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A Batch Arrival Two Types of Bulk Service Queue with Server Breakdown and Modified M-Vacation

Research Article*

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Abstract: In this paper a batch arrival two types of general bulk service queuing system with server breakdown and modified M-vacation is considered. The service is done in bulk with a minimum of 'a' customer and a maximum of 'b' customers. At first the server provides first essential service (FES) bulk service) to the arriving customers and sequentially the server must provide second essential services (SES). At the completion of two types of service, if the server is breakdown with probability π, then the renovation of service station will be consider. After completing the renovation of service station or there is no breakdown of the service with probability $(1-\pi)$, the queue length is less than 'a' then the server will avail of multiple vacations till the queue length reaches 'a'' or consequently he completes M number of vacations, which occurs first, After completing the M-th vacation, if the queue length is still less than a then the server remains in the system till it reaches a (this period is known as dormant period). At a vacation completion epoch or a two type of services completion epoch or during the dormant period, if the queue length ξ is at least a, the server service a batch of minimum (ξ, b) customers where $b \geq a$. The probability generating function of queue science at the random epoch is obtained.

Keywords: Batch arrival, bulk service, breakdown, M-vacation.

1. Introduction

Application of server vacation models can be found in manufacturing systems, designing of local area networks and data communication systems. Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. Very few authours are works on queueing systems with closedown time. An M/G/1 queue is analyzed by Takagi considered closedown time and set up time. It is observed that most of the studies on vacation queue are concentrated only on single server on single arrival and single vacation. Once the arrival occur in bulk one expect that the server can also be done in bulk.

Li and Zhu (1997) investigated a single arrival, single service finite queue with generalized vacations and exhaustive service where arrival rates depend on the number of customers in the system. Batch arrival queueing systems with vacations were developed by several researches such as Lee.et. al (1994), Ke and Chang (2009) etc. Parathasarathy and Sudesh (2010) have discussed a state-dependent queue alternating between arrivals and services, in which they obtained time-dependent system size probabilities and the duration of the busy period in a close form. Wang et.al (2007) have analyzed a single unreliable server in an M^x/M/1 queueing system with multiple vacations, Ke (2007) discussed the operating characteristics

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of an $M^{[x]}$G/1 queueing system under vacation policies with startup/closedown times are generally distributed. Choudhury and Madan (2007) have considered a batch arrival but single service Bernoulli vacation queue, with a random setup time under restricted admissiability policy. Sikdar and Gupta (2008) have discussed on the batch arrival, batch service queue, but finite buffer under server's vacation. $M^{[x]}/M^{[y]}$/1/N queue. Jain and Upadhyaya (2010) considered with the modified Bernoulli vacation schedule for the unreliable server batch arrival queueing system with essential and milti-optional services under N- policy.

1.1. Notations

Let X be the group size random variable of the arrival λ, be the Poisson arrival rate. g_k be the probability that 'k' customers arrive in a batch and $X(z)$ be its probability generating function (PGF). $S(\cdot)$, $V(\cdot)$, $R(\cdot)$ Cumulative distribution function of service time, vacation time, renovation time. $s(x)$, $v(x)$, $r(x)$ Probability density function of S, V and R. $\tilde{S}(\theta)$, $\tilde{V}(\theta)$, $\tilde{R}(\theta)$ be the Laplace-Stieltjes transform of S, V, R, $S^0(t)$, $V^0(t)$, $R^0(t)$ Remaining service time of a batch in service time, vacation time, renovation time at time 't'

$N_s(t)$ = Number of customers in the service at time t

$N_q(t)$ = Number of customers in the queue at time t

The different states of the server at time t are defined as follows

$C(t) = 0$; if the server is busy with FES

1; if the server is busy with SES

2 ; if the server is on vacation

3; if the server is on dormant period

4; if the server is on renovation time

$z(t) = j$, if the server is on j$^{\text{th}}$ vacation starting from the idle period. To obtain the system equations, the following state probabilities are defined;

- $P^1_{i,n}(x,t)\,dt = P\left\{N_s(t) = i,\ N_q(t) = n,\ x \le S^0(t) \le x+dt,\ C(t) = 0\right\},\quad a \le i \le b,\ n \ge 0,$
- $P^2_{i,n}(x,t)\,dt = P\left\{N_s(t) = i,\ N_q(t) = n,\ x \le S^0(t) \le x+dt,\ C(t) = 1\right\},\quad a \le i \le b,\ n \ge 0,$
- $Q_n(x,t)\,dt = P\left\{N_q(t) = n,\ x \le V^0(t) \le x+dt,\ C(t) = 2\right\},\quad n \ge 0,$
- $T_n(t)\,dt = P\{N_q(t) = n,\ C(t) = 3\},\quad 0 \le n \le a-1,$
- $R_n(x,t)\,dt = P\left\{N_q(t) = n,\ x \le R^0(t) \le x+dt,\ C(t) = 4\right\},\quad n \ge 0.$

Now, the following system equations are obtained for the queueing system, using supplementary variable technique:

$$T_0(t+\Delta t) = T_0(t)(1-\lambda\Delta t) + Q_{M,0}(0,t)\,\Delta t$$

$$T_n(t+\Delta t) = T_n(t)(1-\lambda\Delta t) + Q_{M,n}(0,t)\,\Delta t + \sum_{k=1}^{n} T_{n-k}(t)\,\lambda g_k \Delta t,\quad 1 \le n \le a-1,$$

$$P_{i,0}{}^1(x-\Delta t, t+\Delta t) = P_{i,0}{}^1(x,t)(1-\lambda\Delta t) + \sum_{m=a}^{b} P_{i,0}{}^2(0,t)\,s_1(x)\,\Delta t$$
$$+ \sum_{k=1}^{M} Q_{kI}(0,t)\,\lambda g_{i-k} s(x)\,\Delta t + \sum_{m=0}^{a-1} T_m(t)\,\lambda g_{i-m} s_1(x)\Delta t,\quad a \le i \le b$$

$$P_{i,j}{}^1(x-\Delta t, t+\Delta t) = P_{i,j}{}^1(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} P_{i,j-k}{}^1(x,t)\,\lambda g_k \Delta t;\quad a \le i \le b-1; j \ge 1$$

$$P_{b,j}{}^{1}(x-\Delta t,t+\Delta t)=P_{b,j}{}^{1}(x,t)(1-\lambda\Delta t)+\sum_{m=a}^{b}P_{m,b+j}{}^{2}(0,t)s_1(x)\Delta t+\sum_{k=1}^{j}P_{b,j-k}{}^{1}(x,t)\lambda g_k\Delta t$$
$$+\sum_{m=0}^{a-1}T_m(t)\lambda g_{b+j-m}s_1(x)\Delta t+\sum_{j=0}^{M}Q_{l,b+j}(0,t)\lambda g_{b+j-k}s_1(x)\Delta t,\quad j\geq 1$$
$$P_{i,0}{}^{2}(x-\Delta t,t+\Delta t)=P_{i,0}{}^{2}(x,t)(1-\lambda\Delta t)+P_{i,0}{}^{1}(x,t)s_2(x)\Delta t+R_n(0,t)s_2(x)\Delta t,\quad a\leq i\leq b$$
$$P_{i,j}{}^{2}(x-\Delta t,t+\Delta t)=P_{i,j}{}^{2}(x,t)(1-\lambda\Delta t)+P_{i,j}{}^{1}(x,t)s_2(x)\Delta t++\sum_{k=1}^{j}P_{b,j-k}{}^{2}(x,t)\lambda g_k\Delta t+R_n(0,t)s_2\Delta t,\quad j\geq 1$$
$$Q_{l,0}(x-\Delta t,t+\Delta t)=Q_{l,0}(x,t)(1-\lambda\Delta t)+R_0(0,t)v(x)\Delta t$$
$$Q_{l,n}(x-\Delta t,t+\Delta t)=Q_{l,n}(x,t)(1-\lambda\Delta t)+(1-\pi)\sum_{m=a}^{b}P_{m,n}{}^{2}(0,t)v(x)\Delta t$$
$$+\sum_{k=1}^{n}Q_{l,n-k}(x,t)\lambda g_k\ \Delta t+R_n(0,t)v(x)\Delta t,\quad 1\leq n\leq a-1$$
$$Q_{l,n}(x-\Delta t,t+\Delta t)=Q_{l,n}(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{n}Q_{l,n-k}(x,t)g_k\lambda\Delta t+R_n(0,t)v(x)\Delta t;\quad n\geq a$$
$$Q_{j,0}(x-\Delta t,t+\Delta t)=Q_{j,0}(x,t)(1-\lambda\Delta t)+Q_{j-1,0}(0,t)v(x)\Delta t;\quad 2\leq j\leq M$$
$$Q_{j,n}(x-\Delta t,t+\Delta t)=Q_{j,n}(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{n}Q_{j,n-k}(x,t)\lambda g_k\ \Delta t+Q_{j-1,n}(0,t)v(x);\quad 1\leq n\leq a-1,\ 2\leq j\leq M$$
$$Q_{j,n}(x-\Delta t,t+\Delta t)=Q_{j,n}(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{n}Q_{j,n-k}(x,t)\lambda g_k\Delta t;\quad n\geq a,2\leq j\leq M$$
$$R_0(x-\Delta t,t+\Delta t)=R_0(x,t)(1-\lambda\Delta t)+\pi\sum_{m=a}^{b}P_{m,0}{}^{2}(0,t)r(x)\Delta t$$
$$R_n(x-\Delta t,t+\Delta t)=R_n(x,t)(1-\lambda\Delta t)+\pi\sum_{m=a}^{b}P_{m,n}{}^{2}(0,t)r(x)\Delta t+\sum_{k=1}^{j}R_{n-k}(x,t)\lambda g_k\Delta t,\quad n\leq a-1$$
$$R_n(x-\Delta t,t+\Delta t)=R_n(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{j}R_{n-k}(x,t)\lambda g_k\Delta t,\quad n\geq a.$$

2. Steady State Queue Size Distribution

From the above equations, the steady state queue size equations are obtained as follows:

$$0=-\lambda_0T_0+Q_{M,0}(0)\tag{1}$$
$$0=-\lambda_0T_n+Q_{M,n}(0)+\sum_{k=1}^{n}T_{n-k}\lambda_0g_k,\quad 1\leq n\leq a-1\tag{2}$$
$$-\frac{dy}{dx}P_{i,0}^{1}(x)=-\lambda P_{i,0}{}^{1}(x)+\sum_{m=a}^{b}P_{m,i}{}^{2}(0)s_1(x)\Delta t+\sum_{l=1}^{M}Q_k(0)s_1(x)+\lambda\sum_{m=0}^{a-1}T_m\lambda g_{i-m}s_1(x)\quad a\leq i\leq b\tag{3}$$
$$-\frac{dy}{dx}P_{i,j}^{1}(x)=-\lambda P_{i,0}{}^{1}(x)+\sum_{k=1}^{j}P_{i,j-k}{}^{1}(x)\lambda g_k;\quad a\leq i\leq b-1\ ,j\geq 1\tag{4}$$
$$-\frac{dy}{dx}P_{b,j}^{1}(x)=-\lambda P_{b,j}{}^{1}(x)+\sum_{k=1}^{j}P_{b,j-k}{}^{1}(x)\lambda g_k+\sum_{m=a}^{b}P_{m,b+j}{}^{2}(0)s_1(x)$$
$$+\sum_{m=0}^{n-1}T_m\lambda g_{b+j-m}\ s_1(x)+\sum_{i=1}^{M}Q_{l,b+i}(0)s_1(x);\quad j\geq 1\tag{5}$$
$$-\frac{dy}{dx}P_{i,0}^{2}(x)=-\lambda P_{i,0}{}^{2}(x)+P_{i,0}{}^{1}(x,t)s_2(x)+R_i(0)s_2(x),\quad a\leq i\leq b\tag{6}$$
$$-\frac{dy}{dx}P_{i,j}^{2}(x)=P_{i,j}{}^{2}(x,t)+P_{i,j}{}^{1}(x,t)s_2(x)+\sum_{k=1}^{j}P_{b,j-k}{}^{2}(x,t)\lambda g_k+R_{i,j}(0)s_2(x),\quad j\geq 1\tag{7}$$

$$-\frac{dy}{dx}Q_{l,0}(x) = -\lambda Q_{l,0}(x) + (1-\pi)\sum_{m=a}^{b} P_{m,0}{}^{2}(0)\,v(x) + R_0(0)\,v(x) \tag{8}$$

$$-\frac{dy}{dx}Q_{l,n}(x) = -\lambda Q_{l,n}(x) + (1-\pi)\sum_{m=a}^{b} P_{m,0}{}^{2}(0)\,v(x) + \sum_{k=1}^{n} Q_{l,n-k}(x)\,\lambda g_k + R_n(0)\,v(x);\ \ 1\le n\le a-1 \tag{9}$$

$$-\frac{dy}{dx}Q_{l,n}(x) = -\lambda Q_{l,n}(x) + \sum_{k=1}^{n} Q_{l,n-k}(x)\,\lambda g_k + R_n(0)\,v(x);\ \ n\ge a \tag{10}$$

$$-\frac{dy}{dx}Q_{j,0}(x) = -\lambda Q_{j,0}(x) + Q_{j-1,0}(x)\,v(x);\ \ 2\le j\le M \tag{11}$$

$$-\frac{dy}{dx}Q_{j,n}(x) = -\lambda Q_{j,n}(x) + Q_{j-1,0}(x)\,v(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\,\lambda g_k\ ;\ \ 1\le n\le a-1,\ \ 2\le j\le M \tag{12}$$

$$-\frac{dy}{dx}Q_{j,n}(x) = -\lambda Q_{j,n}(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\,\lambda g_k;\ \ n\ge a,\ 2\le j\le M \tag{13}$$

$$-\frac{dy}{dx}R_{0}(x) = -\lambda R_0(x) + \pi\sum_{m=a}^{b} P_{m,0}{}^{2}(0)\,r(x), \tag{14}$$

$$-\frac{dy}{dx}R_{n}(x) = -\lambda R_n(x) + \pi\sum_{m=a}^{b} P_{m,0}{}^{2}(0)\,r(x) + \sum_{k=1}^{n} R_{n-k}(x)\lambda g_k;\ \ 1\le n\le a-1 \tag{15}$$

$$-\frac{dy}{dx}R_{n}(x) = -\lambda R_n(x) + \sum_{k=1}^{n} R_{n-k}(x)\lambda g_k,\ \ n\ge a \tag{16}$$

The Laplace-Stieltjes transforms of $P_{i,n}(x)$ and $Q_j(x)$ are defined as:

$$\tilde{P}_{i,n}^{i}(\theta) = \int_0^{8} e^{-\theta x} P_{i,n}{}^{i}(x)\,dx,\ \ i=1,2;\ \tilde{Q}_j(\theta) = \int_0^{8} e^{-\theta x} Q_j(x)\,dx \tag{17}$$

and $\tilde{R}_n(\theta) = \int_0^{\infty} e^{-\theta x} R_n(x)\,dx$. Taking Laplace-Stieltjes transform on both sides, we get

$$\theta\tilde{P}_{i,0}^{1}(\theta) - P_{i,0}(0) = \lambda\tilde{P}_{i,0}^{1}(\theta) - \left[\sum_{m=a}^{b} P_{m,i}(0) + \sum_{l=1}^{M} Q_{l,i}(0)\,\lambda g_{i-k} + \sum_{m=0}^{a-1} T_m\lambda g_{i-m}\right]\tilde{S}_1(\theta);\ \ a\le i\le b, \tag{18}$$

$$\theta\tilde{P}_{i,j}^{1}(\theta) - P_{i,j}{}^{1}(0) = \lambda\tilde{P}_{i,j}^{1}(\theta) - \lambda\sum_{k=1}^{j}\tilde{P}_{i,j-k}^{1}(\theta)g_k,\ \ a\le i< b-1,\ j\ge 1 \tag{19}$$

$$\begin{aligned}\theta\tilde{P}_{b,j}^{1}(\theta) - P_{b,j}{}^{1}(0) &= \lambda\tilde{P}_{b,j}^{1}(\theta) - \sum_{k=1}^{j}\tilde{P}_{b,j-k}^{1}(\theta)\lambda g_k - \sum_{m=a}^{b} P_{m,b+j}{}^{2}(0)\,\tilde{S}_1(\theta)\\ &\quad - \sum_{m=0}^{n-1} T_m\ \lambda g_{b+j-m}\tilde{S}_1(\theta) - \sum_{l=1}^{M}\tilde{Q}_{l,b+j}(\theta)\,\tilde{S}_1(\theta),\ \ j\ge 1\end{aligned} \tag{20}$$

$$\theta\tilde{P}_{i,0}^{2}(\theta) - P_{i,0}{}^{2}(0) = \lambda\tilde{P}_{i,j}^{1}(\theta) - P_{i,0}{}^{1}(0)\,\tilde{S}_2(\theta) - R_i(0)\,\tilde{S}_2(\theta);\ \ a\le i< b \tag{21}$$

$$\theta\tilde{P}_{i,j}^{2}(\theta) - P_{i,j}{}^{2}(0) = \lambda\tilde{P}_{i,j}^{2}(\theta) - \lambda\sum_{k=1}^{j}\tilde{P}_{i,j-k}^{2}(\theta)g_k - P_{i,0}{}^{2}(0)\,\tilde{S}_2(\theta) - R_i(0)\,\tilde{S}_2(\theta);\ \ a\le i< b,\ j\ge 1 \tag{22}$$

$$\theta\tilde{Q}_{l,0}(\theta) - Q_{l,0}(0) = \lambda\tilde{Q}_{l,0}(\theta) - (1-\pi)\sum_{m=a}^{b} P_{m,0}{}^{2}(0)\,\tilde{V}(\theta) - R_0(0)\,\tilde{V}(\theta); \tag{23}$$

$$\theta\tilde{Q}_{l,n}(\theta) - Q_{l,n}(0) = \lambda\tilde{Q}_{l,n}(\theta) - (1-\pi)\sum_{m=a}^{b} P_{m,0}{}^{2}(0)\,\tilde{V}(\theta) - \sum_{k=1}^{j} Q_{l,n-k}(\theta)\,\lambda g_k - R_n(0)\,\tilde{V}(\theta);\ \ 1\le n\le a-1 \tag{24}$$

$$\theta\tilde{Q}_{l,n}(\theta) - Q_{l,n}(0) = \lambda\tilde{Q}_{l,n}(\theta) - \lambda\sum_{k=1}^{j}\tilde{Q}_{l,n-k}(\theta)g_k,\ \ n\ge a \tag{25}$$

$$\theta\tilde{Q}_{j,0}(\theta) - Q_{j,0}(0) = \lambda\tilde{Q}_{j,0}(\theta) - \tilde{Q}_{j-1,0}(\theta)\,\tilde{V}(\theta);2\le j\le M \tag{26}$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda\tilde{Q}_{j,n}(\theta) - \lambda\sum_{k=1}^{j}\tilde{Q}_{j,n-k}(\theta)g_k - Q_{j-1,n}(0)\,\tilde{V}(\theta);\ \ 2\le j\le M,\ \ 1\le n\le a-1 \tag{27}$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda\tilde{Q}_{j,n}(\theta) - \lambda\sum_{k=1}^{j}\tilde{Q}_{l,n-k}(\theta)g_k;\ \ 2\le j\le M,\ n\ge a \tag{28}$$

$$\theta \tilde{R}_0(\theta) - R_0(0) = -\lambda R_0(\theta) - \pi \sum_{m=a}^{b} P_{m,0}{}^2(0)\, R(\theta), \tag{29}$$

$$\theta \tilde{R}_n(\theta) - R_n(0) = -\lambda R_n(\theta) - \pi \sum_{m=a}^{b} P_{m,0}{}^2(0)\, R(\theta) - \sum_{k=1}^{n} R_{n-k}(\theta)\lambda g_k; \quad 1 \le n \le a-1 \tag{30}$$

$$\theta \tilde{R}_n(\theta) - R_n(0) = -\lambda R_n(\theta) - \sum_{k=1}^{n} R_{n-k}(\theta)\lambda g_k, \quad n \ge a \tag{31}$$

3. System Size Distribution

To obtain the system size distribution let us define PGF's as follows:

$$\tilde{P}_i^l(z,\theta) = \sum_{n=0}^{\infty} \tilde{P}_{i,n}^l(\theta)\, z^n, \ l=1,2; \qquad P_i{}^l(z,0) = \sum_{n=0}^{\infty} P^l{}_{i,n}(0)\, z^n, \ l=1,2; \ a \le i \le b,$$

$$\tilde{Q}_j(z,\theta) = \sum_{n=0}^{\infty} \tilde{Q}_{j,n}(\theta)\, z^n; \qquad Q_j(z,0) = \sum_{j=0}^{a-1} Q_{j,n}(0)\, z^n$$

$$\tilde{C}(z,\theta) = \sum_{n=0}^{\infty} \tilde{C}_n(\theta)\, z^n; \qquad C(z,0) = \sum_{n=0}^{\infty} C_n(0)\, z^n;$$

$$T(z) = \sum_{j=0}^{a-1} T_n z^n \tag{32}$$

The probability generating function P(z) of the number of customers in the queue at an arbitrary time epoch of the proposed model can be obtained using the following equation

$$P(z) = \sum_{i=a}^{b-1} \tilde{P}_i^1(z,0) + \tilde{P}_b^1(z,0) + \sum_{i=a}^{b} \tilde{P}_i^2(z,0) + \sum_{j=1}^{M} \tilde{Q}_j(z,0) + T(z) \tag{33}$$

In order to find the following $\tilde{P}_i(z,\theta), \tilde{P}_b(z,\theta), \tilde{Q}(z,\theta)$ and $\tilde{C}(z,\theta)$ sequence of operations are done. Multiply the equations (21) by z^0, (22) by $z^n(1 < n < a-1)$ and (23) by $z^n\,(n \ge a)$, summing up from $n = 0 to \infty$ and by using (32), we get

$$[\theta - (\lambda - \lambda X(z))]\, \tilde{Q}_l(z,\theta) = Q_l(z,0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} \left[(1-\pi) \sum_{m=a}^{b} P_{m,n}{}^2(0) + R_n(0) \right] z^n \tag{34}$$

Multiply the equations (24) by z^0, (25) by $z^n \quad (1 < n < a-1)$ and (26) by $z^n \quad (n \ge a)$, summing up from $n = 0 to \infty$ and by using (32), we get

$$[\theta - (\lambda - \lambda X(z))]\, \tilde{Q}_j(z,\theta) = Q_j(z,0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n; \quad 2 \le j \le M \tag{35}$$

Multiply the equations (29) by $z^0 \quad (1 < n < a-1)$ and (30) and (31) by $z^n \ (n > a)$ and summing up from $n = 0 to \infty$ and by using (32), we get

$$[\theta - (\lambda - \lambda X(z))]\, \tilde{R}(z,\theta) = R(z,0) - \tilde{R}(\theta)\, \pi \sum_{n=0}^{a-1} \left[\sum_{m=a}^{b} P_{m,n}{}^2(0) \right] z^n \tag{36}$$

Multiply the equations (18) by z^0, (19) by $z^j \quad (j > a)$ and summing up from $n = 0 to \infty$ and using (32), we get

$$[\theta - (\lambda - \lambda X(z))]\, \tilde{P}_i^1(z,\theta) = P_i{}^1(z,0) - \tilde{S}_1(\theta) \left[\sum_{m=a}^{b} P_{m,i}{}^2(0) + R_i(0) + \sum_{l=1}^{M} Q_{l,i}(0)\, \lambda g_{i-k} + \sum_{m=0}^{n-1} T_m \lambda g_{i-m} \right], \ a \le i \le b-1 \tag{37}$$

Multiply the equations (21) by z^0, (22) z^j $(j > a)$ and summing up from $j = 0to\infty$ and using (32), we get

$$z^b\left[\theta-(\lambda-\lambda X(z))\right]\tilde{P}_b^1(z,\theta)=z^b{P_b}^1(z,0)-\tilde{S}_1(\theta)\left[\begin{array}{c}\sum\limits_{m=a}^{b-1}{P_m}^2(z,0)-\sum\limits_{m=a}^{b}\sum\limits_{j=0}^{b-1}{P_{m,j}}^2(0)z^jR(z,0)-\sum\limits_{n=0}^{b-1}R_n(0)z^n\\ +\lambda\left(T(z)X(z)-\sum\limits_{m=0}^{a-1}\left(T_mz^m\sum\limits_{j=1}^{b-m-1}g_jz^j\right)\right)\\ \sum\limits_{l=1}^{M}Q_l(z,0)-\sum\limits_{l=1}^{M}\sum\limits_{j=0}^{b-1}Q_{l,j}(0)z^j\end{array}\right] \tag{38}$$

Multiply the equations (21) by z^0, (22) z^j $(j > a)$ and summing up from $j = 0 \to \infty$ and using (32), we get

$$\left[\theta-(\lambda-\lambda X(z))\right]\tilde{P}_i^2(z,\theta)={P_i}^2(z,0)-\tilde{S}_2(\theta)\left[{P_i}^1(z,0)+R(z,0)-\sum_{n=0}^{b-1}R_n(0)z^n\right] \tag{39}$$

By substituting $\theta = (\lambda - \lambda X(z))$ in the equations (32), (33) we get

$$Q_l(z,0)=\tilde{V}((\lambda-\lambda X(z)))\sum_{n=0}^{a-1}\left[(1-\pi)\sum_{m=a}^{b}{P_{m,n}}^2(0)+R_n(0)\right]z^n \tag{40}$$

$$Q_j(z,0)=\tilde{V}((\lambda-\lambda X(z)))\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n;\ 2\le j\le M \tag{41}$$

By substituting $\theta = (\lambda - \lambda X(z))$ in the equations (34)-(39), we get

$$R(z,0)=\tilde{R}((\lambda-\lambda X(z)))\,\pi\sum_{n=0}^{a-1}\sum_{m=a}^{b}{P_{m,n}}^2(0)z^n; \tag{42}$$

$${P_i}^1(z,0)=\widetilde{S_1}((\lambda-\lambda X(z)))\left[\sum_{m=a}^{b}{P_{m,i}}^2(0)+R_i(0)+\sum_{l=1}^{M}Q_{l,i}(0)+\sum_{m=0}^{a-1}T_m\lambda g_{i-m}\right];\quad a\le i\le b-1 \tag{43}$$

$${P_b}^1(z,0)=\frac{\tilde{S}_1((\lambda-\lambda X(z)))f(z)}{Z^b-\tilde{S}_1((\lambda-\lambda X(z)))\tilde{S}_2((\lambda-\lambda X(z)))} \tag{44}$$

where

$$\begin{aligned}f(z)=&\sum_{m=a}^{b-1}\tilde{S}_2((\lambda-\lambda X(z))){P_m}^1(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}{P_{m,j}}^2(0)z^j+\lambda\left(T(z)X(z)-\sum_{m=0}^{a-1}T_mz^m\sum_{j=1}^{b-m-1}g_jz^j\right)\\&+\sum_{l=1}^{M}\left(\tilde{Q}_l(z,0)-\sum_{j=0}^{b-1}Q_{l,j}(0)z^j\right)\end{aligned} \tag{45}$$

$$\tilde{P}_i^2(z,0)=\tilde{S}_2((\lambda-\lambda X(z)))\left[{P_i}^1(z,0)+R(z,0)-\sum_{n=0}^{b-1}R_n(0)z^n\right] \tag{46}$$

Substituting the expressions for ${P_m}^2(z,0)$, $a\le m\le b-1$ from (43) and $\tilde{Q}_l(z,0)$, $1\le l\le M$ from (40) and (41) in $f(z)$.

$$f(z)=\tilde{S}_2((\lambda-\lambda X(z)))\left\{\begin{array}{c}\sum\limits_{n=a}^{b-1}\left[\left[\sum\limits_{m=a}^{b}{P_{m,n}}^2(0)+R_i(0)+\sum\limits_{l=1}^{M}Q_{l,n}(0)+\sum\limits_{m=0}^{a-1}T_m\lambda g_{n-m}\right]\right]-\sum\limits_{m=a}^{b}\sum\limits_{j=0}^{b-1}{P_{m,j}}^2(0)z^j\\ \tilde{V}((\lambda-\lambda X(z)))\sum\limits_{n=0}^{a-1}\left[(1-\pi)\sum\limits_{m=a}^{b}{P_{m,n}}^2(0)+R_n(0)+\sum\limits_{j=1}^{M}Q_{j,n}(0)\right]z^n\\ -\sum\limits_{j=1}^{M}\sum\limits_{n=0}^{b-1}Q_{j,n}(0)z^n+\tilde{R}((\lambda-\lambda X(z)))\,\pi\sum\limits_{n=0}^{a-1}\sum\limits_{m=a}^{b}{P_{m,n}}^2(0)z^n-\sum\limits_{n=0}^{b-1}R_n(0)z^n\\ +\lambda\left(T(z)X(z)-\sum\limits_{m=0}^{a-1}\left(T_mz^m\sum\limits_{j=1}^{b-m-1}g_jz^j\right)\right)\end{array}\right\}$$

From the equation (34) & (40), we have

$$\tilde{Q}_1(z,\theta)=\frac{\left(\tilde{V}((\lambda-\lambda X(z)))-\tilde{V}(\theta)\right)\sum\limits_{n=0}^{a-1}\left[(1-\pi)\sum\limits_{m=a}^{b}{P_{m,n}}^2(0)+R_n(0)\right]z^n}{(\theta-(\lambda-\lambda X(z)))} \tag{47}$$

From the equation (35) & (41), we have

$$\tilde{Q}_j(z,\theta) = \frac{\left(\tilde{V}((\lambda - \lambda X(z))) - \tilde{V}(\theta)\right)\sum_{n=0}^{b}\sum_{j=1}^{M} Q_{j-1,n}(0) z^n}{(\theta - (\lambda - \lambda X(z)))}, \quad 2 \le j \le M \tag{48}$$

From the equation (36) & (42), we have

$$\tilde{R}(z,\theta) = \frac{\left(\tilde{R}((\lambda - \lambda X(z))) - \tilde{R}(\theta)\right)\pi \sum_{n=0}^{a-1}\sum_{j=1}^{M} P_{m,n}{}^2(0) z^n}{(\theta - (\lambda - \lambda X(z)))} \tag{49}$$

From the equation (37) & (43), we have

$$\tilde{P}_i^1(z,\theta) = \frac{\left(\tilde{S}_1((\lambda - \lambda X(z))) - \tilde{S}_1(\theta)\right)\left[\sum_{m=a}^{b} P_{m,i}{}^2(0) + \sum_{l=1}^{M} Q_{l,i}(0) + \sum_{m=0}^{a-1} T_m \lambda g_{i-m}\right]}{(\theta - (\lambda - \lambda X(z)))}; \quad a \le i \le b-1 \tag{50}$$

From the equation (38) & (44), we have

$$\tilde{P}_b^1(z,\theta) = \frac{\left[\tilde{S}_1((\lambda - \lambda X(z))) - \tilde{S}_1(\theta)\right] f(z)}{(\theta - (\lambda - \lambda X(z)))\left(Z^b - \tilde{S}_1((\lambda - \lambda X(z)))\right)\tilde{S}_2((\lambda - \lambda X(z)))} \tag{51}$$

$$\tilde{P}_i^2(z,\theta) = \frac{\left(\tilde{S}_2((\lambda - \lambda X(z))) - \tilde{S}_2(\theta)\right) P_i{}^1(z,0)}{(\theta - (\lambda - \lambda X(z)))} \tag{52}$$

Let

$$p_i = \sum_{m=a}^{b} P_{m,i}(0), \quad r_i = R_i(0), q_i = \sum_{l=1}^{M} Q_{l,i}(0) \quad \text{and} \quad c_i = p_i + q_i + r_i \tag{53}$$

Using the Equations (47) -(52) in the Equation (53), the probability generating function of the queue size, P(z) at an arbitrary time epoch is obtained as

$$P(z) = \frac{\left\{\begin{array}{c} \left(\tilde{S}_1((\lambda - \lambda X(z)))\tilde{S}_2((\lambda - \lambda X(z))) - 1\right)\sum_{i=a}^{b-1} c_i\left(z^b - z^i\right) + \left(\tilde{V}((\lambda - \lambda X(z))) - 1\right) \\ \left[\left(\left(\tilde{S}_1((\lambda - \lambda X(z)))\tilde{S}_2((\lambda - \lambda X(z)))\right) - 1\right)\right) + \left(Z^b - \tilde{S}_1((\lambda - \lambda X(z)))\tilde{S}_2((\lambda - \lambda X(z)))\right)\right] \\ \sum_{i=0}^{a-1}(c_i - \pi p_i^2) z^i - \pi\left[\left(\tilde{R}((\lambda - \lambda X(z))) - 1\right) + \left(Z^b - \tilde{S}_1((\lambda - \lambda X(z)))\tilde{S}_2((\lambda - \lambda X(z)))\right)\right]\sum_{i=0}^{a-1} p_i{}^2 z^i \\ + \left(\tilde{S}_1((\lambda - \lambda X(z)))\tilde{S}_2((\lambda - \lambda X(z))) - 1\right)\sum_{i=a}^{b-1}\left(z^b - z^i\right)\sum_{m=0}^{a-1} T_m \lambda g_{i-m} + \lambda T(z)(X(z) - 1) \\ \left(z^b - \tilde{S}_1((\lambda - \lambda X(z)))\tilde{S}_2((\lambda - \lambda X(z)))\right) \end{array}\right\}}{(-\lambda + \lambda X(z))\left(Z^b - \tilde{S}_1((\lambda - \lambda X(z)))\tilde{S}_2((\lambda - \lambda X(z)))\right)} \tag{54}$$

3.1. Steady State Condition

The probability generating function $P(z)$ has to satisfy $P(1) = 1$. In order to satisfy the condition, applying L'Hospital's rule and evaluating $\lim_{z \to \infty} P(z)$ and equating the expression to 1, $b - \lambda E(X)[[E(S_1) + E(S_2) + \pi E(R)]] > 0$ is obtained. Define 'ρ' as $\frac{\lambda E(X)[E(S_1)+E(S_2)+\pi E(R)]}{b}$. Thus $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model.

3.2. Computational Aspects of Unknowns Probabilities

Equation (54) gives PGF of the number of customers in the queue,which involves the unknown T_i and $\tilde{Q}_i(\theta)$ are expressed in terms p_i and the known function $\tilde{V}(\lambda)$ respectively. To find the unknown constants, Rouche's theorem of complex variables is used. By Rouche's theorem, it follows that $(z^b - \overline{S}_1(\lambda - \lambda X(z))\ \overline{S}_2(\lambda - \lambda X(z))$ has $b - 1$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic with in the on the unit circle, the numerator of (54) must vanish at these points, which gives b equations with be unknowns. Thus Equation (54) gives the PGF of the number of customers in the queue at an arbitrary time.

4. Conclusion

In this paper a batch arrival two types of general bulk service queuing system with server breakdown and modified M-vacation is considered. Probability generating function of queue size at an arbitrary time epoch .

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