International Journal of Mathematics And its Applications
Volume 4, Issue 4 (2016), 43–58. (Special Issue)
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Stability of a 2 - Variable AC - Mixed Type Functional Equation in Paranormed Spaces: Direct and Fixed Point Methods

Research Article*

M.Arunkumar[1], John M.Rassias[2], S.Murthy[3] and M.Arulselvan[4]

1 Department of Mathematics, Government Arts College, Tiruvannamalai, TamilNadu, India.
2 Pedagogical Department E.E, Section of Mathematics and Informatics, National and Capodistrian University of Athens, Greece.
3 Department of Mathematics, Government Arts College for Men, Krishnagiri, Tamil Nadu, India.
4 SRGDS Matriculation Hr. Sec. School, Tiruvannamalai, TamilNadu, India.

Abstract: In this paper, authors established the generalized Ulam - Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w)=4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w)$$

in paranormed spaces using direct and fixed point methods.

MSC: 39B52, 32B72, 32B82.

Keywords: Additive functional equations, cubic functional equation, Mixed type AC functional equation, Ulam - Hyers stability, Paranormed space.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers theorem was generalized by Aoki [2] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of Rassias theorem was obtained by Gavruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. In 1982, Rassias [21] followed the innovative approach of the Rassias theorem [24] in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p \cdot ||y||^q$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$. In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [27] by considering the summation of both the sum and the product of two $p-$norms in the spirit of Rassias approach. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [5, 12–14, 25].

* *Proceedings : National Conference on* `Recent Trends in Applied Mathematics` *held on 22 & 23.07.2016, organized by Department of Mathematics, St. Joseph's College of Arts & Science, Manjakuppam, Cuddalore (Tamil Nadu), India.*

Over the last six or seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involves only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 5, 10, 12–14, 23, 25, 28, 29, 34]. Recently, M. Arunkumar et al., [3] first time introduced and investigated the solution and generalized Ulam-Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x+y,2z+w)-f(2x-y,2z-w)=4[f(x+y,z+w)-f(x-y,z-w)]-6f(y,w) \quad (1)$$

having solutions

$$f(x,y)=ax+by \quad (2)$$

and

$$f(x,y)=ax^3+bx^2y+cxy^2+dy^3 \quad (3)$$

in Banach space via direct and fixed point approach. The solution of the AC functional equation (1) is given in the following lemmas.

Lemma 1.1 ([3]). *If $f:U^2\to V$ be a mapping satisfying (1) and let $g:U^2\to V$ be a mapping given by*

$$g(x,x)=f(2x,2x)-8f(x,x) \quad (4)$$

for all $x\in U$ then

$$g(2x,2x)=2g(x,x) \quad (5)$$

for all $x\in U$ such that g is additive.

Lemma 1.2 ([3]). *If $f:U^2\to V$ be a mapping satisfying (1) and let $h:U^2\to V$ be a mapping given by*

$$h(x,x)=f(2x,2x)-2f(x,x) \quad (6)$$

for all $x\in U$ then

$$h(2x,2x)=8h(x,x) \quad (7)$$

for all $x\in U$ such that h is cubic.

Remark 1.3 ([3]). *If $f:U^2\to V$ is a mapping satisfying (1) and let $g,h:U^2\to V$ be a mapping defined in (4) and (6) then*

$$f(x,x)=\frac{1}{6}(h(x,x)-g(x,x)) \quad (8)$$

for all $x\in U$.

Very recently, Choonkil Park and Jung Rye Lee [20] investigated the Hyers - Ulam stability of the following additive - quadratic - cubic - quartic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \quad (9)$$

in paranormed spaces.

In this paper, we obtain the generalized Ulam - Hyers stability of a 2 - variable AC - mixed type functional equation (1) in paranormed spaces using direct and fixed point methods. The generalized Ulam-Hyers stability in paranormed using direct and fixed point method is discussed in Section 3 and Section 4, respectively.

2. Basic Concepts on Paranormed Spaces

The concept of statistical convergence for sequences of real numbers was introduced by Fast [6] and Steinhaus [31] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [7, 15, 18, 19, 30]). This notion was defined in normed spaces by Kolk [16]. We recall some basic facts concerning Fréchet spaces.

Definition 2.1 ([33]). *Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that*

(P1) $P(0) = 0$*;*

(P2) $P(-x) = P(x)$*;*

(P3) $P(x + y) \le P(x) + P(y)$ *(triangle inequality);*

(P4) If $\{t_n\}$ *is a sequence of scalars with* $t_n \to t$ *and* $\{x_n\} \subset X$ *with* $P(x_n - x) \to 0$*, then* $P(t_n\, x_n - tx) \to 0$ *(continuity of multiplication).*

The pair (X, P) *is called a* **paranormed space** *if* P *is a* **paranorm** *on* X*.*

Definition 2.2 ([33]). *The paranorm is called total if, in addition, we have*

(P5) $P(x) = 0$ *implies* $x = 0$*.*

Definition 2.3 ([33]). *A Fréchet space is a total and complete paranormed space.*

3. Stability Results in Paranormed Space: Direct Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1) in paranormed using direct method. Through out this section let (U, P) be a Fréchet space and $(V, ||\cdot||)$ be a Banach space. Define a mapping $F : U^2 \to V$ by

$$F(x, y, z, w) = f(2x + y, 2z + w) - f(2x - y, 2z - w) - 4f(x + y, z + w) + 4f(x - y, z - w)] + 6f(y, w)$$

for all $x, y, z, w \in U$.

Theorem 3.1. *Let $j = \pm 1$. Let $f : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{2^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}w) = 0 \tag{10}$$

such that the functional inequality

$$P\left(F(x, y, z, w)\right) \le \alpha(x, y, z, w) \tag{11}$$

for all $x, y, z, w \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ satisfying the functional equation (1) and

$$P\left(f(2x, 2x) - 8f(x, x) - A(x, x)\right) \le \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}x)}{2^{kj}} \tag{12}$$

for all $x \in U$. The mapping $\beta(2^{kj}x)$ and $A(x, x)$ are defined by

$$\beta(2^{kj}x) = 4\ \alpha(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x) + \alpha(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x, 2^{(k+1)j}x) \tag{13}$$

$$P\left(\lim_{n\to\infty}\frac{1}{2^{nj}}\left(f(2^{(n+1)j}x,2^{(n+1)j}x)-8f(2^{nj}x,2^{nj}x)\right)-A(x,x)\right)\to 0 \tag{14}$$

for all $x\in U$.

Proof. Assume $j=1$. Letting (x,y,z,w) by (x,x,x,x) in (11), we obtain

$$P\left(f(3x,3x)-4f(2x,2x)+5f(x,x)\right)\leq\alpha(x,x,x,x) \tag{15}$$

for all $x\in U$. Replacing (x,y,z,w) by $(x,2x,x,2x)$ in (11), we get

$$P\left(f(4x,4x)-4f(3x,3x)+6f(2x,2x)-4f(x,x)\right)\leq\alpha(x,2x,x,2x) \tag{16}$$

for all $x\in U$. Now, from (15) and (16) and using $(P3)$, we have

$$P\left(f(4x,4x)-10f(2x,2x)+16f(x,x)\right)\leq 4\alpha(x,x,x,x)+\alpha(x,2x,x,2x) \tag{17}$$

for all $x\in U$. From (17), we arrive

$$P\left(f(4x,4x)-10f(2x,2x)+16f(x,x)\right)\leq\beta(x) \tag{18}$$

where

$$\beta(x)=4\alpha(x,x,x,x)+\alpha(x,2x,x,2x) \tag{19}$$

for all $x\in U$. It is easy to see from (18) that

$$P\left(f(4x,4x)-8f(2x,2x)-2(f(2x,2x)-8f(x,x))\right)\leq\beta(x) \tag{20}$$

for all $x\in U$. Using (4) in (20), we obtain

$$P\left(g(2x,2x)-2g(x,x)\right)\leq\beta(x) \tag{21}$$

for all $x\in U$. From (21), we arrive

$$P\left(\frac{g(2x,2x)}{2}-g(x,x)\right)\leq\frac{\beta(x)}{2} \tag{22}$$

for all $x\in U$. Now replacing x by $2x$ and dividing by 2 in (22), we get

$$P\left(\frac{g(2^2x,2^2x)}{2^2}-\frac{g(2x,2x)}{2}\right)\leq\frac{\beta(2x)}{2^2} \tag{23}$$

for all $x\in U$. From (22) and (23), we obtain

$$\begin{aligned}P\left(\frac{g(2^2x,2^2x)}{2^2}-g(x,x)\right)&\leq P\left(\frac{g(2x,2x)}{2}-g(x,x)\right)+P\left(\frac{g(2^2x,2^2x)}{2^2}-\frac{g(2x,2x)}{2}\right)\\&\leq\frac{1}{2}\left[\beta(x)+\frac{\beta(2x)}{2}\right]\end{aligned} \tag{24}$$

for all $x \in U$. Proceeding further and using induction on a positive integer n , we get

$$P\left(\frac{g(2^n x, 2^n x)}{2^n} - g(x,x)\right) \leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{\beta(2^k x)}{2^k} \leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\beta(2^k x)}{2^k} \tag{25}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{\frac{g(2^n x, 2^n x)}{2^n}\right\}$, replacing x by $2^m x$ and dividing by 2^m in (25), for any $m, n > 0$, we deduce

$$P\left(\frac{g(2^{n+m}x, 2^{n+m}x)}{2^{(n+m)}} - \frac{g(2^m x, 2^m x)}{2^m}\right) = \frac{1}{2^m}\,P\left(\frac{g(2^n \cdot 2^m x, 2^n \cdot 2^m x)}{2^n} - g(2^m x, 2^m x)\right) \leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\beta(2^{k+m}x)}{2^{k+m}} \to 0 \quad as \;\; m \to \infty$$

for all $x \in U$. This shows that the sequence $\left\{\frac{g(2^n x, 2^n x)}{2^n}\right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $A(x,x) : U^2 \to V$ such that

$$P\left(\lim_{n\to\infty}\frac{g(2^n x, 2^n x)}{2^n} - A(x,x)\right) \to 0 \;\; \forall \;\; x \in U.$$

By continuity of multiplication, we have

$$P\left(\lim_{n\to\infty}\frac{t_n\; g(2^n x, 2^n x)}{2^n} - tA(x,x)\right) \to 0 \;\; \forall \;\; x \in U.$$

Letting $n \to \infty$ in (25) and using (4), we see that (12) holds for all $x \in U$. To show that A satisfies (1), replacing (x,y,z,w) by $(2^n x, 2^n y, 2^n z, 2^n w)$ and dividing by 2^n in (11), we obtain

$$\frac{1}{2^n}P\Big(F(2^n x, 2^n y, 2^n z, 2^n w)\Big) \leq \frac{1}{2^n}\alpha(2^n x, 2^n y, 2^n z, 2^n w)$$

for all $x, y, z, w \in X$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x,x)$, we see that

$$P\left(A(2x+y, 2z+w) - A(2x-y, 2z-w) - 4[A(x+y, z+w) - A(x-y, z-w)] + 6A(y,w)\right) = 0, \tag{26}$$

for all $x, y, z, w \in X$. Using $(P5)$ in (26), we arrive

$$A(2x+y, 2z+w) - A(2x-y, 2z-w) = 4[A(x+y, z+w) - A(x-y, z-w)] - 6A(y,w).$$

Hence A satisfies (1) for all $x, y, z, w \in X$. To prove A is unique 2-variable additive function satisfying (1), we let $B(x,x)$ be another 2-variable additive mapping satisfying (1) and (12), then

$$\begin{aligned}(A(x,x) - B(x,x)) \leq \frac{1}{2^n}\big\{&P\left(A(2^n x, 2^n x) - f(2^{n+1}x, 2^{n+1}x) + 8f(2^n x, 2^n x)\right) \\ &+P\left(f(2^{n+1}x, 2^{n+1}x) - 8f(2^n x, 2^n x) - B(2^n x, 2^n x)\right)\big\} \\ \leq \sum_{k=0}^{\infty}\frac{\beta(2^{k+n}x)}{2^{(k+n)}} &\to 0 \quad as \;\; n \to \infty\end{aligned}$$

for all $x \in X$. Hence A is unique. For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1).

Corollary 3.2. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$P\left(F(x,y,z,w)\right) \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s<1 \quad or \quad s>1; \\ \lambda\ ||x||^s||y||^s||z||^s||w||^s, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \\ \lambda\left\{||x||^s||y||^s||z||^s||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \end{cases} \tag{27}$$

for all $x,y,z,w \in U$, then there exists a unique 2- variable additive function $A : U^2 \to V$ such that

$$P\left(f(2x,2x) - 8f(x,x) - A(x,x)\right) \leq \begin{cases} 5\lambda, \\ \dfrac{(18+2^{s+1})\lambda||x||^s}{|2-2^s|}, \\ \dfrac{(4+2^{2s})\lambda||x||^{4s}}{|2-2^{4s}|} \\ \left(\dfrac{(22+2^{2s})}{|2-2^{4s}|} + \dfrac{2\cdot 2^{4s}}{|2-2^{2s}|}\right)\lambda||x||^{4s} \end{cases} \tag{28}$$

for all $x \in U$.

Corollary 3.3. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x,y,z,w))\| \leq \begin{cases} \lambda, & \\ \lambda\left\{P(x)^s + P(y)^s + P(z)^s + P(w)^s\right\}, & s<1 \quad or \quad s>1; \\ \lambda\ P(x)^sP(y)^sP(z)^sP(w)^s, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \\ \lambda\left\{P(x)^sP(y)^sP(z)^sP(w)^s + \left\{P(x)^{4s} + P(y)^{4s} + P(w)^{4s} + P(z)^{4s}\right\}\right\}, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \end{cases} \tag{29}$$

for all $x,y,z,w \in U$, then there exists a unique 2- variable additive function $A : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \leq \begin{cases} 5\lambda, \\ \dfrac{(18+2^{s+1})\lambda P(x)^s}{|2-2^s|}, \\ \dfrac{(4+2^{2s})\lambda P(x)^{4s}}{|2-2^{4s}|} \\ \left(\dfrac{(22+2^{2s})}{|2-2^{4s}|} + \dfrac{2\cdot 2^{4s}}{|2-2^{2s}|}\right)\lambda P(x)^{4s} \end{cases} \tag{30}$$

for all $x \in U$.

Theorem 3.4. *Let $j = \pm 1$. Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{8^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}w) = 0 \tag{31}$$

such that the functional inequality

$$P\left(F(x,y,z,w)\right) \leq \alpha(x,y,z,w) \tag{32}$$

for all $x,y,z,w \in U$. Then there exists a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1) and

$$P\left(f(2x,2x) - 2f(x,x) - C(x,x)\right) \leq \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}x)}{8^{kj}} \tag{33}$$

for all $x \in U$. The mapping $\beta(2^{kj}x)$ and $C(x,x)$ are defined by

$$\beta(2^{kj}x) = 4\alpha(2^{kj}x, 2^{kj}x, 2^{kj}x, 2^{kj}x) + \alpha(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x, 2^{(k+1)j}x) \tag{34}$$

$$P\left(\lim_{n\to\infty} \frac{1}{8^{nj}}\left(f(2^{(n+1)j}x, 2^{(n+1)j}x) - 2f(2^{nj}x, 2^{nj}x)\right) - C(x,x)\right) \to 0 \tag{35}$$

for all $x \in U$.

Proof. It is easy to see from (18) that

$$P\left(f(4x,4x) - 2f(2x,2x) - 8(f(2x,2x) - 2f(x,x))\right) \leq \beta(x) \tag{36}$$

for all $x \in U$. Using (6) in (36), we obtain

$$P\left(h(2x,2x) - 8h(x,x)\right) \leq \beta(x) \tag{37}$$

for all $x \in U$. From (37), we arrive

$$P\left(\frac{h(2x,2x)}{8} - h(x,x)\right) \leq \frac{\beta(x)}{8} \tag{38}$$

for all $x \in U$. Now replacing x by $2x$ and dividing by 8 in (38), we get

$$P\left(\frac{h(2^2x,2^2x)}{8^2} - \frac{h(2x,2x)}{8}\right) \leq \frac{\beta(2x)}{8^2} \tag{39}$$

for all $x \in U$. From (38) and (39), we obtain

$$\begin{aligned} P\left(\frac{h(2^2x,2^2x)}{8^2} - h(x,x)\right) &\leq P\left(\frac{h(2x,2x)}{8} - h(x,x)\right) + P\left(\frac{h(2^2x,2^2x)}{8^2} - \frac{h(2x,2x)}{8}\right) \\ &\leq \frac{1}{8}\left[\beta(x) + \frac{\beta(2x)}{8}\right] \end{aligned} \tag{40}$$

for all $x \in U$. Proceeding further and using induction on a positive integer n , we get

$$P\left(\frac{h(2^nx,2^nx)}{8^n} - h(x,x)\right) \leq \frac{1}{8}\sum_{k=0}^{n-1}\frac{\beta(2^kx)}{8^k} \leq \frac{1}{8}\sum_{k=0}^{\infty}\frac{\beta(2^kx)}{8^k} \tag{41}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{\frac{h(2^nx,2^nx)}{8^n}\right\}$, replacing x by 2^mx and dividing by 8^m in (41), for any $m, n > 0$, we deduce

$$\begin{aligned} P\left(\frac{h(2^{n+m}x,2^{n+m}x)}{8^{(n+m)}} - \frac{h(2^mx,2^mx)}{8^m}\right) &= \frac{1}{8^m}\,P\left(\frac{h(2^n\cdot 2^mx, 2^n\cdot 2^mx)}{8^n} - h(2^mx,2^mx)\right) \\ &\leq \frac{1}{8}\sum_{k=0}^{n-1}\frac{\beta(2^{k+m}x)}{8^{k+m}} \leq \frac{1}{8}\sum_{k=0}^{\infty}\frac{\beta(2^{k+m}x)}{8^{k+m}} \to 0 \quad as \quad m\to\infty \end{aligned}$$

for all $x \in U$. This shows that the sequence $\left\{\frac{h(2^nx,2^nx)}{8^n}\right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $C(x,x) : U^2 \to V$ such that

$$P\left(\lim_{n\to\infty}\frac{h(2^nx,2^nx)}{8^n} - C(x,x)\right) \to 0 \quad \forall\ x \in U.$$

By continuity of multiplication, we have

$$P\left(\lim_{n\to\infty}\frac{t_n\ h(2^nx,2^nx)}{8^n} - tC(x,x)\right) \to 0 \quad \forall\ x \in U.$$

Letting $n \to \infty$ in (41) and using (6), we see that (33) holds for all $x \in U$. To show that C satisfies (1) and it is unique the proof is similar to that of Theorem 3.1. For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.4 concerning the stability of (1).

Corollary 3.5. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$P\left(F(x,y,z,w)\right) \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s<3 \quad or \quad s>3; \\ \lambda\ ||x||^s||y||^s||z||^s||w||^s, & s<\frac{3}{4} \quad or \quad s>\frac{3}{4}; \\ \lambda\left\{||x||^s||y||^s||z||^s||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s<\frac{3}{4} \quad or \quad s>\frac{3}{4}; \end{cases} \tag{42}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable cubic function $C : U^2 \to V$ such that

$$P\left(f(2x,2x) - 2f(x,x) - C(x,x)\right) \leq \begin{cases} \dfrac{5\lambda}{7}, \\ \dfrac{(18+2^{s+1})\lambda||x||^s}{7|8-2^s|}, \\ \dfrac{(4+2^{2s})\lambda||x||^{4s}}{7|8-2^{4s}|} \\ \left(\dfrac{(22+2^{2s})}{7|8-2^{4s}|} + \dfrac{2\cdot 2^{4s}}{7|8-2^{2s}|}\right)\lambda||x||^{4s} \end{cases} \tag{43}$$

for all $x \in U$.

Corollary 3.6. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x,y,z,w)\| \leq \begin{cases} \lambda, & \\ \lambda\left\{P(x)^s + P(y)^s + P(z)^s + P(w)^s\right\}, & s<3 \quad or \quad s>3; \\ \lambda\ P(x)^sP(y)^sP(z)^sP(w)^s, & s<\frac{3}{4} \quad or \quad s>\frac{3}{4}; \\ \lambda\left\{P(x)^sP(y)^sP(z)^sP(w)^s + \left\{P(x)^{4s} + P(y)^{4s} + P(w)^{4s} + P(z)^{4s}\right\}\right\}, & s<\frac{3}{4} \quad or \quad s>\frac{3}{4}; \end{cases} \tag{44}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable cubic function $C : U^2 \to V$ such that

$$P\left(f(2x,2x) - 2f(x,x) - C(x,x)\right) \leq \begin{cases} \dfrac{5\lambda}{7}, \\ \dfrac{(18+2^{s+1})\lambda P(x)^s}{7|8-2^s|}, \\ \dfrac{(4+2^{2s})\lambda P(x)^{4s}}{7|8-2^{4s}|} \\ \left(\dfrac{(22+2^{2s})}{7|8-2^{4s}|} + \dfrac{2\cdot 2^{4s}}{7|8-2^{2s}|}\right)\lambda P(x)^{4s} \end{cases} \tag{45}$$

for all $x \in U$.

Now, we are ready to prove our main stability results.

Theorem 3.7. *Let $j = \pm 1$. Let $F : U^2 \to V$ be a mapping for which there exist a function $\alpha : U^4 \to [0, \infty)$ with the condition given in (10) and (31) respectively, such that the functional inequality*

$$P\left(F(x,y,z,w)\right) \leq \alpha(x,y,z,w) \tag{46}$$

for all $x, y, z, w \in U$. Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1) and

$$P\left(f(x,x) - A(x,x) - C(x,x)\right) \leq \frac{1}{6}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{2^{kj}} + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{8^{kj}}\right\} \tag{47}$$

for all $x \in U$. The mapping $\beta(2^{kj}x)$, $A(x,x)$ and $C(x,x)$ are respectively defined in (13), (14) and (35) for all $x \in U$.

Proof. By Theorems 3.1 and 3.4, there exists a unique 2-variable additive function $A_1 : U^2 \to V$ and a unique 2-variable cubic function $C_1 : U^2 \to V$ such that

$$P\left(f(2x, 2x) - 8f(x, x) - A_1(x, x)\right) \leq \frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{2^{kj}} \tag{48}$$

and

$$P\left(f(2x, 2x) - 2f(x, x) - C_1(x, x)\right) \leq \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{8^{kj}} \tag{49}$$

for all $x \in U$. Now from (48) and (49), one can see that

$$\begin{aligned}
&P\left(f(x, x) + \frac{1}{6}A_1(x, x) - \frac{1}{6}C_1(x, x)\right)\\
&= P\left(\left\{-\frac{f(2x, 2x)}{6} + \frac{8f(x, x)}{6} + \frac{A_1(x, x)}{6}\right\} + \left\{\frac{f(2x, 2x)}{6} - \frac{2f(x, x)}{6} - \frac{C_1(x, x)}{6}\right\}\right)\\
&\leq \frac{1}{6}\left\{P\left(f(2x, 2x) - 8f(x, x) - A_1(x, x)\right) + P\left(f(2x, 2x) - 2f(x, x) - C_1(x, x)\right)\right\}\\
&\leq \frac{1}{6}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{2^{kj}} + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{8^{kj}}\right\}
\end{aligned}$$

for all $x \in U$. Thus we obtain (49) by defining $A(x, x) = \frac{-1}{6}A_1(x, x)$ and $C(x, x) = \frac{1}{6}C_1(x, x)$, $\beta(2^{kj}x)$, $A(x, x)$ and $C(x, x)$ are respectively defined in (13), (14) and (35) for all $x \in U$. □

The following corollary is the immediate consequence of Theorem 3.7, using Corollaries 3.2 and 3.5 concerning the stability of (1).

Corollary 3.8. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$P\left(F(x, y, z, w)\right) \leq \begin{cases}\lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s\right\}, & s < 1 \quad or \quad s > 1;\\ \lambda\ ||x||^s||y||^s||z||^s||w||^s, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4};\\ \lambda\left\{||x||^s||y||^s||z||^s||w||^s + \left\{||x||^{4s} + ||y||^{4s} + ||w||^{4s} + ||z||^{4s}\right\}\right\}, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4};\end{cases} \tag{50}$$

for all $x, y, z, w \in U$, then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ such that

$$P\left(f(x, x) - A(x, x) - C(x, x)\right) \leq \begin{cases}\frac{5\lambda}{6}\left(1 + \frac{1}{7}\right),\\ \frac{(18 + 2^{s+1})}{6}\left(\frac{1}{|2 - 2^s|} + \frac{1}{7|8 - 2^s|}\right)\lambda||x||^s,\\ \frac{(4 + 2^{2s})}{6}\left(\frac{1}{|2 - 2^{4s}|} + \frac{1}{7|8 - 2^{4s}|}\right)\lambda||x||^{4s}\\ \frac{(22+2^{2s})}{6}\left(\left(\frac{1}{|2-2^{4s}|} + \frac{1}{7|8-2^{4s}|}\right) + \frac{(2\cdot 2^{4s})}{6}\left(\frac{1}{|2 - 2^{2s}|} + \frac{1}{7|8 - 2^{2s}|}\right)\right)\lambda||x||^{4s}\end{cases} \tag{51}$$

for all $x \in U$.

Corollary 3.9. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x, y, z, w)\| \leq \begin{cases}\lambda, & \\ \lambda\left\{P(x)^s + P(y)^s + P(z)^s + P(w)^s\right\}, & s < 1 \quad or \quad s > 1;\\ \lambda\ P(x)^sP(y)^sP(z)^sP(w)^s, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4};\\ \lambda\left\{P(x)^sP(y)^sP(z)^sP(w)^s + \left\{P(x)^{4s} + P(y)^{4s} + P(w)^{4s} + P(z)^{4s}\right\}\right\}, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4};\end{cases} \tag{52}$$

for all $x, y, z, w \in U$, then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ such that

$$\|f(x,x) - A(x,x) - C(x,x)\| \leq \begin{cases} \frac{5\lambda}{6}\left(1+\frac{1}{7}\right), \\ \frac{(18+2^{s+1})}{6}\left(\frac{1}{|2-2^s|}+\frac{1}{7|8-2^s|}\right)\lambda P(x)^s, \\ \frac{(4+2^{2s})}{6}\left(\frac{1}{|2-2^{4s}|}+\frac{1}{7|8-2^{4s}|}\right)\lambda P(x)^{4s} \\ \frac{(22+2^{2s})}{6}\left(\left(\frac{1}{|2-2^{4s}|}+\frac{1}{7|8-2^{4s}|}\right)+\frac{(2\cdot 2^{4s})}{6}\left(\frac{1}{|2-2^{2s}|}+\frac{1}{7|8-2^{2s}|}\right)\right)\lambda P(x)^{4s} \end{cases} \quad (53)$$

for all $x \in U$.

4. Stability Results: Fixed Point Method

In this section, we apply a fixed point method for achieving stability of the 2-variable AC functional equation (1). Now, we present the following theorem due to B. Margolis and J.B. Diaz [17] for fixed point Theory.

Theorem 4.1 ([17])**.** *Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \quad \forall \quad n \geq 0,$$

or there exists a natural number n_0 such that

(A1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(A2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T

(A3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;

(A4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of (1).

Through out this section let (U, P) be a Fréchet space and V be a Banach space. Define a mapping $F : U^2 \to V$ by

$$F(x,y,z,w) = f(2x+y, 2z+w) - f(2x-y, 2z-w) - 4f(x+y, z+w) + 4f(x-y, z-w)] + 6f(y,w)$$

for all $x, y, z, w \in U$.

Theorem 4.2. *Let $F : U^2 \to V$ be a mapping for which there exists a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^n} \alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w) = 0 \quad (54)$$

where $\mu_i = 2$ if $i = 0$ and $\mu_1 = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$P\left(F(x,y,z,w)\right) \leq \alpha(x,y,z,w) \quad (55)$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \frac{1}{2}\,\beta\left(x\right),$$

has the property

$$\gamma(x) \le L\ \mu_i\ \gamma\left(\mu_i x\right). \tag{56}$$

Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ satisfying the functional equation (1) and

$$P\left(f(2x,2x) - 8f(x,x) - A(x,x)\right) \le \frac{L^{1-i}}{1-L}\gamma(x) \tag{57}$$

for all $x \in U$. The mapping $\beta(x)$ and $A(x,x)$ are defined in (19) and (14) respectively for all $x \in U$.

Proof. Consider the set $\Omega = \{p/p : U^2 \to V,\ p(0,0) = 0\}$ and introduce the generalized metric on Ω, $d(p,q) = d_\gamma(p,q) = \inf\{K \in (0,\infty) : P\left(p(x,x) - q(x,x)\right) \le K\gamma(x), x \in U\}$. It is easy to see that (Ω, d) is complete. Define $T : \Omega^2 \to \Omega$ by $Tp(x,x) = \frac{1}{\mu_i}p(\mu_i x, \mu_i x)$, for all $x \in U$. Now $p, q \in \Omega$, implies $d(Tp,Tq) \le Ld(p,q)$, for all $p,q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L. From (21), we arrive

$$P\left(\frac{g(2x,2x)}{2} - g(x,x)\right) \le \frac{\beta(x)}{2} \tag{58}$$

for all $x \in U$. Using (56) for the case $i = 0$ it reduces to

$$P\left(\frac{g(2x,2x)}{2} - g(x,x)\right) \le\ L\gamma(x)$$

for all $x \in U$,

$$\text{i.e.,}\quad d_\gamma(g,Tg) \le L \Rightarrow d(g,Tg) \le L \le L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (58), we get,

$$P\left(g(x,x) - 2g\left(\frac{x}{2},\frac{x}{2}\right)\right) \le \beta\left(\frac{x}{2}\right) \tag{59}$$

Using (56) for the case $i = 1$ it reduces to

$$P\left(g(x,x) - 2g\left(\frac{x}{2},\frac{x}{2}\right)\right) \le\ \gamma(x)$$

for all $x \in U$,

$$\text{i.e.,}\quad d_\gamma(g,Tg) \le 1 \Rightarrow d(g,Tg) \le 1 \le L^0 < \infty.$$

In both cases, we arrive $d(g,Tg) \le L^{1-i}$. Therefore $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point A of T in Ω such that

$$P\left(\lim_{n\to\infty}\frac{1}{\mu_i^n}(f(\mu_i^{(n+1)}x, \mu_i^{(n+1)}x) - 8f(\mu_i^n x, \mu_i^n x)) - A(x,x)\right) \to 0 \tag{60}$$

for all $x \in U$.

To prove $A : U^2 \to V$ is additive. Replacing (x,y,z,w) by $(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n)$ in (55) and dividing by μ_i^n, it follows from (54) that

$$P\left(A(x,y,z,w)\right) = \lim_{n\to\infty} P\left(\frac{1}{\mu_i^n}F(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n)\right) \le \lim_{n\to\infty}\frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)}{\mu_i^n} = 0$$

for all $x,y,z,w \in U$, i.e., A satisfies the functional equation (1). By $(A3)$, A is the unique fixed point of T in the set $\Delta = \{A \in \Omega : d(f,A) < \infty\}, A$ is the unique function such that $P\left(f(2x,2x) - 8f(x,x) - A(x,x)\right) \le K\gamma(x)$ for all $x \in U$ and $K > 0$. Finally by $(A4)$, we obtain $d(f,A) \le \frac{1}{1-L}d(f,Tf)$ this implies $d(f,A) \le \frac{L^{1-i}}{1-L}$ which yields $P\left(f(2x,2x) - 8f(x,x) - A(x,x)\right) \le \frac{L^{1-i}}{1-L}\gamma(x)$ this completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 4.2 concerning the stability of (1).

Corollary 4.3. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x,y,z,w)\| \leq \begin{cases} \lambda, & \\ \lambda\left\{P(x)^s + P(y)^s + P(z)^s + P(w)^s\right\}, & s<1 \quad or \quad s>1; \\ \lambda\ P(x)^s P(y)^s P(z)^s P(w)^s, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \\ \lambda\left\{P(x)^s P(y)^s P(z)^s P(w)^s + \left\{P(x)^{4s} + P(y)^{4s} + P(w)^{4s} + P(z)^{4s}\right\}\right\}, & s<\frac{1}{4} \quad or \quad s>\frac{1}{4}; \end{cases} \tag{61}$$

for all $x,y,z,w \in U$, then there exists a unique 2- variable additive function $A : U^2 \to V$ such that

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \leq \begin{cases} \frac{2^{(s-1)}(18+2^{s+1})\lambda P(x)^s}{|2-2^s|}, \\ \frac{2^{(4s-1)(4+2^{2s})}\lambda P(x)^{4s}}{|2-2^{4s}|} \\ \frac{2^{(4s-1)}\left(22+2^{2s}+2\cdot 2^{4s}\right)\lambda P(x)^{4s}}{2-2^{4s}} \end{cases} \tag{62}$$

for all $x \in U$.

Proof. Setting

$$\alpha(x,y,z,w) = \begin{cases} \lambda\left\{P(x)^s + P(y)^s + P(z)^s + P(w)^s\right\}, \\ \lambda\ P(x)^s P(y)^s P(z)^s P(w)^s \\ \lambda\left\{P(x)^s P(y)^s P(z)^s P(w)^s + \left\{P(x)^{4s} + P(y)^{4s} + P(w)^{4s} + P(z)^{4s}\right\}\right\} \end{cases}$$

for all $x,y,z,w \in U$. Now

$$\frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w)}{\mu_i^n} = \begin{cases} \frac{\lambda}{\mu_i^n}\{P(\mu_i^n x)^s + P(\mu_i^n y)^s + P(\mu_i^n z)^s + P(\mu_i^n w)^s\}, \\ \frac{\lambda}{\mu_i^n}\ \ P(\mu_i^n x)^s\ P(\mu_i^n y)^s\ P(\mu_i^n z)^s\ P(\mu_i^n w)^s \\ \frac{\lambda}{\mu_i^n}\ \left\{P(\mu_i^n x)^s\ P(\mu_i^n y)^s\ P(\mu_i^n z)^s\ P(\mu_i^n w)^s\right. \\ \left.+\left\{P(\mu_i^n x)^{4s} + P(\mu_i^n y)^{4s} + P(\mu_i^n z)^{4s} + P(\mu_i^n w)^{4s}\right\}\right\} \end{cases} \begin{cases} \to 0 \text{ as } n\to\infty, \\ \to 0 \text{ as } n\to\infty, \\ \to 0 \text{ as } n\to\infty. \end{cases}$$

Thus, (54) is holds. But we have $\gamma(x) = \frac{1}{2}\ \beta(x)$ has the property $\gamma(x) \leq L\cdot\mu_i\ \gamma(\mu_i x)$ for all $x \in U$. Hence

$$\gamma(x) = \frac{1}{2}\ \beta(x) = \frac{1}{2}\left(4\alpha(x,x,x,x) + \alpha(x,2x,x,2x)\right) = \begin{cases} \frac{\lambda}{2}\left(18P(x)^s + 2P(2x)^s\right), \\ \frac{\lambda}{2}\left(4+2^{2s}\right)P(x)^{4s}, \\ \frac{\lambda}{2}\left(22+2^{2s}+2\cdot 2^{4s}\right)P(x)^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i}\gamma(\mu_i x) = \begin{cases} \mu_i^{s-1}\frac{\lambda}{2}\left(18+2^{s+1}\right)P(x)^s, \\ \mu_i^{4s-1}\frac{\lambda}{2}\left(4+2^{2s}\right)P(x)^{4s}, \\ \mu_i^{4s-1}\frac{\lambda}{2}\left(22+2^{2s}+2\cdot 2^{4s}\right)P(x)^{4s}. \end{cases} \begin{cases} \mu_i^{s-1}\beta(x), \\ \mu_i^{4s-1}\beta(x), \\ \mu_i^{4s-1}\beta(x). \end{cases}$$

Hence the inequality (56) holds either, $L = 2^{s-1}$ for $s<1$ if $i=0$ and $L = \frac{1}{2^{s-1}}$ for $s>1$ if $i=1$. Now from (57), we prove the following cases for condition (i).

Case:1 $L = 2^{s-1}$ for $s<1$ if $i=0$

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \leq \frac{\left(2^{(s-1)}\right)^{1-0}}{1-2^{(s-1)}}\left\{\frac{18+2^{(s+1)}}{2}\right\}\ \lambda P(x)^s \leq \frac{2^{(s-1)}\left(18+2^{(s+1)}\right)\lambda P(x)^s}{2-2^s}$$

Case:2 $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\|f(2x,2x) - 8f(x,x) - A(x,x)\| \leq \frac{\left(\frac{1}{2^{(s-1)}}\right)^{1-1}}{1 - \frac{1}{2^{(s-1)}}} \left\{ \frac{18 + 2^{(s+1)}}{2} \right\} \lambda P(x)^s \leq \frac{2^{(s-1)} \left(18 + 2^{(s+1)}\right) \lambda P(x)^s}{2^s - 2}$$

The rest of the cases is similar to that of Case 1 and 2 Hence the proof is complete □

Theorem 4.4. *Let $F : U^2 \to V$ be a mapping for which there exists a function $\alpha : U^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^{3n}} \alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n w) = 0 \tag{63}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_1 = \frac{1}{2}$ if $i = 1$ such that the functional inequality

$$P\left(F(x,y,z,w)\right) \leq \alpha(x,y,z,w) \tag{64}$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function $x \to \gamma(x) = \frac{1}{2}\ \beta(x)$, has the property

$$\gamma(x) \leq L\ \mu_i^3\ \gamma\left(\mu_i x\right). \tag{65}$$

Then there exists a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1) and

$$P\left(f(2x,2x) - 2f(x,x) - C(x,x)\right) \leq \frac{L^{1-i}}{1-L}\gamma(x) \tag{66}$$

for all $x \in U$. The mapping $\beta(x)$ and $C(x,x)$ are defined in (19) and (35) respectively for all $x \in U$.

Corollary 4.5. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x,y,z,w)\| \leq \begin{cases} \lambda \left\{P(x)^s + P(y)^s + P(z)^s + P(w)^s\right\}, & s < 1 \quad or \quad s > 1; \\ \lambda\ P(x)^s P(y)^s P(z)^s P(w)^s, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \\ \lambda \left\{P(x)^s P(y)^s P(z)^s P(w)^s + \left\{P(x)^{4s} + P(y)^{4s} + P(w)^{4s} + P(z)^{4s}\right\}\right\}, & s < \frac{3}{4} \quad or \quad s > \frac{3}{4}; \end{cases} \tag{67}$$

for all $x, y, z, w \in U$, then there exists a unique 2- variable cubic function $C : U^2 \to V$ such that

$$\|f(2x,2x) - 2f(x,x) - C(x,x)\| \leq \begin{cases} \frac{2^{(s-1)}(18 + 2^{s+1})\lambda P(x)^s}{7|8 - 2^s|}, \\ \frac{2^{(4s-1)}(4 + 2^{2s})\lambda P(x)^{4s}}{7|8 - 2^{4s}|} \\ \frac{2^{(4s-1)}\left(22 + 2^{2s} + 2 \cdot 2^{4s}\right)\lambda P(x)^{4s}}{7|8 - 2^{4s}|} \end{cases} \tag{68}$$

for all $x\ \in\ U$.

Now, we are ready to prove the main fixed point stability results.

Theorem 4.6. *Let $F : U^2 \to V$ be a mapping for which there exists a function $\alpha : U^4 \to [0, \infty)$ with the conditions (54) and (63) where $\mu_i = 2$ if $i = 0$ and $\mu_1 = \frac{1}{2}$ if $i = 1$ such that the functional inequality*

$$P\left(F(x,y,z,w)\right) \leq \alpha(x,y,z,w) \tag{69}$$

for all $x, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function $x \to \gamma(x) = \frac{1}{2}\ \beta(x)$, has the properties (56) and (65). Then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ satisfying the functional equation (1) and

$$P\left(f(x,x) - A(x,x) - C(x,x)\right) \leq \frac{1}{3}\frac{L^{1-i}}{1-L}\gamma(x) \tag{70}$$

for all $x \in U$. The mapping $\beta(x), A(x,x)$ and $C(x,x)$ are defined in (19), (14) and (35) respectively for all $x \in U$.

Proof. By Theorems 4.2 and 4.4, there exists a unique 2-variable additive function $A_1 : U^2 \to V$ and a unique 2-variable cubic function $C_1 : U^2 \to V$ such that

$$P\left(f(2x,2x) - 8f(x,x) - A_1(x,x)\right) \leq \frac{L^{1-i}}{1-L}\gamma(x) \text{ and} \tag{71}$$

$$P\left(f(2x,2x) - 2f(x,x) - C_1(x,x)\right) \leq \frac{L^{1-i}}{1-L}\gamma(x) \tag{72}$$

for all $x \in U$. Now from (71) and (72), one can see that

$$\begin{aligned} P\left(f(x,x) + \frac{1}{6}A_1(x,x) - \frac{1}{6}C_1(x,x)\right) &\leq \frac{1}{6}\left\{P\left(f(2x,2x) - 8f(x,x) - A_1(x,x)\right) + P\left(f(2x,2x) - 2f(x,x) - C_1(x,x)\right)\right\} \\ &\leq \frac{1}{6}\left\{\frac{L^{1-i}}{1-L}\gamma(x) + \frac{L^{1-i}}{1-L}\gamma(x)\right\} \end{aligned}$$

for all $x \in U$. Thus we obtain (70) by defining $A(x,x) = \frac{-1}{6}A_1(x,x)$ and $C(x,x) = \frac{1}{6}C_1(x,x)$,$\beta(x)$, $A(x,x)$ and $C(x,x)$ are respectively defined in (19), (14) and (35) for all $x \in U$. □

The following Corollary is an immediate consequence of Theorem 4.6, using Corollaries 4.3 and 4.5 concerning the stability of (1).

Corollary 4.7. *Let $F : U^2 \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|F(x,y,z,w)\| \leq \begin{cases} \lambda\left\{P(x)^s + P(y)^s + P(z)^s + P(w)^s\right\}, & s < 1 \quad or \quad s > 1; \\ \lambda \; P(x)^s P(y)^s P(z)^s P(w)^s, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \\ \lambda\left\{P(x)^s P(y)^s P(z)^s P(w)^s + \left\{P(x)^{4s} + P(y)^{4s} + P(w)^{4s} + P(z)^{4s}\right\}\right\}, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \end{cases} \tag{73}$$

for all $x,y,z,w \in U$, then there exists a unique 2-variable additive mapping $A : U^2 \to V$ and a unique 2-variable cubic mapping $C : U^2 \to V$ such that

$$\|f(x,x) - A(x,x) - C(x,x)\| \leq \begin{cases} \dfrac{2^{(s-1)}(18+2^{s+1})}{3}\left(\dfrac{1}{|2-2^s|} + \dfrac{1}{7|8-2^s|}\right)\lambda P(x)^s, \\ \dfrac{2^{(4s-1)}(4+2^{2s})}{3}\left(\dfrac{1}{3|2-2^s|} + \dfrac{1}{7|8-2^s|}\right)\lambda P(x)^{4s}, \\ \dfrac{2^{(4s-1)}\left(22+2^{2s}+2\cdot 2^{4s}\right)}{3}\left(\dfrac{1}{|2-2^s|} + \dfrac{1}{7|8-2^s|}\right)\lambda P(x)^{4s} \end{cases} \tag{74}$$

for all $x \in U$.

References

[1] J.Aczel and J.Dhombres, *Functional Equations in Several Variables*, Cambridge Univ, Press, (1989).

[2] T.Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2(1950), 64-66.

[3] M.Arunkumar, Matina J.Rassias and Yanhui Zhang, *Ulam - Hyers stability of a 2- variable AC - mixed type functional equation: direct and fixed point methods*, Journal of Modern Mathematics Frontier (JMMF), 1(3)(2012), 10-26.

[4] J.H.Bae and W.G.Park, *A functional equation originating from quadratic forms*, J. Math. Anal. Appl., 326(2007), 1142-1148.

[5] S.Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, (2002).

[6] H.Fast, *Sur la convergence statistique*, Colloq Math., 2(1951), 241-244.

[7] JA.Fridy, *On statistical convergence*, Analysis, 5(1985), 301-313.

[8] P.Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184(1994), 431-436.

[9] M.Eshaghi Gordji and H.Khodaie, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, arxiv: 0812. 2939v1 Math FA, 15 Dec 2008.

[10] M.Eshaghi Gordji, H.Khodaei and J.M.Rassias, *Fixed point methods for the stability of general quadratic functional equation*, Fixed Point Theory, 12(1)(2011), 71-82.

[11] D.H.Hyers, *On the stability of the linear functional equation*, Proc.Nat. Acad.Sci., U.S.A., 27(1941), 222-224.

[12] D.H.Hyers, G.Isac and Th.M.Rassias, *Stability of functional equations in several variables*, Birkhauser, Basel, (1998).

[13] S.M.Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, (2001).

[14] Pl.Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, (2009).

[15] S.Karakus, *Statistical convergence on probabilistic normed spaces*, Math Commun., 12(2007), 11-23.

[16] E.Kolk *The statistical convergence in Banach spaces*, Tartu Ul Toime., 928(1991), 41-52.

[17] B.Margoils and J.B.Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., 126(74)(1968), 305-309.

[18] M.Mursaleen, *λ-statistical convergence*, Math Slovaca., 50(2000), 111-115.

[19] M.Mursaleen and SA.Mohiuddine, *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J Comput Anal Math., 233(2009), 142-149.

[20] C.Park and J.R.Lee, *An AQCQ-functional equation in paranormed spaces*, Advances in Difference Equations, doi: 10.1186/1687-1847-2012-63.

[21] J.M.Rassias, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, 46(1982), 126-130.

[22] J.M.Rassias, K.Ravi, M.Arunkumar and B.V.Senthil Kumar, *Ulam Stability of Mixed type Cubic and Additive functional equation*, Functional Ulam Notions (F.U.N) Nova Science Publishers, (2010), Chapter 13, 149-175.

[23] J.M.Rassias, E.Son and H.M.Kim, *On the Hyers-Ulam stability of 3D and 4D mixed type mappings*, Far East J. Math. Sci., 48(1)(2011), 83-102.

[24] Th.M.Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72(1978), 297-300.

[25] Th.M.Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht, Bostan London, (2003).

[26] K.Ravi and M.Arunkumar, *Stability of a 3-variable Quadratic Functional Equation*, Journal of Quality Measurement and Analysis, 4(1)(2008), 97-107.

[27] K.Ravi, M.Arunkumar and J.M.Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, 3(08)(2008), 36-47.

[28] K.Ravi, J.M.Rassias, M.Arunkumar and R.Kodandan, *Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation*, J. Inequal. Pure Appl. Math., 10(4)(2009), Article 114.

[29] S.M.Jung and J.M.Rassias, *A fixed point approach to the stability of a functional equation of the spiral of Theodorus*, Fixed Point Theory Appl., 2008(2008), Article ID 945010.

[30] T.Salat, *On the statistically convergent sequences of real numbers*, Math Slovaca., 30(1980), 139-150.

[31] H.Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq Math., 2(1951), 73-34.

[32] S.M.Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, NewYork, (1964).

[33] A.Wilansky, *Modern Methods in Topological Vector Space*, McGraw-Hill International Book Co., New York, (1978).

[34] T.Z.Xu, J.M.Rassias and W.X.Xu, *A fixed point approach to the stability of a general mixed AQCQ-functional equation in non-Archimedean normed spaces*, Discrete Dyn. Nat. Soc., 2010(2010), Article ID 812545.