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Solution of Fuzzy Game Problem Using Hexagonal Fuzzy Numbers

Research Article*

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Abstract: In this paper, we consider a two person zero sum game with imprecise values in payoff matrix. The imprecise values are assumed to be hexagonal fuzzy numbers. An approach for solving problems by using ranking of the fuzzy numbers has been considered to solve the fuzzy game problem. By using ranking to the payoffs we convert the fuzzy value game problem to crisp valued game problem, which can be solved using the traditional method minimax-maximin principle.

Keywords: Fuzzy Set, Fuzzy Numbers, Hexagonal Fuzzy Numbers, Ranking of Fuzzy Numbers, Fuzzy Game Problem, pay off matrix.

1. Introduction

The mathematical treatment of the Game theory was made available in 1944, when John Von Newmann and Oscar Morgenstern [11] published the famous article 'Theory of games and economic Behavior'. The problem of Game theory defined as a body of knowledge that deals with making decisions when two or more intelligent and rational opponents are involved under conditions of conflict and competition. When we apply the Game theory to model some practical problems which we encounter in real situations, we have to know the values of payoffs exactly. However, it is difficult to know the exact values of payoffs and we could only know the values of payoffs approximately. In such situations, it is useful to model the problems as games with fuzzy payoffs. In a fuzzy game problem, all parameters are fuzzy numbers. Fuzzy numbers may be normal (or) abnormal, triangular or trapezoidal or hexagonal or octagonal. Basirzadeh [5] have proposed a method for ranking fuzzy numbers using α-cuts in which he has given a ranking for triangular and trapezoidal fuzzy numbers. A ranking using α-cuts was introduced on hexagonal fuzzy numbers. Using this ranking, the fuzzy Game problem is converted to a crisp value problem, which can be solved using the traditional method.

2. Preliminaries

Definition 2.1. *Let X be a nonempty set. A fuzzy set "A" in X is characterized by its membership function $A : X \to [0,1]$ and $A(x)$ is interpreted as the degree of membership of element x in fuzzy A for each $x \in X$. The value zero is used to represent complete non-membership; the value one is used to represent complete membership and values in between are used to represent intermediate degrees of membership. The mapping A is also called the membership function of fuzzy set A.*

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Definition 2.2. *A crisp set is a special case of a fuzzy set, in which the membership function only takes two values, commonly defined as 0 and 1.*

Definition 2.3. *A Fuzzy number "A" is a convex normalized fuzzy set on the real line R such that:*

1. *There exist at least one $x_0 \in R$ with $\mu_A(x_0) = 1$.*

2. *$\mu_A(x)$ is piecewise continuous.*

Definition 2.4. *A triangular fuzzy number $\tilde{A}$ denoted by (a_1, a_2, a_3), and the membership function is defined as*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & for\ a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & for\ a_2 \le x \le a_3 \\ 0, & otherwise \end{cases}$$

Definition 2.5. *A trapezoidal fuzzy number $\tilde{A}$ can be defined as (a_1, a_2, a_3, a_4), and the membership function is defined as*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & for\ a_1 \le x \le a_2 \\ 1, & for\ a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3}, & for\ a_3 \le x \le a_4 \\ 0, & elsewhere \end{cases}$$

3. Hexagonal Fuzzy Numbers [6, 7]

A fuzzy number is a hexagonal fuzzy number denoted by $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ where a_1, a_2, a_3, a_4, a_5, a_6 are real numbers and its membership function $\mu_{\tilde{A}_H}(x)$ is given below.

$$\mu_{\tilde{A}_H}(x) = \begin{cases} 0, & for\ x < a_1 \\ \frac{1}{2}\frac{(x-a_1)}{(a_2-a_1)}, & for\ a_1 \le x \le a_2 \\ \frac{1}{2} + \frac{1}{2}\frac{(x-a_2)}{(a_3-a_2)}, & for\ a_2 \le x \le a_3 \\ 1, & for\ a_3 \le x \le a_4 \\ 1 - \frac{1}{2}\frac{(x-a_4)}{(a_5-a_4)}, & for\ a_4 \le x \le a_5 \\ \frac{1}{2}\frac{(a_6-x)}{(a_6-a_5)}, & for\ a_5 \le x \le a_6 \\ 0, & for\ x > a_6 \end{cases}$$

Definition 3.1. *A Hexagonal fuzzy number denoted by $\tilde{A}_H$ is defined as $\tilde{A}_w = (P_1(u), Q_1(v), Q_2(v), P_2(u))$ for $u \in [0, 0.5]$ and $v \in [0.5, w]$ where,*

(1). $P_1(u)$ is a bounded left continuous non decreasing function over [0, 0.5].

(2). $Q_1(v)$ is a bounded left continuous non decreasing function over [0.5, w].

(3). $Q_2(v)$ is a bounded continuous non increasing function over [w, 0.5].

(4). $P_2(u)$ is a bounded left continuous non increasing function over [0.5, 0].

Remark 3.2. *If $w = 1$, then the hexagonal fuzzy number is called a normal hexagonal fuzzy number.*

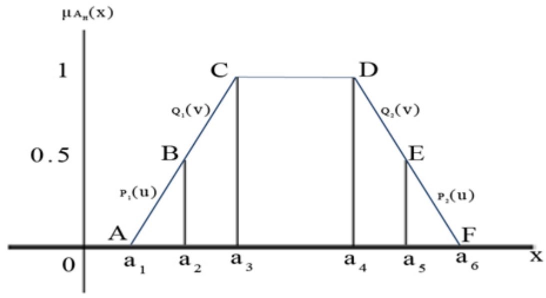

Figure 1. **Graphical representation of a normal hexagonal fuzzy number for $x \in [0,1]$**

Remark 3.3. *Hexagonal fuzzy number $\tilde{A}_H$ is the ordered quadruple $P_1(u)$, $Q_1(v)$, $Q_2(v)$, $P_2(u)$ for $u \in [0, 0.5]$ and $v \in [0.5, w]$, where*

$$P_1(u) = \frac{1}{2}\frac{(u - a_1)}{(a_2 - a_1)}; \qquad Q_1(v) = \frac{1}{2} + \frac{1}{2}\frac{(x - a_2)}{(a_3 - a_2)};$$
$$P_2(u) = \frac{1}{2}\frac{(a_6 - x)}{(a_6 - a_5)}; \qquad Q_2(v) = 1 - \frac{1}{2}\frac{(x - a_4)}{(a_5 - a_4)}.$$

Remark 3.4. *Membership function $\mu_{\tilde{A}_H}(x)$ are continuous functions.*

Definition 3.5. *A positive hexagonal fuzzy number $\tilde{A}_H$ is denoted as $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ where all $a_i > 0$ for all $i = 1, 2, 3, 4, 5, 6$. For example $\tilde{A}_H = (1, 2, 3, 4, 5, 6)$.*

Definition 3.6. *A negative hexagonal fuzzy number $\tilde{A}_H$ is denoted as $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ where all $a_i < 0$ for all $i = 1, 2, 3, 4, 5, 6$. For example $\tilde{A}_H = (-8, -7, -6, -4, -3, -2)$.*

Note 3.7. *A negative hexagonal fuzzy number can be written as the negative multiplication of a positive hexagonal fuzzy number. For example $\tilde{A}_H = (-2, -4, -6, -8, -10, -12)$. Then $\tilde{A}_H = -(2, 4, 6, 8, 10, 12)$.*

Definition 3.8. *Let $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $\tilde{B}_H = (b_1, b_2, b_3, b_4, b_5, b_6)$ be two hexagonal fuzzy number, If $\tilde{A}_H$ is identically equal to $\tilde{B}_H$ only if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$, $a_5 = b_5$, $a_6 = b_6$.*

Definition 3.9 (Symmetric Image)**.** *If $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ is the hexagonal fuzzy number then $-\tilde{A}_H = (-a_6, -a_5, -a_4, -a_3, -a_2, -a_1)$ which is the symmetric image of $\tilde{A}_H$ is also a hexagonal fuzzy number.*

Example 3.10. *If $\tilde{A}_H = (1, 2, 3, 5, 6, 7)$ then $-\tilde{A}_H = (-7, -6, -5, -3, -2, -1)$ which is again a hexagonal fuzzy number.*

4. Alpha Cut

The classical set $\tilde{A}_\alpha$ called alpha cut set is the set of elements whose degree of membership is the set of elements whose degree of membership in $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ is no less than α, it is defined as

$$\tilde{A}_\alpha = \{x \in X / \mu_{\tilde{A}_H}(x) \geq \alpha\}$$
$$= \begin{cases} [P_1(\alpha), P_2(\alpha)], & for\ \alpha \in [0,\ 0.5) \\ [Q_1(\alpha), Q_2(\alpha)], & for\ \alpha \in (0.5,\ 1] \end{cases}$$

Definition 4.1 (Operations of Hexagonal Fuzzy numbers). *Following are the three operations that can be performed on hexagonal fuzzy numbers.*

Suppose $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ *and* $\tilde{B}_H = (b_1, b_2, b_3, b_4, b_5, b_6)$ *are two hexagonal fuzzy numbers then*

Addition:

$$\tilde{A}_H(+)\tilde{B}_H = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6)$$

Subtraction:

$$\tilde{A}_H(-)\tilde{B}_H = (a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4, a_5 - b_5, a_6 - b_6)$$

Multiplication:

$$\tilde{A}_H(*)\tilde{B}_H = (a_1 * b_1, a_2 * b_2, a_3 * b_3, a_4 * b_4, a_5 * b_5, a_6 * b_6)$$

Definition 4.2 (α-cut of a normal hexagonal fuzzy number). *The* α*-cut of a normal hexagonal fuzzy number* $\tilde{A}_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ *given by the definition (i.e.)* $w = 1$ *for all* $\alpha \in [0, 1]$ *is*

$$A_\alpha = \begin{cases} [2\alpha\,(a_2 - a_1) + a_1, -2\alpha(a_6 - a_5) + a_6], & for\ \alpha \in [0,\ 0.5) \\ [2\alpha(a_3 - a_2) - a_3 + 2a_2, -2(a_5 - a_4) + 2a_5 - a_4], & for\ \alpha \in [0.5,\ 1] \end{cases}$$

5. Ranking Of Hexagonal Fuzzy Numbers

Definition 5.1 ([1]). *The measure of a hexagonal fuzzy number is obtained by the average of the two fuzzy side areas, left side area and right side area, from membership function to* α*-axis.*

Definition 5.2 ([13]). *Let* $\tilde{A}$ *be a normal hexagonal fuzzy number. The value* $M_0^{hex}(\tilde{A})$*, called the measure of* $\tilde{A}$ *is calculated as follows:*

$$M_0^{hex}(\tilde{A}) = \frac{3\sqrt{3}}{4}\left[(a_1 + a_3 + a_6)\,(k) + (a_2 + a_4 + a_5)\,(1 - k)\right], \quad where\ \ 0 \le k \le 1.$$

Definition 5.3 (Pure strategy [1]). *Pure strategy is a decision making rule in which one particular course of action is selected.*

For fuzzy games the min-max principle is described by Nishizaki. The course of the fuzzy game is determined by the desire of A to maximize his gain and that of restrict his loss to a minimum.

Definition 5.4 (Saddle point [1]). *If in a game, the max-min value equals the mini-max value, then the game is said to have a saddle point and the corresponding strategies which give the saddle point are called optimal strategies. The amount of payoff at an equilibrium point is called the crisp game value of the game matrix.*

6. Solution of All 2×2 Matrix Game [2]

Consider the general 2×2 game matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. To solve this game we proceed as follows:

- Test for a saddle point.
- If there is no saddle point, solve by finding equalizing strategies.

The Optimal mixed strategies for player $A = (p_1, p_2)$ and for player $B = (q_1, q_2)$, where

$$p_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}; \quad p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}; \quad q_2 = 1 - q_1 \text{ and}$$

$$\text{Value of the game V} = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Example 6.1. *Consider the following fuzzy game problem.*

$$\begin{array}{cc} & \textit{Player B} \\ \textit{Player A} & \begin{pmatrix} (2,4,6,8,10,14) & (3,5,11,12,15,18) \\ (-6,-5,6,8,9,12) & (-2,-1,0,1,2,4) \end{pmatrix} \end{array}$$

Solution: By definition of hexagonal fuzzy number $\tilde{A}$ is calculated as

Step 1: Convert the given fuzzy problem into a crisp value problem. This problem is done by taking the value of k as 0.2, we obtain the value of $M_0^{hex}(a_{ij})$.

$a_{11} = (2,\ 4,\ 6,\ 8,\ 10,\ 14)$	$M_0^{hex}(a_{ij}) = \frac{3\sqrt{3}}{4}[(2+6+14)(0.2) + (4+8+10)(0.8)]$ $= \frac{3\sqrt{3}}{4}[22(0.2) + 22(0.8)] = 28.5$
$a_{12} = (3,\ 5,\ 11,\ 12,\ 15,\ 18)$	$M_0^{hex}(a_{ij}) = \frac{3\sqrt{3}}{4}[(3+11+18)(0.2) + (5+12+15)(0.8)]$ $= \frac{3\sqrt{3}}{4}[32(0.2) + 32(0.8)] = 41.5$
$a_{21} = (-6, -5,\ 6,\ 8,\ 9,\ 12)$	$M_0^{hex}(a_{ij}) = \frac{3\sqrt{3}}{4}[(-6+6+12)(0.2) + (-5+8+9)(0.8)]$ $= \frac{3\sqrt{3}}{4}[12(0.2) + 12(0.8)] = 15.5$
$a_{22} = (-2, -1,\ 0,\ 1,\ 2,\ 4)$	$M_0^{hex}(a_{ij}) = \frac{3\sqrt{3}}{4}[(3+11+18)(0.2) + (5+12+15)(0.8)]$ $= \frac{3\sqrt{3}}{4}[2(0.2) + 2(0.8)] = 2.5$

Since the condition $a_1 + a_3 + a_6 = a_2 + a_4 + a_5$ is satisfied by all the hexagonal numbers for any value of k. We will get the same matrix as below.

Step 2: The pay-off matrix is

$$\begin{array}{cc} & \textit{Player B} \\ \textit{Player A} & \begin{pmatrix} 28.5 & 41.5 \\ 15.5 & 2.5 \end{pmatrix} \end{array}$$

Minimum of 1st row = 28.5; Maximum of 1st column = 28.5

Minimum of 2nd row = 2.5; Maximum of 2nd column = 41.5

max(min) = 28.5; min(max) = 28.5

It has saddle Point. The Crisp solution to the problem is saddle point = (A_1, B_1). Therefor the **Value of the game** = 28.5.

Example 6.2. *Consider the following fuzzy game problem.*

$$\begin{array}{cc} & \textit{Player B} \\ \textit{Player A} & \begin{pmatrix} (-4,-3,1,2,3,5) & (8,9,10,11,12,14) \\ (4,5,6,7,10,12) & (0,1,2,3,8,10) \end{pmatrix} \end{array}$$

Solution: By definition of hexagonal fuzzy number $\tilde{A}$ is calculated as

Step 1: Convert the given fuzzy problem into a crisp value problem. This problem is done by taking the value of k as 0.2, we obtain the value of $M_0^{hex}(a_{ij})$.

$a_{11} = (-4,\ -3,\ 1,\ 2,\ 3,\ 5)$	$M_0^{hex}(a_{ij}) = \frac{3\sqrt{3}}{4}[(-4+1+5)(0.2)+(-3+2+3)(0.8)$ $= \frac{3\sqrt{3}}{4}[2(0.2)+2(0.8)] = 2.5$
$a_{12} = (8,\ 9,\ 10,\ 11,\ 12,\ 14)$	$M_0^{hex}(a_{ij}) = \frac{3\sqrt{3}}{4}[(8+10+14)(0.2)+(9+11+12)(0.8)]$ $= \frac{3\sqrt{3}}{4}[32(0.2)+32(0.8)] = 41.5$
$a_{21} = (4,\ 5,\ 6,\ 7,\ 10,\ 12)$	$M_0^{hex}(a_{ij}) = \frac{3\sqrt{3}}{4}[(4+6+12)(0.2)+(5+7+10)(0.8)]$ $= \frac{3\sqrt{3}}{4}[22(0.2)+22(0.8)] = 28.5$
$a_{22} = (0,\ 1,\ 2,\ 3,\ 8,\ 10)$	$M_0^{hex}(a_{ij}) = \frac{3\sqrt{3}}{4}[(0+2+10)(0.2)+(1+3+8)(0.8)]$ $= \frac{3\sqrt{3}}{4}[12(0.2)+12(0.8)] = 15.5$

Since the condition $a_1 + a_3 + a_6 = a_2 + a_4 + a_5$ is satisfied by all the hexagonal numbers for any value of k. We will get the same matrix as below.

Step 2: The pay-off matrix is

$$\begin{array}{cc} & \textit{Player B} \\ \textit{Player A} & \begin{pmatrix} 2.5 & 41.5 \\ 28.5 & 15.5 \end{pmatrix} \end{array}$$

Minimum of 1st row = 2.5; Maximum of 1st column = 28.5

Minimum of 2nd row = 15.5; Maximum of 2nd column = 41.5

max(min) = 15.5; min(max) = 28.5

It has no saddle Point.

Step 3: To find Optimum mixed strategy and value of the game. Here $a_{11} = 2.5$, $a_{12} = 41.5$, $a_{21} = 28.5$, $a_{22} = 15.5$.

$$p_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{15.5-41.5}{(2.5+15.5)-(41.5+28.5)} = \frac{1}{2}; \qquad p_2 = 1-p_1 = 1-\frac{1}{2} = \frac{1}{2};$$

$$q_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{15.5-28.5}{(2.5+15.5)-(41.5+28.5)} = \frac{1}{4}; \qquad q_2 = 1-q_1 = 1-\frac{1}{4} = \frac{3}{4}$$

$$\text{Strategy for player A} = (p_1, p_2) = \frac{1}{2}, \frac{1}{2}$$

$$\text{Strategy for player B} = (q_1, q_2) = \frac{1}{4}, \frac{3}{4}$$

$$\begin{aligned} \text{Value of the game V} &= \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \\ &= \frac{(2.5)(15.5)-(41.5)(28.5)}{(2.5+15.5)-(41.5+28.5)} \\ &= \frac{38.5-1182.75}{18-70} \\ &= \frac{-1144}{-52} \end{aligned}$$

Value of the game V = 22.

Remark 6.3. *If the hexagonal numbers are slightly modified so that the condition* $a_1 + a_3 + a_6 \neq a_2 + a_4 + a_5$ *is not satisfied, then for such a problem the solution for different values of k* $(0 \leq k \leq 1)$ *can be easily checked to lie in a finite interval.*

Remark 6.4. *In the above two examples we have considered only* 2×2 *fuzzy games but the method applied here can be used to solve any* $m \times n$ *fuzzy game.*

7. Conclusion

- In this paper, a method of solving fuzzy game problem using ranking of fuzzy numbers has been considered.
- The parameter k can be modified suitably by the decision maker to get the desired result.
- We may get different fuzzy game value for different values of k for the same fuzzy game.

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