrix} \end{aligned}$$

Whose characteristic equation is given by $\lambda^2 - A\lambda + BC = 0$, where $BC > 0$ and we have $A > 0$ if $R_0^y > \frac{2}{(1+\theta)(1-m)}$ and $A < 0$ if $R_0^y < \frac{2}{(1+\theta)(1-m)}$. This indicates that the eigenvalues of $J^z(x^*, z^*)$ are $\lambda_1 = \frac{A - \sqrt{A^2 - 4BC}}{2}$ and $\lambda_2 = \frac{A + \sqrt{A^2 - 4BC}}{2}$ when $A^2 > 4BC$ (or) $\lambda_1 = \frac{A - i\sqrt{4BC - A^2}}{2}$ and $\lambda_2 = \frac{A + i\sqrt{4BC - A^2}}{2}$ when $A^2 < 4BC$. Therefore, $E_8^y = (x^*, y^*)$ exists and is locally asymptotically stable if $\frac{1}{1-m} < R_0^y < \min\left\{\frac{1}{\theta(1-m)}, \frac{2}{(1-m)(1+\theta)}\right\} = \frac{2}{(1-m)(1+\theta)}$. □

2.2. The Random Refuge Model with Allee Effect

The Mathematical model is written as

$$\begin{aligned} \frac{dx}{dt} &= rx\,(x-\theta)\,(1-x) - a(x-m)(\mathrm{y}+\mathrm{z}) \\ \frac{dy}{dt} &= b\,(x-m)\,y + yz - \mu y \\ \frac{dz}{dt} &= b\,(x-m)\,z - \mathrm{yz} - \sigma z \end{aligned} \tag{6}$$

Dynamics of submodels : In order to understand the full dynamics of system (6), we should have a complete picture of the dynamics of the following two submodels:

Sub-model III : The plant-herbivore system in the absence of the disease in (6) is represented as

$$\begin{aligned} \frac{dx}{dt} &= x\,(x-\theta)\,(1-x) - a(x-m)z \\ \frac{dz}{dt} &= b(x-m)z - \sigma z \end{aligned} \tag{7}$$

Equilibrium points : The system (7) has four equilibrium points which are

1. $E_9^z = (0,0)$
2. $E_{10}^z = (\theta,0)$
3. $E_{11}^z = (1,0)$
4. $E_{12}^z = (x^*,z^*)$, where $x^* = \frac{1}{R_0^z} + m$ and $z^* = \frac{R_0^z}{a}\left(\frac{1}{R_0^z}+m\right)\left(\frac{1}{R_0^z}+m-\theta\right)\left(1-\frac{1}{R_0^z}-m\right)$.

Dynamical behaviour : In this section, we study the local behaviour of the system (7) about each equilibrium points. The Jacobian matrix is given by

$$J^z\,(x,z) = \begin{bmatrix} (x-\theta)\,(1-x) + x\,(1-2x+\theta) - az & -a(x-m) \\ bz & b(x-m)-\sigma \end{bmatrix} \tag{8}$$

Proposition 2.9. *The equilibrium point $E_9^z = (0,0)$ is locally asymptotically stable.*

Proof. The Jacobian matrix evaluated at E_9^z is given by

$$J^z\,(0,0) = \begin{bmatrix} -\theta & am \\ 0 & -bm-\sigma \end{bmatrix}$$

The equilibrium point $E_9^z = (0,0)$ is always locally asymptotically stable since its eigenvalues are $\lambda_1 = -\theta\,(<0)$, $\lambda_2 = -bm-\sigma(<0)$. □

Proposition 2.10. *The equilibrium point $E_{10}^z = (\theta,0)$ is a*

(a). Saddle if $R_0^z < \frac{1}{(\theta-m)}$.

(b). Source if $R_0^z > \frac{1}{(\theta-m)}$.

Proof. The Jacobian matrix evaluated at E^z_{10} is given by

$$J^z(\theta,0)=\begin{bmatrix}\theta(1-\theta) & -\text{a}(\theta-m)\\ 0 & \text{b}(\theta-m)-\sigma\end{bmatrix}$$

The eigenvalues of $J^z(\theta,0)$ are $\lambda_1=\theta(1-\theta)>0$ and $\lambda_2=b(\theta-m)-\sigma=\begin{cases}<0 \ \ if\ R^z_0<\frac{1}{(\theta-m)}\\ >0\ \ if\ R^z_0>\frac{1}{(\theta-m)}\end{cases}$. Therefore, $E^y_{10}=(\theta,0)$ is a saddle if $R^z_0<\frac{1}{(\theta-m)}$ and is a source if $R^z_0>\frac{1}{(\theta-m)}$. □

Proposition 2.11. *The equilibrium point $E^z_{11}=(1,0)$ is a*

(a). Stable if $R^z_0<\frac{1}{1-m}$.

(b). Saddle if $R^z_0>\frac{1}{1-m}$.

Proof. The Jacobian matrix evaluated at E^z_{11} is given by

$$J^z(1,0)=\begin{bmatrix}\theta-1 & -a(1-m)\\ 0 & \text{b}(1-\text{m})-\sigma\end{bmatrix}$$

The eigenvalues of $J^z(1,0)$ are $\lambda_1=(\theta-1)<0$ and $\lambda_2=b(1-m)-\sigma=\begin{cases}<0\ \ if\ R^z_0<\frac{1}{1-m}\\ >0\ \ if\ R^z_0>\frac{1}{1-m}\end{cases}$. Therefore, $E^z_{11}=(1,0)$ is locally asymptotically stable if $R^z_0<\frac{1}{1-m}$ and is a saddle if $R^z_0>\frac{1}{1-m}$. □

Sub-model IV : The plant-herbivore system in the absence of the predation in (6) is represented as

$$\begin{aligned}\frac{dx}{dt}&=x(x-\theta)(1-x)-a(x-m)y\\ \frac{dz}{dt}&=b(x-m)y-\mu z\end{aligned}\tag{9}$$

Equilibrium points : The system (9) has four equilibrium points which are

1. $E^y_{13}=(0,0)$
2. $E^y_{14}=(\theta,0)$
3. $E^y_{15}=(1,0)$
4. $E^y_{16}=(x^*,z^*)$, where $x^*=\frac{1}{R^y_0}+m$ and $z^*=\frac{R^y_0}{a}\left(\frac{1}{R^y_0}+m\right)\left(\frac{1}{R^y_0}+m-\theta\right)\left(1-\frac{1}{R^y_0}-m\right)$.

Dynamical behaviour : In this section, we study the local behaviour of the system (9) about each equilibrium points. The Jacobian matrix is given by

$$J^y(x,z)=\begin{bmatrix}(x-\theta)(1-x)+x(1-2x+\theta)-ay & -ax\\ by & b(x-m)-\mu\end{bmatrix}\tag{10}$$

Proposition 2.12. *The equilibrium point $E^y_{13}=(0,0)$ is locally asymptotically stable.*

Proof. The Jacobian matrix evaluated at E^y_{13} is given by

$$J^y(0,0) = \begin{bmatrix} -\theta & 0 \\ 0 & -bm-\mu \end{bmatrix}$$

The equilibrium point $E^y_{13} = (0,0)$ is always locally asymptotically stable since its eigenvalues are $\lambda_1 = -\theta\,(<0)$, $\lambda_2 = -bm-\mu(<0)$. □

Proposition 2.13. *The equilibrium point* $E^y_{14} = (\theta, 0)$ *is a*

(a). Saddle if $R^y_0 < \frac{1}{(\theta-m)}$.

(b). Source if $R^y_0 > \frac{1}{(\theta-m)}$.

Proof. The Jacobian matrix evaluated at E^y_{14} is given by

$$J^y(\theta,0) = \begin{bmatrix} \theta(1-\theta) & -\mathrm{a}\theta \\ 0 & \mathrm{b}(\theta-m)-\mu \end{bmatrix}$$

The eigenvalues of $J^y(\theta,0)$ are $\lambda_1 = \theta\,(1-\theta) > 0$ and $\lambda_2 = b(\theta-m)-\mu = \begin{cases} <0 \ \ if \ R^y_0 < \frac{1}{(\theta-m)} \\ >0 \ \ if \ R^y_0 > \frac{1}{(\theta-m)} \end{cases}$. Therefore, $E^y_{14} = (\theta,0)$ is a saddle if $R^y_0 < \frac{1}{(\theta-m)}$ and is a source if $R^y_0 > \frac{1}{(\theta-m)}$. □

Proposition 2.14. *The equilibrium point* $E^y_{15} = (1, 0)$ *is a*

(a). Stable if $R^y_0 < \frac{1}{1-m}$.

(b). Saddle if $R^y_0 > \frac{1}{1-m}$.

Proof. The Jacobian matrix evaluated at E^y_{15} is given by

$$J^y(1,0) = \begin{bmatrix} \theta-1 & -a \\ 0 & \mathrm{b}(1-\mathrm{m})-\mu \end{bmatrix}$$

The eigenvalues of $J^y(1,0)$ are $\lambda_1 = (\theta-1) < 0$ and $\lambda_2 = b\,(1-m)-\mu = \begin{cases} <0 \ \ if \ R^y_0 < \frac{1}{1-m} \\ >0 \ \ if \ R^y_0 > \frac{1}{1-m} \end{cases}$. Therefore, $E^y_{15} = (1,0)$ is locally asymptotically stable if $R^y_0 < \frac{1}{1-m}$ and is a saddle if $R^y_0 > \frac{1}{1-m}$. □

3. Numerical Analysis

In this section, we depict the analytical findings with the help of numerical simulations by using MATLAB programming. Time series plots for system (2), (4), (7) and (10) to prove the theoretical results with hypothetical set of data are presented.

Case 1 : Figure: 1 and 2 shows the time series graph for constant refuge model on plant-herbivore system with Allee effect. We shall consider $r = 1$, $\theta = 0.5$, $a = 0.4$, $m = 0.108$, $b = 0.3$, $\sigma = 0.4$ and $\mu = 0.5$.

1. At equilibrium point (0, 0, 0), the eigen values are $\lambda_1 = -0.5$, $\lambda_2 = -0.4$, $\lambda_1 = -0.5$ so that $|_{1,2,3}| < 1$. Hence the trivial equilibrium point is stable.

2. At equilibrium point (0.5, 0, 0), the eigen values are $\lambda_1 = 0.25$, $\lambda_2 = -0.2662$, $\lambda_1 = -0.3662$ so that $|_{1,2,3}| < 1$. Hence the trivial equilibrium point is stable.

3. At equilibrium point (1, 0, 0), the eigen values are $\lambda_1 = -0.5$, $\lambda_2 = -0.1324$, $\lambda_1 = -0.2324$ so that $|_{1,2,3}| < 1$. Hence the trivial equilibrium point is stable.

Case 2 : Figure: 3 and 4 shows the time series graph for constant refuge model on plant-herbivore system with Allee effect. We shall consider $r = 1$, $\theta = 0.5$, $a = 0.4$, $m = 0.02$, $b = 0.3$, $\sigma = 0.4$ and $\mu = 0.5$.

1. At equilibrium point (0, 0, 0), the eigen values are $\lambda_1 = -0.5$, $\lambda_2 = -0.406$, $\lambda_1 = -0.506$ so that $|_{1,2,3}| < 1$. Hence the trivial equilibrium point is stable

2. At equilibrium point (0.5, 0, 0), the eigen values are $\lambda_1 = 0.25$, $\lambda_2 = -0.256$, $\lambda_1 = -0.356$ so that $|_{1,2,3}| < 1$. Hence the trivial equilibrium point is stable.

3. At equilibrium point (1, 0, 0), the eigen values are $\lambda_1 = -0.5$, $\lambda_2 = -0.106$, $\lambda_1 = -0.206$ so that $|_{1,2,3}| < 1$. Hence the trivial equilibrium point is stable.

4. Conclusion

The mathematical model describes the spread of a disease in a plant-herbivore system with Allee effect has been discussed and the stability of different equilibrium points has been thoroughly examined. Numerical simulations were used to find the effect of the disease and Allee effect on the species and diagrams were presented which are supporting our results.

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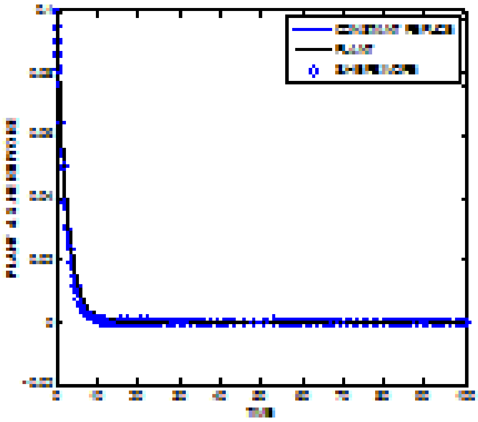

Figure 1: Time series graph for constant refuge model on plant and susceptible herbivore with Allee effect ($\theta = 0.5$, $a = 0.4$, $m = 0.108$, $b = 0.3$, $\sigma = 0.4$)

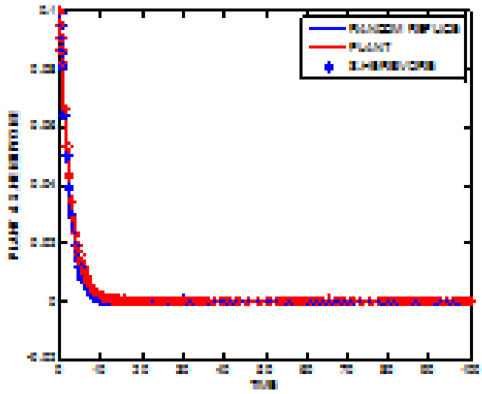

Figure 2: Time series graph for constant refuge model on plant and infected herbivore with Allee effect ($\theta = 0.5$, $a = 0.4$, $m = 0.108$, $b = 0.3$, $\mu = 0.5$)

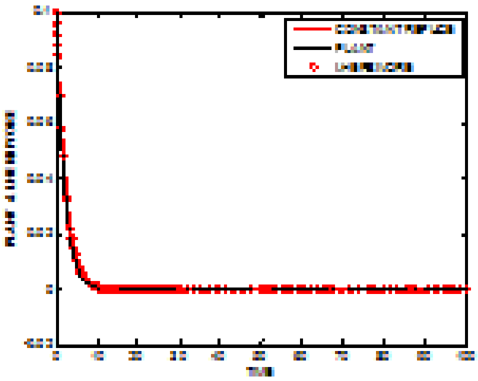

Figure 3: Time series graph for random refuge model on plant and susceptible herbivore with Allee effect ($\theta = 0.5$, $a = 0.4$, $m = 0.02$, $b = 0.3$, $\sigma = 0.4$)

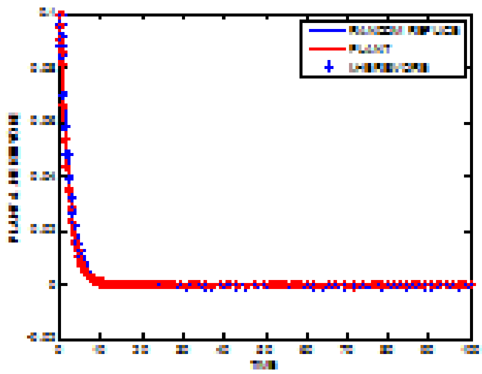

Figure 4: Time series graph for random refuge model on plant and infected herbivore with Allee effect ($\theta = 0.5$, $a = 0.4$, $m = 0.02$, $b = 0.3$, $\mu = 0.5$)