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Fundamental Operation of Differential Transformation Method

Research Article*

T.Henson[1] and G.Arunagiri[1]

1 P.G. and Research Department of Mathematics, St. Joseph's College of Arts and Science, Cuddalore, Tamilnadu, India.

Abstract: This study deals about the application with fundamental operation of differential transformation method and is used to find the approximate solution for SIR model. SIR model are nonlinear system with fundamental operation of ordinary differential equation that has no analytic solution. DTM uses the fundamental operations of differential transformation function of the original nonlinear system. The result reveals that this method is accurate and efficient for solving systems of ordinary differential equations.

Keywords: Fundamental operations of differential transformation method, Epidemiology and SIR model.

1. Introduction

Epidemiology studies the spread of diseases in population and primarily the human population. Often the work of a mathematical epidemiologist consists of model building, estimation of parameters and investigation of the sensitivity of models to changes in the parameters and numerical simulations. All these activities are expected to tell us something about the spread of the disease in the population, the possibility to control this spread and maybe how to make the disease disappear from the population these diseases modeled most often are the so called infectious diseases, i.e. disease that are contagious and can be transferred from one individual to another through contact. Examples of such diseases are the childhood diseases: measles, rubella, chicken pox and mumps the sexually transmitted diseases: HIV/AIDS, gonorrhea, syphilis and others. Although epidemiologic approaches can be applied to all types of disease, injury, and health conditions, the chain of infection for infectious diseases is better understood. In addition, infectious diseases remain an important focus of state and local public health department activities. Description the some of the key concepts of infectious disease epidemiology are presented below. These concepts are rooted in infectious disease, but are also relevant to noninfectious diseases.

2. S-I-R Model

The S-I-R model was introduced by W.O.Kermack and has played a major role in mathematical epidemiology. In the model, a population is divided into three groups: the susceptible(s), the infective (i), and the recovered (r), with numbers s, i and

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r respectively. The total population (n) $isn = s + I + r$ the susceptible are those who are not infected and not immune, the infective are those who are infected and can transmit the disease, and the recovered are those who have been infected, have recovered and are permanently immune. Consider an epidemic that occurs on a timescale that is much shorter than that of the population ,in other words regard the population as having a constant size and ignore births and deaths then we have the following system of nonlinear ordinary differential equation

$$\frac{ds}{dt} = -c\beta s(t)\frac{i(t)}{n}$$
$$\frac{di}{dt} = c\beta s(t)\frac{i}{n} - \gamma i(t)(a)$$
$$\frac{dr}{dt} = \gamma i(t)$$

This model is appropriate to viral diseases such as meals, mumps and rubella. The SIR model has no analytical solution, but we can conduct numerical simulation for the approximate solution. Including births and deaths in the standard S-I-R model (1) for epidemics the resulting model will allow us to look at events of longer duration. We consider a model with a constant birth rate and a constant per-capita death rate. We assume that all births are into the susceptible and the death rate is equal for members of all three classes. The birth and death rates are equal so that population is stationary then we have the system.

$$\frac{ds}{dt} = \mu(s + i + r) - \alpha s(t)i(t) - \mu s(t)$$
$$\frac{di}{dt} = \alpha s(t)i(t) - (\gamma + \mu)i(t)(b)$$
$$\frac{dr}{di} = \gamma i(t) - \mu r(t)$$

Notation for Epidemic and Network Parameters

S	the fractional of population susceptible
I	the fractional of population infectious
R	the fractional of population recovered
$S(t)$	the number of susceptible at time t respectively
$I(t)$	the number of infective at time t respectively
$R(t)$	the number of recovered at time t respectively
C	the number of contacts per unit time
β	the probability of disease transmission per contact
γ	the per-capita recovery rate
μ	the per-capita removal rate
-α	the transitivity

The equation (a) and (b) are solving using differential transformation method for numerical comparison.The purpose of this paper is to employ the differential transformation method to systems of differential equations which are often encountered in many branches of physics, chemical and engineering. The differential transformation technique is one of the numerical methods for ordinary differential equations whichuse the form of polynomials as the approximation to the exact solution. The concept of differential transformation was first proposed by zhou [2], (see Ref. [3], [4], [5] and [6]), and it was applied to solve linear and non-linear initial value problems in circuit analysis. Abbasov et all [8] used the used the method of differential transform to obtain approximate solutions of the linear and non-linear equations related to engineering problems and observed that the numerical results are in good agreement with the analytical solutions . In this paper, the differential transformation technique is applied to solve the linear, non-linear and stiff systems of differential equations, the method can be used to evaluate approximating solution by the transformed equations obtained from the original equation using the fundamental operations of differential transformation. The fundamental differential transformation technique can be used to obtain both numerical and analytical solutions of both linear and non-linear differential equations.

The present paper has been organized as following in section 3, basic definition of the differential transformations are introduced. The operational properties of the differential transformation are shown in section 3, Numerical examples have been presented in section 4, to illustrate the effectiveness of the proposed method

3. Basic Definitions

With reference to articles [9], [5], [6] and [10] we introduce in this section the basic definition of the differential transformation.

Definition 3.1. *It $u(t)$ is analytic in the domain T, then it will be differentiated continuously with respective time t*

$$\frac{d^k U(t)}{dt^k} = \varphi(t.k), \quad for\ all\ \ t \in T \tag{1}$$

For $t = t_i$. Then $\varphi(t.k) = \varphi(t_j.k)$, where k belongs to the set of non-negative integers, denoted as the K-domain. Therefore, equation (1) can be rewritten as

$$U(k) = \varphi(t_i.k) = [\frac{d^k U(t)}{dt^k}]_{t=ti}, \tag{2}$$

where U(K) is called the spectrum of U(t) at $t = t_j$.

Definition 3.2. *If u(t) can be expressed by Taylor' is series, then U(t) can be represented as*

$$u(t) = \sum_{k=0}^{\infty} [\frac{(t-t_i)^k}{k!}] U(k) \tag{3}$$

Equation (3) is called the inverse of U(K). Using the symbol D denoting the differential transformation process and combining (2) and (3), it is obtained that

$$u(t) = \sum_{k=0}^{\infty} [\frac{(t-t_i)^k}{k!}] U(k) \cong D^{-} U(k) \tag{4}$$

Using the differential transformation, a differential equation in the domain of interest can be transformed to an algebraic equation in the K-domain and the u(t) can be obtained by a finite –term of Taylor's series plus a remainder thus

$$u(t) = \sum_{k=0}^{n} [\frac{(t-t_i)^k}{k!}] U(k) + R_{n+1}(t) \tag{5}$$

4. The Operation Properties of Differential Transformation Method

It $u(t)$ and $v(t)$ are two uncorrelated functions with time t where $U(K)$ and $V(K)$ are the transformed functions corresponding to $u(t)$ and $v(t)$ then we can easily proof the fundamental mathematics operation performed by differential transformation and are listed as follows see [9], [5] and [6]:

Formulae

1. It $z(t) = U(t) \pm V(t)$ then,

$$Z(K) = U(K) + V(K) \tag{6}$$

2. It $z(t) = \alpha U(t)$, then

$$Z(K) = \alpha U(K) \tag{7}$$

3. If $z(t) = \frac{du(t)}{dt}$ then,

$$Z(K) = (K+1)U(K+1) \tag{8}$$

4. If $z(t) = \frac{d^2U(t)}{dt^2}$ then,

$$Z(K) = (K+1)(K+2)U(K+2) \tag{9}$$

5. If $z(t) = \frac{d^mU(t)}{dt^n}$ then

$$Z(k) = (k+1)(k+2)\ldots(k+m)U(k+m) \tag{10}$$

6. If $z(t) = U(t)V(t)$ then,

$$Z(K) = \sum V(l)U(K-l), \tag{11}$$

7. If $z(t) = t^m$ then,

$$Z(K) = \delta(K-m),\ \delta(K-m) = 0, \text{ if } K \neq m \tag{12}$$

8. If $z(t) = exp(\lambda t)$ then,

$$Z(K) = \frac{\lambda^k}{K!} \tag{13}$$

9. if $z(t) = (1+t)^m$ then,

$$Z(K) = \frac{m(m-1)\ldots(m-K+1)}{K!} \tag{14}$$

10. If $z(t) = \sin(wt + \alpha)$ then,

$$Z(k) = \frac{w^k}{k!}\sin(\pi K/2) + (\alpha) \tag{15}$$

5. Numerical Examples

In this section the differential transformation technique is applied to solve different systems linear, non-linear and stiff of differential equations, we consider the following examples:

Example 5.1. *We first start by considering the following non-linear systems*

$$\frac{dy_1(x)}{dx} = -y_1(x) \tag{16}$$

$$\frac{dy_2(x)}{dx} = y_1(x) - y_2^2(x) \tag{17}$$

$$\frac{dy_3(x)}{dx} = y_2^2(x) \tag{18}$$

With initial conditions, $y_1(0) = 1$, $y_2(0) = 0$ and $y_3(0) = 0$. Taking the differential transformation of equations (16), (17) and (18) respectively. We get

$$Y_1(K+1) = -(1/K+1)[y_1(K)] \tag{19}$$

$$Y_2(K+1) = (1/K+1)[y_1(k) - \sum y_2 l y_2(K-l)] \tag{20}$$

$$Y_3(K+1) = (1/K+1)[\sum y_2 l y_2(K-l)] \tag{21}$$

With

$$Y_1(0) = 1, \tag{22}$$

$$Y_2(0) = 0, \tag{23}$$

$$Y_3(0) = 0, \tag{24}$$

From (2), (19), (20), (21), (22), (23) and (24), The set of values $y_1(K+1)$, $y_2(K+1)$, $y_3(K+1)$ of $y_1(x)$, $y_2(x)$ and $y_3(x)$ with the differential transformation method are presented in table respectively.

K	$Y_1(K+1)$	$Y_2(K+1)$	$Y_3(K+1)$
0	-1	1	0
1	$\frac{1}{2}$	$-\frac{1}{2}$	0
2	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
3	$\frac{1}{24}$	$\frac{5}{24}$	$-\frac{1}{4}$
4	$-\frac{1}{120}$	$\frac{1}{40}$	$-\frac{1}{60}$
5	$\frac{1}{720}$	$-\frac{71}{720}$	$\frac{7}{72}$

Table 1. Fundamental transformation values of example 1 for K = 0.1.2

From (3) and all values in table 1 for K=0.1.2.n. . . , the solution of the given non-linear differential systems (16), (17) and (18) can be written in the closed form as follows:

$$Y_1(x) = 1 - x + (1/2) - x^2 - (1/6)x^3 + (1/24)x^4 - (1/120)x^5 + (1/720)x^6 \tag{25}$$

$$Y_2(x) = x - (1/2)x^2 - (1/6)x^3 + (5/24)x^4 + (1/40)x^5 - 71/720x^6 \tag{26}$$

$$Y_3(x) = y_3x^3 - (1/4)x^4 - (1/60)x^5 + (7/72)x^6 \tag{27}$$

Gives the numerical results (25), (26) and (27) for Example 5.1

6. Application of Fundamental Operation Transformation Method

The fundamental differential transformations function of the original function in table 2. We obtained the recurrence relation of SIR model equation (a) as

$$S(k+1) = \frac{1}{k+1}\left[\frac{-c\beta}{n}\sum_{m=0}^{k} s\,(m)\,I(k-m)\right]$$

$$I(k+1) = \frac{1}{k+1}\left[\frac{c\beta}{n}\sum_{m=0}^{k} s\,(m)\,I\,(k-m) - \gamma I(k)\right]$$

$$R(k+1) = \frac{1}{k+1}[\gamma I(k)]$$

With Initial condition $s(0) = 1000$, $i(0) = 200$, $r(0) = 100$ and parameter $c = 5$, $\beta = 0.03$, $\gamma = 0.2$, $n = 2000$ then $S(0) = 1000$, $I(0) = 200$, $R(0) = 100$.

$S(1) = -15$, $S(2) = 0.075938$, $S(3) = -0.04595807$, $S(4) = 11.48361844$

$I(1) = -25$, $I(2) = 1.45$, $I(3) = 612.4924958$, $I(4) = 1406.7207$

$R(1) = 40$, $R(2) = -2.5$, $R(3) = 0.09657$, $R(4) = 54.586335$

Then the closed form of the solution where $k = 4$ can be written as

$$S(t) = \sum_{m=0}^{k} s(k)t^k = 1000 - 15t + 0.075938t^2 - 0.0459587t^3 + 11.48361844t^4$$
$$I(t) = \sum_{m=0}^{k} I(k)t^k = 200 - 25t + 1.45t^2 + 612.4924958t^3 + 1406.2767t^4$$
$$R(t) = \sum_{m=0}^{k} R(k)t^k = 100 + 40t - 2.5t^2 + 0.09657t^3 + 54.5863t^4$$

Also, using the transformation of SIR model equation (b) gives

$$S(k+1) = \frac{1}{k+1}\left[\mu\left(S\left(k\right) + I\left(k\right) + R(k)\right) - \alpha \sum_{m=0}^{k} S\left(m\right) I\left(k-m\right) - \mu S(k)\right]$$
$$I(k+1) = \frac{1}{k+1}\left[\propto \sum_{m=0}^{k} S\left(m\right) I(k-m) - \gamma I\left(k\right) - \mu I(k)\right]$$
$$R(k+1) = \frac{1}{k+1}[\gamma I\left(k\right) - \mu R(k)]$$

With initial condition $s(0) = 1000$, $i(0) = 20$, $r(0) = 0$ and parameter $\alpha = 0.004$, $\gamma = 3$, $\mu = 0.04$, $n = 2000$. From the initial conditions $S(0) = 1000$, $I(0) = 20$, $R(0) = 0$ then Equation (8) gives

$S(1) = -43.6,$	$S(2) = -0.9584,$	$S(3) = -0.313309843,$	$S(4) = -1.925934$
$I(1) = -20.8,$	$I(2) = -0.688,$	$I(3) = 0.733388544,$	$I(4) = 0.4029758$
$R(1) = 60,$	$R(2) = -2,$	$R(3) = -0.431568,$	$R(4) = 0.3710107$

Therefore, the closed from of the solution where k=4 can be written as

$$S(t) = \sum_{m=0}^{k} s(k)t^k = 1000 - 43.6t - 0.9584t^2 - 0.313309843t^3 - 1.925934t^4$$
$$i(t) = \sum_{m=0}^{k} I(k)t^k = 20 - 20.8t - 0.688t^2 + 0.733388544t^3 + 0.4029758t^4$$
$$r(t) = \sum_{m=0}^{k} R(k)t^k = 0 + 60t - 2t^2 - 0.431568t^3 + 0.3710107t^4$$

The result above indicates that both methods are in complete agreement which implies that both are powerful tools for solving system of differential equations.

7. Conclusion

In this paper, differential transformation method has been successfully used and obtained the approximate solution of SIR model with some initial condition. We have also discussed the fundamental operation on differential transformation method. We have discussed some numerical examples also.

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