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A Batch Arrival Two Types of Bulk Service Queue with Vacation Interrupted, Optional Service and Closedown

Research Article*

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Abstract: In this paper a batch arrival two types of general bulk service queuing system with vacation interrupted, optional service and closedown is considered. The service is done in bulk with a minimum of 'a' customer and a maximum of 'b' customers. At first the server provides first essential service (FES) (bulk service) to the arriving customers and sequentially the server must provide second essential services (SES). At the completion of two types of service the leaving batch of customers may request for optional service with probability π. If no request for optional service is made after the completion of two type of service or after completion of optional service and number of customers in the queue is less than 'a' then the server closed down its work. After that the server goes for single vacation. If the queue size reaches 'a' during the vacation, the server terminates the vacation abruptly and resumes for FES. On completion of the vacation the server remains in the system (dormant period) until the queue length reaches 'a'. After the completion of optional service with π or after completion of two type of service with $(1-\pi)$ and the number of customer in the queue is greater than 'a' the server will continue the batch service with general bulk service rule. The probability generating function of queue science at the random epoch is obtained.

Keywords: Batch Arrival, Vacation Interrupted, Optional Service, Closedown.

1. Introduction

In recent years, Bulk service queueing system with server vacations have been developed for a wide range applications in production, communication systems, bank services and etc. Li and Tian (2007) studied the discrete-time GI/Geo/1 queue with working vacation and vacation interruption. Ji-Hong Li et al (2008) studied GI/M/1 queue with working vacations and vacation interruption. Zhang and Shi (2009) provided a study on the M/M/1 queue with Bernoulli-Schedule-Controlled vacation and vacation interruption. Mian Zhang and Zhengting Hou (2010) studied an M/G/1 queue with working vacations and vacation interruption. Yutaka BABA (2010) studied the M/PH/1 queue with working vacations and vacation interruption. Mian Zhang and Zhengting Hou (2011) studied an MAP/G/1 queue with working vacations and vacation interruption.

Madan and Baklizi (2002) considered an M/G/1 queueing model, in which the server performs first essential service to all arriving customers. As soon as the first service is over, they may leave the system with the probability $(1-\pi)$ and second optional service is provided with probability π. Madan et al (2004) analyzed a single server bulk arrival queue,

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in which the leaving batch of customers might opt for re-service. Arumuganathan and Judeth Malliga (2006) analyzed a bulk queue with repair of service station and set up time. S.Jayakumar and R.Arumuganathan (2011) discussed a bulk service queue with multiple vacation and request for re-service. Ke (2007) discussed the operating characteristics of an $M^{[x]}G/1$ queueing system under vacation policies with startup/closedown times are generally distributed. Choudhury and Madan (2007) have considered a batch arrival but single service Bernoulli vacation queue , with a random setup time under restricted admissibility policy. Sikdar and Gupta (2008) have discussed on the batch arrival, batch service queue ,but finite buffer under server's vacation. $M^{[x]}/M^{[y]}/1/N$ queue. Jain and Upadhyaya (2010) considered with the modified Bernoulli vacation schedule for the unreliable server batch arrival queueing system with essential and milti-optional services under N-policy.

1.1. Notations

Let X be the group size random variable of the arrival, λ be the Poisson arrival rate, g_k be the probability that 'k' customers arrive in a batch and $X(z)$ be its probability generating function (PGF). $S_1(.)$, $S_2(.)$, $V(.)$, $C(.)$, $R(.)$ Cumulative distribution function of FSE, SES service time, vacation time, closedown work, optional service time. $s_1(x)$, $s_2(x)$, $v(x)$, $c(x)$, $r(x)$ Probability density function of S_1, S_2, V, C and R.
$\tilde{S}_1(\theta)$, $\tilde{S}_2(\theta)$, $\tilde{V}(\theta)$, $\tilde{C}(\theta)$, $\tilde{R}(\theta)$ be the Laplace-Stieltjes transform of S_1, S_2, V, R. $S_1^0(t)$, $S_2^0(t)$, $V^0(t)$, $C^0(t)$, $R^0(t)$. Remaining service time of a batch in FSE, SES service time, vacation time, closedown work, optional service at time 't'
$N_s(t) =$ Number of customers in the service at time t
$N_q(t) =$ Number of customers in the queue at time t
The different states of the server at time t are defined as follows

$$Y(t) = \begin{cases} 0, & \text{if the server is busy with FES;} \\ 1, & \text{if the server is busy with SES;} \\ 2, & \text{if the server is on vacation ;} \\ 3, & \text{if the server is busy with optional service;} \\ 4, & \text{if the server is on closedown work;} \\ 5, & \text{if the server is on idle period.} \end{cases}$$

To obtain the system of Equations, the following state probabilities are defined:

$$P^1_{i,n}(x,t)dt = P\{N_s(t) = i,\ \ N_q(t) = n,\ \ x \le S^0(t) \le x + dt,\ \ Y(t) = 0\},\ \ a \le i \le b, n \ge 0,$$
$$P^2_{i,n}(x,t)dt = P\{N_s(t) = i,\ \ N_q(t) = n,\ \ x \le S^0(t) \le x + dt,\ \ Y(t) = 1\},\ \ a \le i \le b, n \ge 0,$$
$$Q_n(x,t)dt = P\{N_q(t) = n,\ \ x \le V^0(t) \le x + dt,\ \ Y(t) = 2\},\ \ n \ge 0$$
$$R_n(x,t)dt = P\{N_q(t) = n,\ \ x \le R^0(t) \le x + dt,\ \ Y(t) = 3\},\ \ n \ge 0$$
$$C_n(x,t)dt = P\{N_q(t) = n,\ \ x \le C^0(t) \le x + dt,\ \ Y(t) = 4\},\ \ n \ge 0$$
$$T_n(t)dt = P\{N_q(t) = n,\ \ Y(t) = 5\},\ \ 0 \le n \le a - 1.$$

Now, the following system equations are obtained for the queueing system, using supplementary variable technique:

$$T_0(t + \Delta t) = T_0(t)(1 - \lambda\Delta t) + Q_0(0,t)\Delta t$$
$$T_n(t + \Delta t) = T_n(t)(1 - \lambda\Delta t) + Q_n(0,t)\Delta t + \sum_{k=1}^{n} T_{n-k}(t)\lambda g_k\Delta t,\ \ 1 \le n \le a - 1,$$

$$P^1_{i,0}(x-\Delta t,t+\Delta t)=P^1_{i,0}(x,t)(1-\lambda\Delta t)+\sum_{m=a}^{b}P^2_{i,0}(0,t)s_1(x)\Delta t+\sum_{k=1}^{n}Q_k(0,t)\lambda g_{i-k}s(x)\Delta t$$
$$+\sum_{m=0}^{a-1}T_m(t)\lambda g_{i-m}s_1(x)\Delta t;\quad a\le i\le b$$
$$P^1_{i,j}(x-\Delta t,t+\Delta t)=P^1_{i,j}(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{j}P^1_{i,j-k}(x,t)\lambda g_k\Delta t;\ a\le i\le b-1;\ j\ge 1$$
$$P^1_{b,j}(x-\Delta t,t+\Delta t)=P^1_{b,j}(x,t)(1-\lambda\Delta t)+\sum_{m=a}^{b}P^2_{m,b+j}(0,t)s_1(x)\Delta t+\sum_{k=1}^{j}P^1_{b,j-k}(x,t)\lambda g_k\Delta t$$
$$+\sum_{m=0}^{a-1}T_m(t)\lambda g_{b+j-m}s_1(x)\Delta t+\sum_{k=0}^{a-1}Q_k(x,t)\lambda g_{b+j-k}s_1(x)\Delta t;\quad j\ge 1$$
$$P^2_{i,0}(x-\Delta t,t+\Delta t)=P^2_{i,0}(x,t)(1-\lambda\Delta t)+P^1_{i,0}(x,t)s_2(x)\Delta t+C_i(0,t)s_1(x)\Delta t+R_i(0,t)s_2(x)\Delta t,\quad a\le i\le b$$
$$P^2_{i,j}(x-\Delta t,t+\Delta t)=P^2_{i,j}(x,t)(1-\lambda\Delta t)+P^1_{i,j}(x,t)s_2(x)\Delta t++R_{b+j}(0,t)s_2(x)\Delta t+C_{b+i}(0,t)s_2(x)\Delta t$$
$$+\sum_{k=1}^{j}P^2_{b,j-k}(x,t)\lambda g_k\Delta t+R_n(0,t)s_2\Delta t,\quad j\ge 1$$
$$Q_0(x-\Delta t,t+\Delta t)=Q_0(x,t)(1-\lambda\Delta t)+C_0(0)v(x)\Delta t;$$
$$Q_n(x-\Delta t,t+\Delta t)=Q_n(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{n}Q_{n-k}(x,t)\lambda g_k\Delta t;+C_0(0)v(x)\Delta t;\quad 1\le n\le a-1$$
$$C_0(x-\Delta t,t+\Delta t)=C_0(x,t)(1-\lambda\Delta t)+\sum_{m=a}^{b}P^2_{m,0}(0,t)c(x)$$
$$C_n(x-\Delta t,t+\Delta t)=C_n(x,t)(1-\lambda\Delta t)+\sum_{m=a}^{b}P^2_{m,n}(0,t)c(x)+\lambda\sum_{k=1}^{n}C_{n-k}(x,t)g_k;\quad 1\le n\le a-1,$$
$$C_n(x-\Delta t,t+\Delta t)=C_n(x,t)(1-\lambda\Delta t)+\lambda\sum_{k=1}^{n}C_{n-k}(x,t)g_k;\quad n\ge a.$$
$$R_0(x-\Delta t,t+\Delta t)=R_0(x,t)(1-\lambda\Delta t)+\pi\sum_{m=a}^{b}P^2_{m,0}(0,t)r(x)$$
$$R_n(x-\Delta t,t+\Delta t)=R_n(x,t)(1-\lambda\Delta t)+\pi\sum_{m=a}^{b}P^2_{m,n}(0,t)r(x)+\lambda\sum_{k=1}^{n}R_{n-k}(x,t)g_k;\quad 1\le n\le a-1,$$
$$R_n(x-\Delta t,t+\Delta t)=R_n(x,t)(1-\lambda\Delta t)+\lambda\sum_{k=1}^{n}R_{n-k}(x,t)g_k;\quad n\ge a.$$

2. Steady State Queue Size Distribution

From the above equations, the steady state queue size equations are obtained as follows:

$$0=-\lambda_0T_0+Q_{M,0}(0) \tag{1}$$
$$0=-\lambda_0T_n+Q_{M,n}(0)+\sum_{k=1}^{n}T_{n-k}\lambda_0g_k,\quad 1\le n\le a-1 \tag{2}$$

$$-\frac{dy}{dx}P^1_{i,0}(x)=-\lambda P^1_{i,0}(x)+\sum_{m=a}^{b}P^2_{m,i}(0)\,s_1(x)\,\Delta t+\sum_{l=1}^{M}Q_k(0)\,s_1(x)+\lambda\sum_{m=0}^{a-1}T_m\lambda g_{i-m}\,s_1(x);\quad a\le i\le b \tag{3}$$
$$-\frac{dy}{dx}P^1_{i,j}(x)=-\lambda P^1_{i,0}(x)+\sum_{k=1}^{j}P^1_{i,j-k}(x)\,\lambda g_k;\quad a\le i\le b-1,\quad j\ge 1 \tag{4}$$
$$-\frac{dy}{dx}P^1_{b,j}(x)=-\lambda P^1_{b,j}(x)+\sum_{k=1}^{j}P^1_{b,j-k}(x)\,\lambda g_k+\sum_{m=a}^{b}P^2_{m,b+j}(0)\,s_1(x)+\sum_{m=0}^{n-1}T_m\lambda g_{b+j-m}s_1(x)+\sum_{k=1}^{a-1}Q_k(0)\,s_1(x) \tag{5}$$

$$-\frac{dy}{dx}P_{i,0}^{2}\left(x\right)=-\lambda P_{i,0}^{2}\left(x\right)+P_{i,0}^{1}\left(x,t\right)s_{2}\left(x\right)+R_{i}\left(0\right)s_{2}\left(x\right),\quad a\leq i\leq b \tag{6}$$

$$-\frac{dy}{dx}P_{i,j}^{2}\left(x\right)=P_{i,j}^{2}\left(x,t\right)+P_{i,j}^{1}\left(x,t\right)s_{2}\left(x\right)+\sum_{k=1}^{j}P_{b,j-k}^{2}\left(x,t\right)\lambda g_{k}+R_{i,j}\left(0\right)s_{2}\left(x\right),\quad j\geq 1 \tag{7}$$

$$-\frac{dy}{dx}Q_{0}\left(x\right)=-\lambda Q_{j,0}\left(x\right)+C_{0}\left(x\right)v\left(x\right); \tag{8}$$

$$-\frac{dy}{dx}Q_{n}\left(x\right)=-\lambda Q_{n}\left(x\right)+\sum_{k=1}^{n}Q_{n-k}\left(x\right)\lambda g_{k}+C_{n}\left(x\right)v\left(x\right);\quad 1\leq n\leq a-1 \tag{9}$$

$$-\frac{dy}{dx}C_{0}\left(x\right)=-\lambda C_{0}\left(x\right)+\sum_{m=a}^{b}P_{m,0}^{2}(0)c\left(x\right) \tag{10}$$

$$-\frac{dy}{dx}C_{n}\left(x\right)=-\lambda C_{n}\left(x\right)+\sum_{m=a}^{b}P_{m,0}^{2}(0)c\left(x\right)+\lambda\sum_{k=1}^{n}C_{n-k}\left(x\right)g_{k}+R_{n}\left(x\right)c\left(x\right);\quad 1\leq n\leq a-1, \tag{11}$$

$$-\frac{dy}{dx}C_{n}\left(x\right)=-\lambda C_{n}\left(x\right)+\lambda\sum_{k=1}^{n-a}C_{n-k}\left(x\right)g_{k};\quad n\geq a \tag{12}$$

$$-\frac{dy}{dx}R_{0}\left(x\right)=-\lambda R_{0}\left(x\right)+\pi\sum_{m=a}^{b}P_{m,0}^{2}\left(0\right)r\left(x\right), \tag{13}$$

$$-\frac{dy}{dx}R_{n}\left(x\right)=-\lambda R_{n}\left(x\right)+\pi\sum_{m=a}^{b}P_{m,0}^{2}\left(0\right)r\left(x\right)+\sum_{k=1}^{n}R_{n-k}\left(x\right)\lambda g_{k};\quad 1\leq n\leq a-1 \tag{14}$$

$$-\frac{dy}{dx}R_{n}\left(x\right)=-\lambda R_{n}\left(x\right)+\sum_{k=1}^{n}R_{n-k}\left(x\right)\lambda g_{k},\quad n\geq a \tag{15}$$

The Laplace-Stieltjes transforms of $P_{i,n}\left(x\right)$, $Q_{j}\left(x\right)$, $C_{n}\left(x\right)$ and $R_{n}\left(x\right)$ are defined as:

$$\begin{aligned}\tilde{P}_{i,n}^{j}\left(\theta\right)&=\int_{0}^{\infty}e^{-\theta x}P_{i,n}^{j}\left(x\right)dx,\quad j=1,2\ \text{ and }\ \tilde{Q}_{j}\left(\theta\right)=\int_{0}^{\infty}e^{-\theta x}Q_{j}\left(x\right)dx\\ \tilde{C}_{n}\left(\theta\right)&=\int_{0}^{\infty}e^{-\theta x}C_{n}\left(x\right)dx\ \text{ and }\qquad\qquad \tilde{R}_{n}\left(\theta\right)=\int_{0}^{\infty}e^{-\theta x}R_{n}\left(x\right)dx\end{aligned} \tag{16}$$

Taking Laplace-Stieltjes transform on both sides, we get

$$\theta\tilde{P}_{i,0}^{1}\left(\theta\right)-P_{i,0}\left(0\right)=\lambda\tilde{P}_{i,0}^{1}\left(\theta\right)-\left[\sum_{m=a}^{b}P_{m,i}\left(0\right)+\sum_{k=1}^{a-1}Q_{k}\left(0\right)\lambda g_{i-k}+\sum_{m=0}^{a-1}T_{m}\lambda g_{i-m}\right]\tilde{S}_{1}\left(\theta\right);\quad a\leq i\leq b, \tag{17}$$

$$\theta\tilde{P}_{i,j}^{1}\left(\theta\right)-P_{i,j}^{1}\left(0\right)=\lambda\tilde{P}_{i,j}^{1}\left(\theta\right)-\lambda\sum_{k=1}^{j}\tilde{P}_{i,j-k}^{1}\left(\theta\right)g_{k},\quad a\leq i<b-1,\quad j\geq 1 \tag{18}$$

$$\theta\tilde{P}_{b,j}^{1}\left(\theta\right)-P_{b,j}^{1}\left(0\right)=\lambda\tilde{P}_{b,j}^{1}\left(\theta\right)-\sum_{k=1}^{j}\tilde{P}_{b,j-k}^{1}\left(\theta\right)\lambda g_{k}-\sum_{m=a}^{b}P_{m,b+j}^{2}\left(0\right)\tilde{S}_{1}\left(\theta\right)-\sum_{m=0}^{n-1}T_{m}\lambda g_{b+j-m}\tilde{S}_{1}\left(\theta\right)-\sum_{k=1}^{a-1}\tilde{Q}_{k}\left(\theta\right)\tilde{S}_{1}\left(\theta\right), \tag{19}$$

$$\theta\tilde{P}_{i,0}^{2}\left(\theta\right)-P_{i,0}^{2}\left(0\right)=\lambda\tilde{P}_{i,j}^{1}\left(\theta\right)-P_{i,0}^{1}\left(0\right)\tilde{S}_{2}\left(\theta\right)-R_{i}\left(0\right)\tilde{S}_{2}\left(\theta\right)-C_{i}\left(0\right)\tilde{S}_{2}\left(\theta\right);\quad a\leq i<b \tag{20}$$

$$\theta\tilde{P}_{i,j}^{2}\left(\theta\right)-P_{i,j}^{2}\left(0\right)=\lambda\tilde{P}_{i,j}^{2}-\lambda\sum_{k=1}^{j}\tilde{P}_{i,j-k}^{2}\left(\theta\right)g_{k}-P_{i,0}^{2}\left(0\right)\tilde{S}_{2}\left(\theta\right)-R_{i}\left(0\right)\tilde{S}_{2}\left(\theta\right)R_{i}\left(0\right)\tilde{S}_{2}\left(\theta\right);\quad a\leq i<b,\quad j\geq 1 \tag{21}$$

$$\theta\tilde{C}_{n}\left(\theta\right)-C_{n}\left(0\right)=\lambda\tilde{C}_{n}\left(\theta\right)-\left[\left(1-\pi\right)\sum_{m=a}^{b}P_{m,n}\left(0\right)+R_{n}\left(0\right)\right]C\left(\theta\right)-\lambda\sum_{k=1}^{j}\tilde{C}_{n-k}\left(\theta\right)g_{k};\quad 1\leq n\leq a-1 \tag{22}$$

$$\theta\tilde{C}_{n}\left(\theta\right)-C_{n}\left(0\right)=\lambda\tilde{C}_{n}\left(\theta\right)-\lambda\sum_{k=1}^{j}\tilde{C}_{n-k}\left(\theta\right)g_{k};\quad n\geq a \tag{23}$$

$$\theta\tilde{Q}_{0}\left(\theta\right)-Q_{0}\left(0\right)=\lambda\tilde{Q}_{0}\left(\theta\right)-\left(1-\pi\right)\sum_{m=a}^{b}{P_{m,0}}^{2}\left(0\right)\tilde{V}\left(\theta\right)-R_{0}\left(0\right)\tilde{V}\left(\theta\right)-C_{0}\left(0\right)\tilde{V}\left(\theta\right); \tag{24}$$

$$\theta\tilde{Q}_{n}\left(\theta\right)-Q_{n}\left(0\right)=\lambda\tilde{Q}_{n}\left(\theta\right)-\left(1-\pi\right)\sum_{m=a}^{b}{P_{m,0}}^{2}\left(0\right)\tilde{V}\left(\theta\right)-\sum_{k=1}^{j}Q_{n-k}\left(\theta\right)\lambda g_{k}-R_{n}\left(0\right)\tilde{V}\left(\theta\right)-C_{n}\left(0\right)\tilde{V}\left(\theta\right) \tag{25}$$

$$\theta\tilde{R}_0(\theta) - R_0(0) = -\lambda R_0(\theta) - \pi\sum_{m=a}^{b} P_{m,0}^{2}(0)\,R(\theta)\,, \tag{26}$$

$$\theta\tilde{R}_n(\theta) - R_n(0) = -\lambda R_n(\theta) - \pi\sum_{m=a}^{b} P_{m,0}^{2}(0)\,R(\theta) - \sum_{k=1}^{n} R_{n-k}(\theta)\lambda g_k;\ \ 1 \le n \le a-1 \tag{27}$$

$$\theta\tilde{R}_n(\theta) - R_n(0) = -\lambda R_n(\theta) - \sum_{k=1}^{n} R_{n-k}(\theta)\lambda g_k,\ \ n \ge a \tag{28}$$

3. System Size Distribution

To obtain the system size distribution let us define PGF's as follows:

$$\begin{aligned}
&\tilde{P}_i^{l}(z,\theta) = \sum_{n=0}^{\infty}\tilde{P}_{i,n}^{l}(\theta)\,z^n,\ \ l=1,2;\ \ P_i^{l}(z,0) = \sum_{n=0}^{\infty}P_{i,n}^{l}(0)\,z^n,\ \ l=1,2;\ \ a \le i \le b,\\
&\tilde{Q}(z,\theta) = \sum_{n=0}^{\infty}\tilde{Q}_n(\theta)\,z^n;\ \ \tilde{Q}(z,0) = \sum_{j=0}^{a-1}Q_n(0)\,z^n\\
&\tilde{C}(z,\theta) = \sum_{n=0}^{\infty}\tilde{C}_n(\theta)\,z^n;\ \ C(z,0) = \sum_{n=0}^{\infty}C_n(0)\,z^n;\ \ T(z) = \sum_{j=0}^{a-1}T_n z^n
\end{aligned} \tag{29}$$

The probability generating function $P(z)$ of the number of customers in the queue at an arbitrary time epoch of the proposed model can be obtained using the following equation

$$P(z) = \sum_{i=a}^{b-1}\tilde{P}_i^{1}(z,0) + \tilde{P}_b^{1}(z,0) + \sum_{i=a}^{b}\tilde{P}_i^{2}(z,0) + \sum_{j=1}^{a-1}\tilde{Q}_j(z,0) + T(z) + C(z,0) + R(z,0) \tag{30}$$

In order to find the following $\tilde{P}_i(z,\theta)$, $\tilde{P}_b(z,\theta)$, $\tilde{Q}(z,\theta)$, $\tilde{C}(z,\theta)$ and $\tilde{R}(z,\theta)$ sequence of operations are done. Multiply (22) by z^0 (23) by $z^n(n=1)$ and summing up from $n=0\to\infty$ and by using (29), we get

$$[\theta - \beta(\lambda - \lambda X(z))]\,\tilde{C}(z,\theta) = C(z,0) - \tilde{C}(\theta)(1-\pi)\left[\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}(0)z^n + \sum_{n=0}^{a-1}R_n(0)\,z^n\right] \tag{31}$$

Multiply the equations (24) by z^0, (25) by z^n $(1<n<a-1)$ and (23) by z^n $(n=a)$, summing up from $n=0\to\infty$ and by using (29), we get

$$[\theta - (\lambda - \lambda X(z))]\,\tilde{Q}(z,\theta) = Q(z,0) - \tilde{V}(\theta)\sum_{n=0}^{a-1}C_n(0)z^n \tag{32}$$

Multiply the equations (26) by z^0 $(1<n<a-1)$ and (27) and (28) by z^n $(n>a)$ and summing up from $n=0\to\infty$ and by using (29), we get

$$[\theta - (\lambda - \lambda X(z))]\,\tilde{R}(z,\theta) = R(z,0) - \tilde{R}(\theta)\,\pi\sum_{n=0}^{a-1}\left[\sum_{m=a}^{b}P_{m,n}^{2}(0)\right]z^n \tag{33}$$

Multiply the equations (17) by z^0, (19) by z^j $(j>a)$ and summing up from $n=0\to\infty$ and using (29), we get

$$[\theta - (\lambda - \lambda X(z))]\,\tilde{P}_i^{1}(z,\theta) = P_i^{1}(z,0) - \tilde{S}_1(\theta)\left[\sum_{m=a}^{b}P_{m,i}^{2}(0) + R_i(0) + \sum_{k=1}^{a-1}Q_k(0)\,\lambda g_{i-k} + \sum_{m=0}^{n-1}T_m\lambda g_{i-m}\right]; \tag{34}$$

$a \le i \le b-1$,. Multiply the equations (21) by z^0 (22) z^j $(j>a)$ and summing up from $j=0\to\infty$ and using (29), we get

$$z^b\left[\theta-(\lambda-\lambda X(z))\right]\tilde{P}_b^1(z,\theta)=z^bP_b^1(z,0)$$
$$-\tilde{S}_1(\theta)\left[\begin{array}{c}\sum_{m=a}^{b-1}P_m{}^2(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}{}^2(0)z^j\\+R(z,0)-\sum_{n=0}^{b-1}R_n(0)z^n+\lambda\left(T(z)X(z)-\sum_{m=0}^{a-1}\left(T_mz^m\sum_{j=1}^{b-m-1}g_jz^j\right)\right)\\+\lambda\left(X(z)\sum_{i=0}^{a-1}\tilde{Q}_i(\theta)z^i-\sum_{i=0}^{a-1}\left(\tilde{Q}_i(\theta)z^i\sum_{j=1}^{b-i-1}g_jz^j\right)\right)+C(z,0)-\sum_{n=0}^{b-1}C_n(0)z^n\end{array}\right]\tag{35}$$

Multiply the equations (21) by z^0 (22) z^j $(j>a)$ and summing up from $j=0\to\infty$ and using (29), we get

$$\left[\theta-(\lambda-\lambda X(z))\right]\tilde{P}_i^2(z,\theta)=P_i{}^2(z,0)-\tilde{S}_2(\theta)\left[P_i{}^1(z,0)+R(z,0)-\sum_{n=0}^{b-1}R_n(0)z^n\right]\tag{36}$$

By substituting $\theta=(\lambda-\lambda X(z))$ in the equations (32), (33) we get

$$C(z,0)=\tilde{C}(\beta(\lambda-\lambda X(z)))\sum_{n=0}^{a-1}\left[(1-\pi)\sum_{m=a}^{b}P_{m,n}(0)+R(0)\right]z^n\tag{37}$$
$$Q(z,0)=\tilde{V}(\lambda-\lambda(z))\sum_{n=0}^{a-1}C_n(0)z^n\tag{38}$$

By substituting $\theta=(\lambda-\lambda X(z))$ in the equations (34)-(39), we get

$$R(z,0)=\tilde{R}((\lambda-\lambda X(z)))\pi\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^2(0)z^n;\tag{39}$$
$$P_i{}^1(z,0)=\widetilde{S_1}((\lambda-\lambda X(z)))\left[\sum_{m=a}^{b}P_{m,i}^2(0)+R_i(0)+\sum_{k=1}^{a-1}Q_k(0)+\sum_{m=0}^{a-1}T_m\lambda g_{i-m}\right];\quad a\le i\le b-1\tag{40}$$
$$P_b^1(z,0)=\frac{\tilde{S}_1((\lambda-\lambda X(z)))f(z)}{Z^b-\tilde{S}_1((\lambda-\lambda X(z)))\tilde{S}_2((\lambda-\lambda X(z)))}\tag{41}$$

where

$$\begin{aligned}f(z)=&\sum_{m=a}^{b-1}\tilde{S}_2((\lambda-\lambda X(z)))P_m^1(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}^2(0)z^j+\lambda\left(T(z)X(z)-\sum_{m=0}^{a-1}T_mz^m\sum_{j=1}^{b-m-1}g_jz^j\right)\\&+\lambda\left(X(z)\sum_{i=0}^{a-1}\tilde{Q}_i(\lambda-\lambda X(z))z^i-\sum_{i=0}^{a-1}\left(\tilde{Q}_i(\lambda-\lambda X(z))z^i\sum_{j=1}^{b-i-1}g_jz^j\right)\right)\end{aligned}\tag{42}$$

$$\tilde{P}_i^2(z,0)=\tilde{S}_2((\lambda-\lambda X(z)))\left[P_i^1(z,0)+R(z,0)-\sum_{n=0}^{b-1}R_n(0)z^n\right]\tag{43}$$

Substituting the expressions for $P_m^2(z,0)$, $a\le m\le b-1$ from (43) and $\tilde{Q}(z,\theta)$ from (43) and (41) in $f(z)$

$$f(z)=\widetilde{S_2}((\lambda-\lambda X(z)))\left\{\begin{array}{c}\sum_{n=a}^{b-1}\left[\left[\sum_{m=a}^{b}P_{m,n}^2(0)+R_i(0)+\sum_{l=1}^{n}Q_l(0)+\sum_{m=0}^{a-1}T_m\lambda g_{n-m}\right]\right]-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}^2(0)z^j\\\tilde{V}((\lambda-\lambda X(z)))\sum_{n=0}^{a-1}\left[(1-\pi)\sum_{m=a}^{b}P_{m,n}^2(0)+R_n(0)+\sum_{j=1}^{n}Q_j(0)\right]z^n\\-\sum_{n=0}^{b-1}Q_{,n}(0)z^n\\+\tilde{R}((\lambda-\lambda X(z)))\pi\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^2(0)z^n-\sum_{n=0}^{b-1}R_n(0)z^n\\+\lambda\left(T(z)X(z)-\sum_{m=0}^{a-1}\left(T_mz^m\sum_{j=1}^{b-m-1}g_jz^j\right)\right)+C(z,0)-\sum_{n=0}^{b-1}C_n(0)z^n\end{array}\right.\tag{44}$$

From the equation (32) & (38), we have

$$\tilde{Q}(z,\theta)=\frac{\left(\tilde{V}((\lambda-\lambda X(z)))-\tilde{V}(\theta)\right)\sum_{n=0}^{a-1}\left[(1-\pi)\sum_{m=a}^{b}P_{m,n}^{2}(0)+R_{n}(0)\right]z^{n}}{(\theta-(\lambda-\lambda X(z)))} \tag{45}$$

From the equation (31) & (37), we have

$$\tilde{C}(z,\theta)=\frac{\left(\tilde{C}((\lambda-\lambda X(z)))-\tilde{C}(\theta)\right)\sum_{n=0}^{a-1}[(1-)p_{n}+r_{n}]z^{n}}{(\theta-(\lambda-\lambda X(z)))} \tag{46}$$

From the equation (33) & (39), we have

$$\tilde{R}(z,\theta)=\frac{\left(\tilde{R}((\lambda-\lambda X(z)))-\tilde{R}(\theta)\right)\pi\sum_{n=0}^{a-1}\sum_{j=a}^{b}P_{m,n}^{2}(0)z^{n}}{(\theta-(\lambda-\lambda X(z)))} \tag{47}$$

From the equation (34) & (40), we have

$$\tilde{P}_{i}^{1}(z,\theta)=\frac{\left(\tilde{S}_{1}((\lambda-\lambda X(z)))-\tilde{S}_{1}(\theta)\right)\left[\sum_{m=a}^{b}P_{m,i}^{2}(0)+\sum_{k=0}^{a-1}Q_{k}(0)+\sum_{m=0}^{a-1}T_{m}\lambda g_{i-m}\right]}{(\theta-(\lambda-\lambda X(z)))};\quad a\leq i\leq b-1 \tag{48}$$

From the equation (35) & (41), we have

$$\tilde{P}_{b}^{1}(z,\theta)=\frac{\left[\tilde{S}_{1}((\lambda-\lambda X(z)))-\tilde{S}_{1}(\theta)\right]\ f(z)}{(\theta-(\lambda-\lambda X(z)))\left(Z^{b}-\tilde{S}_{1}((\lambda-\lambda X(z)))\right)\tilde{S}_{2}((\lambda-\lambda X(z)))} \tag{49}$$

$$\tilde{P}_{i}^{2}(z,\theta)=\frac{\left(\tilde{S}_{2}((\lambda-\lambda X(z)))-\tilde{S}_{2}(\theta)\right)P_{i}^{1}(z,0)}{(\theta-(\lambda-\lambda X(z)))} \tag{50}$$

Let $p_i=\sum_{m=a}^{b}P_{m,i}(0)$, $r_i=R_i(0)$ and $c_i=p_i+q_i+r_i$. Using the Equations (45) -(50) in the Equation (30), the probability generating function of the queue size, $P(z)$ at an arbitrary time epoch is obtained as

$$P(z)=\frac{\left\{\begin{array}{c}\left(\tilde{S}_{1}((\lambda-\lambda X(z)))\tilde{S}_{2}((\lambda-\lambda X(z)))-1\right)\sum_{i=a}^{b-1}c_{i}\left(z^{b}--z^{i}\right)+\left(\tilde{V}((\lambda-\lambda X(z)))-1\right)\\ \left[\left(\left(\tilde{S}_{1}((\lambda-\lambda X(z)))\tilde{S}_{2}((\lambda-\lambda X(z)))\right)-1\right))+\left(Z^{b}-\tilde{S}_{1}((\lambda-\lambda X(z)))\tilde{S}_{2}((\lambda-\lambda X(z)))\right)\right]\sum_{i=0}^{a-1}(c_{i}-\pi p_{i}^{2})z^{i}\\ -\pi\left[\left(\tilde{R}((\lambda-\lambda X(z)))-1\right)+\left(Z^{b}-\tilde{S}_{1}((\lambda-\lambda X(z)))\tilde{S}_{2}((\lambda-\lambda X(z)))\right)\right]\sum_{i=0}^{a-1}p_{i}{}^{2}z^{i}\\ +\left[\left(\tilde{C}((\lambda-\lambda X(z)))-1\right)+\left(Z^{b}-\tilde{S}_{1}((\lambda-\lambda X(z)))\tilde{S}_{2}((\lambda-\lambda X(z)))\right)\right]\sum_{i=0}^{a-1}p_{i}{}^{2}z^{i}\\ +\left(\tilde{S}_{1}((\lambda-\lambda X(z)))\tilde{S}_{2}((\lambda-\lambda X(z)))-1\right)\sum_{i=a}^{b-1}\left(z^{b}-z^{i}\right)\sum_{m=0}^{a-1}T_{m}\lambda g_{i-m}\\ +\lambda T(z)\left(X(z)-1\right)\left(z^{b}-\tilde{S}_{1}((\lambda-\lambda X(z)))\tilde{S}_{2}((\lambda-\lambda X(z)))\right)\end{array}\right\}}{(-\lambda+\lambda X(z))\left(Z^{b}-\tilde{S}_{1}((\lambda-\lambda X(z)))\tilde{S}_{2}((\lambda-\lambda X(z)))\right)} \tag{51}$$

3.1. Steady State Condition

The probability generating function P(z) has to satisfy $P(1)=1$. In order to satisfy the condition, applying L'Hospital's rule and evaluating $\lim_{z\to\infty}P(z)$ and equating the expression to 1,

$$b-\lambda E(X)\left[\left[E(S_1)+E(S_2)+\pi E(R)\right]\right]>0$$

is obtained. Define 'ρ' as $\frac{\lambda E(X)[E(S_1)+E(S_2)+\pi E(R)]}{b}$. Thus $\rho<1$ is the condition to be satisfied for the existence of steady state for the model.

3.2. Computational Aspects of Unknowns Probabilities

Equation (51) gives PGF of the number of customers in the queue,which involves the unknown T_i and $\tilde{Q}_i(\theta)$ are expressed in terms p_i and the known function $\tilde{V}(\lambda)$ respectively. To find the unknown constants, Rouche's theorem of complex variables is used. By Rouche's theorem, it follows that $(z^b - \tilde{S}_1(\lambda - \lambda X(z))\tilde{S}_2(\lambda - \lambda X(z))$ has $b-1$ zeros inside and one on the unit circle $|z| = 1$. Since P(z) is analytic with in the on the unit circle, the numerator of (51) must vanish at these points, which gives b equations with be unknowns. Thus Equation (51) gives the PGF of the number of customers in the queue at an arbitrary time.

4. Conclusion

In this paper In this paper a batch arrival two types of general bulk service queuing system with vacation interrupted, optional service and closedown is considered. Probability generating function of queue size at an arbitrary time epoch.

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