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Solution and Stability of a Functional Equation Originating From Consecutive Terms of a Geometric Progression

Research Article*

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Abstract: In this paper, the author has proved the generalized Ulam-Hyers stability of a new type of the functional equation

$$\ell(uv) + \ell\left(\frac{u}{v}\right) = 2\ell(u)$$

with $v \neq 0$ which is originating from consecutive terms of a geometric progression. An application of this functional equation is also studied.

MSC: 39B52, 32B72, 32B82.

Keywords: Logarithmic functional equations, generalized Ulam- Hyers stability.

1. Introduction

In 1940, S. M. Ulam [26] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these the following question was concerning the stability of homomorphisms. Let G_1 be a group and let G_2 be a metric group with a metric $d(.,.)$. Given $\kappa > 0$, does there exist a $\rho > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(x+y), f(x)+f(y)) < \rho$ for all $x, y \in G_1$, then there exists a homomorphism $A : G_1 \to G_2$ with $d(f(x), A(x)) < \kappa$ for all $x \in G_1$?

In 1941, Hyers [10] gave the first partial solution to Ulam's question for the case of approximate additive mappings for Banach spaces. Then, Aoki [2] and Bourgin [6] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [21] generalized the theorem of Hyers [10] by considering the stability problem with unbounded Cauchy differences. In 1991, Gajda [8] following the same approach as by Rassias [21] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [8] as well as by Rassias and Semrl [23], that one cannot prove a Rassias-type theorem when $p = 1$. Gavruta [9] obtained the generalized result of Rassias's theorem which allows the Cauchy difference to be controlled by a general unbounded function.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [24] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach.

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During the past two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, $k-$additive mappings, invariant means, multiplicative mappings, bounded nth differences, convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations (see[15–19, 23, 25]).

There is a saying in Tamil, **paruppilla kalyanama** i.e., **marriage without lentils**. (In the North American context, can one think of Thanks giving without turkey?) Similarly, can one say a article on functional equations without Cauchys equations? As Cauchys equations permeate functional equations and are fundamental in nature,

The Cauchy functional equations are

$$A(x+y) = A(x) + A(y) \quad \text{(Additive)}, \tag{1}$$

$$E(x+y) = E(x)E(y) \quad \text{(Exponential)}, \tag{2}$$

$$L(xy) = L(x) + L(y) \quad \text{(Logarithmic)}, \tag{3}$$

$$M(xy) = M(x)M(y) \quad \text{(Multiplicative)}. \tag{4}$$

In 1821, they were first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an Cauchy functional equations in honor of Cauchy (see [14]).

M. Arunkumar, C. Leela Sabari [3] proved the solution and Hyers - Ulam - Rassias stability of the additive functional equation

$$f(x) + f(x) = f(2x) \tag{5}$$

which is originating from a chemical equation. Also, M. Arunkumar et. al., [4] investigated the generalized Ulam-Hyers stability of a functional equation

$$f(y) = \frac{f(y+z) + f(y-z)}{2} \tag{6}$$

which is originating from arithmetic mean of consecutive terms of an arithmetic progression using direct and fixed point methods. Infact, M. Arunkumar and Matina J. Rassias [5] established the solution and generalized Ulam - Hyers stability of a new type of reciprocal functional equation

$$f(2y) = \frac{f(y+z)f(y-z)}{f(y+z) + f(y-z)} \tag{7}$$

which originates from consecutive terms of an harmonic progression.

In this paper, the author has proved the generalized Ulam-Hyers stability of a new type of the functional equation

$$\ell(uv) + \ell\left(\frac{u}{v}\right) = 2\ell(u) \tag{8}$$

with $v \neq 0$ which is originating from consecutive terms of a geometric progression. In Section 2, the general solution of the functional equation (8) is given. In Section 3, 4 the Hyers - Ulam - Rassias stability of the logarithmic functional equation (8) is proved using direct and fixed point method. An application of the logarithmic functional equation (8) is also discussed in Section 5.

2. General Solution

In this section, the general solution of the functional equation (8) is given.

Theorem 2.1. *If a mapping $\ell : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation*

$$\ell(uv) + \ell\left(\frac{u}{v}\right) = 2\ell(u) \tag{9}$$

with $\ell(1) = 0$ then ℓ is logarithmic.

Proof. Let $\ell : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (8). Setting $v = u$ in (8), we get

$$\ell(u^2) = 2\ell(u) \tag{10}$$

for all $u \in \mathbb{R}$. Hence ℓ is logarithmic. □

3. Stability Results

In this section, the generalized Ulam - Hyers stability of the functional equation (8) is provided.

Theorem 3.1. *Let $j = \pm 1$. Let $\ell : \mathbb{R} \to \mathbb{R}$ be a mapping for which there exist a function $\zeta : U^2 \to [0, \infty)$ with the condition*

$$\sum_{n=0}^{\infty} \frac{\zeta\left(u^{2^{nj}}, v^{2^{nj}}\right)}{2^{nj}} \quad \text{converges in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} \frac{\zeta\left(u^{2^{nj}}, v^{2^{nj}}\right)}{2^{nj}} = 0 \tag{11}$$

such that the functional inequality

$$\left|\ell(uv) + \ell\left(\frac{u}{v}\right) - 2\ell(u)\right| \leq \zeta\left(u, v\right) \tag{12}$$

for all $u, v \in \mathbb{R}$. Then there exists a unique logarithmic function $L : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation (8) and

$$|\ell(u) - L(u)| \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta\left(u^{2^{kj}}, u^{2^{kj}}\right) + |\ell(1)|}{2^{kj}} \tag{13}$$

for all $u \in \mathbb{R}$. The mapping $L(u)$ is defined by

$$L(u) = \lim_{n\to\infty} \frac{\ell\left(u^{2^{nj}}\right)}{2^{nj}} \tag{14}$$

for all $u \in \mathbb{R}$.

Proof. Assume $j = 1$. Replacing v by u in (12) and dividing by 2, we get

$$\left|\frac{\ell(u^2)}{2} - \ell(u)\right| \leq \frac{1}{2}\left[\zeta\left(u, u\right) + |\ell(1)|\right] \tag{15}$$

for all $u \in \mathbb{R}$. Now replacing u by u^2 and dividing by 2 in (15), we get

$$\left|\frac{\ell(u^4)}{4} - \frac{\ell(u^2)}{2}\right| \leq \frac{1}{2^2}\left[\zeta\left(u^2, u^2\right) + |\ell(1)|\right] \tag{16}$$

for all $u \in \mathbb{R}$. From (15) and (16), we obtain

$$\begin{aligned}\left|\frac{\ell(u^4)}{4} - \ell(u)\right| &\leq \left|\frac{\ell(u^4)}{4} - \frac{\ell(u^2)}{2}\right| + \left|\frac{\ell(u^2)}{2} - \ell(u)\right| \\ &\leq \frac{1}{2}\left[\zeta(u,u) + +|\ell(1)| + \frac{\zeta(u^2,u^2) + |\ell(1)|}{2}\right] \end{aligned} \tag{17}$$

for all $u \in \mathbb{R}$. Proceeding further and using induction on a positive integer n , we get

$$\begin{aligned}\left|\frac{\ell\left(u^{2^n}\right)}{2^n} - \ell(u)\right| &\leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{\zeta\left(u^{2^k},u^{2^k}\right) + |\ell(1)|}{2^k} \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\zeta\left(u^{2^k},u^{2^k}\right) + |\ell(1)|}{2^k}\end{aligned} \tag{18}$$

for all $u \in \mathbb{R}$. In order to prove the convergence of the sequence

$$\left\{\frac{\ell(u^{2^n})}{2^n}\right\},$$

replace u by u^{2^m} and dividing by 2^m in (18), for any $m, n > 0$, we deduce

$$\begin{aligned}\left|\frac{\ell(u^{2^{(n+m)}})}{2^{(n+m)}} - \frac{\ell(u^{2^m})}{2^m}\right| &= \frac{1}{2^m}\left|\frac{\ell(u^{2^n}\cdot u^{2^m})}{2^n} - \ell(u^{2^m})\right| \\ &\leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{\zeta\left(u^{2^{k+m}},u^{2^{k+m}}\right) + |\ell(1)|}{2^{k+m}} \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\zeta\left(u^{2^{k+m}},u^{2^{k+m}}\right) + |\ell(1)|}{2^{k+m}} \\ &\to 0 \quad as \ \ m \to \infty\end{aligned}$$

for all $u \in \mathbb{R}$. Hence the sequence $\left\{\frac{\ell(u^{2^n})}{2^n}\right\}$ is Cauchy sequence. Since $\mathbb{R}$ is complete, there exists a mapping $L : \mathbb{R} \to \mathbb{R}$ such that

$$L(U) = \lim_{n\to\infty}\frac{\ell\left(u^{2^n}\right)}{2^n} \quad \forall \ u \in U.$$

Letting $n \to \infty$ in (18) we see that (13) holds for all $u \in \mathbb{R}$. To prove that L satisfies (8), replacing (u, v) by (u^{2^n}, v^{2^n}) and dividing by 2^n in (12), we obtain

$$\frac{1}{2^n}\left|\ell(u^{2^n}\cdot v^{2^n}) + \ell\left(\frac{u^{2^n}}{v^{2^n}}\right) - 2\ell(u^{2^n})\right| \leq \frac{1}{2^n}\zeta(u^{2^n}, v^{2^n})$$

for all $u, v \in \mathbb{R}$. Letting $n \to \infty$ in the above inequality and using the definition of $L(u)$, we see that

$$L(uv) + L\left(\frac{u}{v}\right) = 2L(u)$$

Hence L satisfies (8) for all $u, v \in \mathbb{R}$. To prove that L is unique, let $M(u)$ be another logarithmic mapping satisfying (8) and (13), then

$$\begin{aligned}|L(u) - M(u)| &= \frac{1}{2^n}\left|L(u^{2^n}) - M(v^{2^n})\right| \\ &\leq \frac{1}{2^n}\left\{\left|L(u^{2^n}) - \ell(v^{2^n})\right| + \left|\ell(u^{2^n}) - M(v^{2^n})\right|\right\} \\ &\leq \sum_{k=0}^{\infty}\frac{\zeta(u^{2^{n+k}}, u^{2^{n+k}}) + |\ell(1)|}{2^{(k+n)}} \\ &\to 0 \quad as \ \ n \to \infty\end{aligned}$$

for all $u \in \mathbb{R}$. Hence L is unique. For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [10], Ulam-Hyers-Rassias [21], Ulam-Gavruta-Rassias [20] and Ulam-JRassias [24] stabilities of (8).

Corollary 3.2. *Let ρ and s be nonnegative real numbers. Let a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality with $\ell(1) = 0$*

$$\left|\ell(uv) + \ell\left(\frac{u}{v}\right) - 2\ell(u)\right| \leq \begin{cases} \rho, \\ \rho\left\{|u|^s + |v|^s\right\}, \\ \rho\ |u|^s|v|^s, \\ \rho\left\{|u|^s|v|^s + \left\{|u|^{2s} + |v|^{2s}\right\}\right\}, \end{cases} \tag{19}$$

for all $u, v \in \mathbb{R}$. Then there exists a unique logarithmic function $L : \mathbb{R} \to \mathbb{R}$ such that

$$|\ell(u) - L(u)| \leq \begin{cases} \rho, \\ \dfrac{2\rho}{|2 - u^{2^s}|}, \\ \dfrac{\rho}{|2 - u^{2^{2s}}|} \\ \dfrac{3\rho}{|2 - u^{2^{2s}}|} \end{cases} \tag{20}$$

for all $u \in \mathbb{R}$.

4. The Alternative of Fixed Point

Now we will recall the fundamental results in fixed point theory.

Theorem 4.1 (Banach's contraction principle)**.** *Let $(\mathbb{R}, d)$ be a complete metric space and consider a mapping $T : \mathbb{R} \to \mathbb{R}$ which is strictly contractive mapping, that is*

(A_1) $d(Tx, Ty) \leq Ld(x, y)$

for some (Lipschitz constant) $L < 1$. Then,

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii) The fixed point for each given element x^* is globally attractive, that is

(A_2) $\lim_{n\to\infty} T^n x = x^*$,

for any starting point $x \in \mathbb{R}$;

(iii) One has the following estimation inequalities:

(A_3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ \ d(T^n x, T^{n+1}x), \forall\ \ n \geq 0, \forall\ \ x \in \mathbb{R}$;

(A_4) $d(x, x^*) \leq \frac{1}{1-L}\ \ d(x, x^*), \forall\ \ x \in X$.

Theorem 4.2 (The alternative of fixed point [?])**.** *Suppose that for a complete generalized metric space $(\mathbb{R}, d)$ and a strictly contractive mapping $T : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant L. Then, for each given element $x \in \mathbb{R}$, either*

(B_1) $d(T^n x, T^{n+1}x) = \infty \quad \forall\ \ n \geq 0$,

or

(B_2) *there exists a natural number n_0 such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in \mathbb{R} : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L}\ \ d(y, Ty)$ *for all* $y \in \mathbb{R}$.

5. Stability Using Fixed Point Method

In this section, we investigate the stability of the given functional equation using the alternative fixed point. Let $\mathbb{R}$ be a real Banach space.

Theorem 5.1. *Let* $\alpha : \mathbb{R}^2 \to [0, \infty)$ *be a function such that*

$$\lim_{k\to\infty} \frac{\alpha(u^{\mu_i^k},\ v^{\mu_i^k})}{\mu_i^k} = 0 \tag{21}$$

for all $u, v \in \mathbb{R}$, *where* $\mu_i = 2$ *if* $i = 0$ *and* $\mu_i = \dfrac{1}{2}$ *if* $i = 1$. *If* $f : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\left|\ell(uv) + \ell\left(\frac{u}{v}\right) - 2\ell(u)\right| \leq \alpha(u, v) \tag{22}$$

for all $u, v \in \mathbb{R}$. *If there exists* $\kappa = \kappa(i) < 1$ *such that the function with* $\ell(1) = 0$,

$$x \to \beta(u) = \alpha\left(u^{\frac{1}{2}}, u^{\frac{1}{2}}\right),$$

has the property

$$\frac{1}{\mu_i}\beta(u^{\mu_i}) = \kappa\ \beta\,(u) \tag{23}$$

for all $u \in \mathbb{R}$. *Then there exists unique logarthamic function* $L : \mathbb{R} \to \mathbb{R}$ *such that*

$$|L(u) - \ell(u)| \leq \frac{\kappa^{1-i}}{1-\kappa}\beta(u) \tag{24}$$

holds for all $u \in \mathbb{R}$.

Proof. Consider the set $X = \{\ell/\ell : \mathbb{R} \to \mathbb{R},\ \ell(0) = 0\}$ and introduce the generalized metric on X,

$$d(y_1, y_2) = \inf\{K \in (0, \infty) : |y_1(u) - y_2(u)| \leq K\beta(u), u \in \mathbb{R}\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$ by $T\ell(u) = \frac{1}{\mu_i}\ell(u^{\mu_i})$, for all $u \in \mathbb{R}$. Now $y_1, y_2 \in X$,

$$\begin{aligned}
d(y_1, y_2) \leq K \Rightarrow & |y_1(u) - y_2(u)| \leq K\beta(u), \in \mathbb{R}.\\
& \Rightarrow \left|\frac{1}{\mu_i}y_1(u^{\mu_i}) - \frac{1}{\mu_i}y_2(u^{\mu_i})\right| \leq \frac{1}{\mu_i}K\beta(u^{\mu_i}), u \in \mathbb{R},\\
& \Rightarrow \left|\frac{1}{\mu_i}y_1(u^{\mu_i}) - \frac{1}{\mu_i}y_2(u^{\mu_i})\right| \leq \kappa K\beta(u), u \in \mathbb{R},\\
& \Rightarrow |Ty_1(u) - Ty_2(u)| \leq \kappa K\beta(u), u \in \mathbb{R},\\
& \Rightarrow d_\beta(y_1, y_2) \leq \kappa K.
\end{aligned}$$

This implies $d(Ty_1, Ty_2) \leq \kappa d(y_1, y_2)$, for all $y_1, y_2 \in X$. i.e., T is a strictly contractive mapping on X with Lipschitz constant κ. Replacing v by u in (22) we get,

$$\left|\ell(u^2) - \ell(1) - 2\ell(u)\right| \leq\ \alpha(u, u). \tag{25}$$

Hence,

$$\begin{aligned} &\left|\ell(u^2) - 2\ell(u)\right| \leq \alpha(u,u) \\ \Rightarrow \quad &\left|\ell(u^2) - 2\ell(u)\right| \leq \alpha(u,u) \\ \Rightarrow \quad &\left|\frac{\ell(u^2)}{2} - \ell(u)\right| \leq \frac{1}{2}\alpha(u,u). \end{aligned}$$

Using (23) for the case $i = 0$ it reduces to

$$\begin{aligned} \left|\frac{\ell(u^2)}{\mu_i} - \ell(u)\right| &\leq \frac{1}{\mu_i}\beta(u^{\mu_i}) \\ |T_\ell(u) - \ell(u)| &\leq \kappa\beta(u) \end{aligned}$$

for all $u \in \mathbb{R}$, i.e., $d(\ell, T_\ell) \leq \kappa = \kappa^1 < \infty$. Again replacing $u = u^{\frac{1}{2}}$ in (25), we get,

$$\begin{aligned} \left|\ell(u) - \ell(1) - 2\ell\left(u^{\frac{1}{2}}\right)\right| &\leq \alpha\left(u^{\frac{1}{2}}, u^{\frac{1}{2}}\right) \\ \left|\ell(u) - 2\ell\left(u^{\frac{1}{2}}\right)\right| &\leq \alpha\left(u^{\frac{1}{2}}, u^{\frac{1}{2}}\right). \end{aligned}$$

Using (23) for the case $i = 1$ it reduces to

$$\begin{aligned} \left|\ell(u) - \frac{1}{\mu_i}\ell\left(u^{\mu_i}\right)\right| &\leq \beta(u) \\ |\ell(u) - T_\ell(u)| &\leq \beta(u) \end{aligned}$$

for all $u \in \mathbb{R}$, i.e., $d(\ell, T\ell) \leq 1 = \kappa^0 < \infty$. Now applying fixed point alternative in above cases and also using $\lim_{n\to\infty} d(T^n\ell, L) = 0$, it follows that there exists a fixed point L of T in X such that

$$L(x) = \lim_{k\to\infty} \frac{\ell(u^{\mu_i^k})}{\mu_i^k} \quad \forall \quad u \in \mathbb{R}. \tag{26}$$

In order to prove $L : \mathbb{R} \to \mathbb{R}$ is logarthamic, replace (u, v) by $\left(u^{\mu_i^k}, v^{\mu_i^k}\right)$ in (22) and dividing by μ_i^k, it follows from (21) and (26), L satisfies (22) for all $u, v \in \mathbb{R}$. i.e., L is logarthamic. Since L is the unique fixed point of T in the set $Y = \{y \in X : d(y_1, y_2) < \infty\}$, using the fixed point alternative result L is the unique function such that

$$|\ell(u) - L(u)| \leq K\beta(u)$$

for all $u \in \mathbb{R}$ and $K > 0$. Again using the fixed point alternative, we obtain

$$d(\ell, L) \leq \frac{1}{1-\kappa} d(\ell, T\ell)$$

this implies

$$d(\ell, L) \leq \frac{\kappa^{1-i}}{1-\kappa}.$$

Hence we conclude that

$$\| \ell(u) - L(u) \| \leq \frac{\kappa^{1-i}}{1-\kappa}\beta(u).$$

This completes the proof of the theorem. □

From the above theorem, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability for the functional equation (22).

Corollary 5.2. *Suppose that a function $\ell : \mathbb{R} \to \mathbb{R}$ satisfies the inequality*

$$\left|\ell(uv) - \ell\left(\frac{u}{v}\right) - \ell(2u)\right| \leq \epsilon\left(|u|^s + |v|^s\right) \tag{27}$$

for all $u, v \in \mathbb{R}$, where $\epsilon > 0, s \neq 2$ are constants. Then there exists a unique logarthamic mapping $L : \mathbb{R} \to \mathbb{R}$ such that

$$|\ell(u) - L(x)| \leq \frac{\epsilon}{1 - |u^{2^s}|} \tag{28}$$

for all $x \in \mathbb{R}$.

Proof. Let $\alpha(u, v) = \epsilon\left(|u|^s + |v|^s\right)$ for all $u, v \in \mathbb{R}$. Then for $s < 2,\ i = 0$ and for $s > 2,\ i = 1$, we obtain

$$\frac{\alpha(u^{\mu_i^k}, v^{\mu_i^k})}{\mu_i^k} = \frac{\epsilon}{\mu_i^k}\left(|u^{\mu_i^k}|^s + |v^{\mu_i^k}|^s\right) = 0 \qquad \text{as} \quad k \to \infty.$$

Thus, (21) holds for all $u, v \in \mathbb{R}$. But we have

$$\beta(u) = \alpha\left(u^{\frac{1}{2}}, v^{\frac{1}{2}}\right).$$

Also it has the property $\frac{1}{\mu_i}\beta(u^{\mu_i}) = \kappa\beta(u)$ for all $u \in X$. Hence

$$\beta(u) = \epsilon\left(|u|^s + |u|^s\right) = 2\epsilon\,|u|^s$$

for all $u \in \mathbb{R}$. Replace u by u^{μ_i} and divide by μ_i, we get

$$\begin{aligned}\frac{1}{\mu_i}\beta(u^{\mu_i}) &= \frac{2\epsilon}{\mu_i}\,|u^{\mu_i}|^s \\ &= \mu_i^{-1}\left\{2\epsilon\,|u^{\mu_i}|^s\right\} \\ &= \begin{cases} u^{2^s}\epsilon & for \quad i = 0 \\ u^{-2^s}4\epsilon & for \quad i = 1 \end{cases}\end{aligned}$$

for all $u \in \mathbb{R}$. Hence (23) holds for $\kappa = u^{\mu_i}$, It follows from (24), we have

$$\begin{aligned}|L(u) - \ell(u)| &\leq \frac{\kappa^{1-i}}{1-\kappa}\beta(u) \\ &= \begin{cases} \frac{(u^{2^s})^{1-0}}{1-u^{2^s}}\epsilon & for \quad i = 0 \\ \frac{(u^{-2^s})^{1-1}}{1-u^{-2^s}}\epsilon & for \quad i = 1 \end{cases} \\ &= \begin{cases} \frac{1}{u^{2^s}-1}\epsilon & for \quad i = 0 \\ \frac{1}{1-u^{-2^s}}\epsilon & for \quad i = 1 \end{cases}\end{aligned}$$

□

6. Application

Consider the logarithmic functional equation (8), that is

$$\ell(uv) + \ell\left(\frac{u}{v}\right) = 2\ell(u)$$

This functional equation can be used to find the consecutive terms of a geometric progression. Since $f(u) = logu$ is the solution of the functional equation, the above equation is written as follows

$$log(uv) + log\left(\frac{u}{v}\right) = 2log(u)$$

Now, let us take the variables as consecutive terms, we note that, **the twice log value of middle term of any three consecutive terms of an geometric progression is always the sum of log values of other two terms.** Any two consecutive terms of an geometric progression differ by the common difference, d. So any three consecutive terms of an geometric progression can be written as

$$\frac{b}{d}, b, bd.$$

The middle term b can be represented by

$$2\ log\ b = log\left(\frac{b}{d}\right) + log(bd)$$

i.e., twice log value of b is the sum of log values of $\frac{b}{d}$ and bd.

References

[1] J.Aczel and J.Dhombres, *Functional Equations in Several Variables*, Cambridge Univ, Press, (1989).

[2] T.Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2(1950), 64-66.

[3] M.Arunkumar and C.Leela Sabari, *Solution and stability of a functional equation originating from a chemical equation*, International Journal Mathematical Sciences and Engineering Applications, 5(II)(2011), 1-8.

[4] M.Arunkumar, S.Hema latha and C.Devi Shaymala Mary, *Functional equation originating from arithmetic Mean of consecutive terms of an arithmetic Progression are stable in banach space: Direct and fixed point method*, JP Journal of Mathematical Sciences, 3(1)(2012), 27-43.

[5] M.Arunkumar and Matina J.Rassias, *Solution and stability of a functional equation Originating from consecutive terms of an Harmonic progression*, Bulletin of the Institute of Mathematics, Academia Sinica, (Submitted).

[6] D.G.Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., 57(1951), 223-237.

[7] S.Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, (2002).

[8] Z.Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci., 14(3)(1991), 431-434.

[9] P.Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184 (1994), 431-436.

[10] D.H.Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., U.S.A., 27(1941), 222-224.

[11] D.H.Hyers, G.Isac and Th.M.Rassias, *Stability of functional equations in several variables*, Birkhauser, Basel, (1998).

[12] S.M.Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, (2001).

[13] Pl.Kannappan, *Quadratic functional equation inner product spaces*, Results Math., 27(3-4)(1995), 368-372.

[14] Pl.Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, (2009).

[15] K.W.Jun, S.M.Jung and Y.H.Lee, A generalization of the Hyers-Ulam-Rassias stability of a functional equation of Davison, J. Korean Math. Soc., 41(3)(2004), 501-511.

[16] E.H.Lee, I.S.Chang and Y.S.Jung, *On stability of the functional equations having relation with a multiplicative derivation*, Bull. Korean Math. Soc., 44(1)(2007), 185-194.

[17] Y.H.Lee and K.W.Jun, *A note on the Hyers-Ulam-Rassias stability of Pexider equation*, J. Korean Math. Soc., 37(1)(2000), 111-124.

[18] Y.W.Lee, *Stability of a generalized quadratic functional equation with Jensen type*, Bull. Korean Math. Soc., 42(1)(2005), 57-73.

[19] T.Miura, S.M.Jung and S.E.Takahasi, *Hyers-Ulam-Rassias stability of the Banach space valued linear differential equations* $y' = \rho y$, J. Korean Math. Soc., 41(6)(2004), 995-1005.

[20] J.M.Rassias, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, 46(1982), 126-130.

[21] Th.M.Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc., 72(1978), 297-300.

[22] Th.M.Rassias and P.Semrl, *On the behavior of mappings which do not satisfy Hers- Ulam stability*, Proc. Amer. Math. Soc., 114(1992), 989-993.

[23] Th.M.Rassias and P.Semrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc., 114(4)(1992), 989-993.

[24] K.Ravi, M.Arunkumar and J.M.Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, 3(08)(2008), 36-47.

[25] J.Rho and H.J.Shin, *Approximation of Cauchy additive mappings*, Bull. Korean Math. Soc., 44(4)(2007), 851-860.

[26] S.M.Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, NewYork, (1964).