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The Automatic Continuity of Epimorphisms in Certain Classes of Topological Algebras

Research Article

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Abstract: In this papers, we are interested in the study of automatic continuity in *semi -simples Complete p-normed *-algebras. To do this, we extend the results known in the case of Banach spaces with p-normed complete spaces.

Keywords: *-algebra, *simple algebra, *semi-simple algebra, automatic continuity, separting space.

1. Introduction

The characterization of the semi-simplicity of an associative unital algebra A brings back its left ideals under investigation. Indeed, it is well-known which A is semi-simple if, and only if, the intersection as of its maximum left ideals is reduced $\{0\}$. While being based on the fact that the property to be locally bounded, is more essential, for an Banach algebra, that locally convexity, we were interested in the p-normalized algebras $(0 < p < 1)$ [1]. In this work, we define for an algebra in involution * $(A, *)$ a concept that we call * semi simplicity, it rests on the study of certain ideals. The interest thus is to restrict with a family of the ideals instead of considering all the left ideals. This concept of *semi simplicity will contribute also under investigation of the problem of the automatic continuity of the linear operators in the topological algebras. The results of this paper are divided into two sections: In the first section, we give various characterizations of *-simple algebras (respectively *-semi-simple.) we show by example that if A is *-simple unital algebra that is not simple, then there exists an simple unital algebra I such that $A = I \oplus I^*$ (propsition 2.4). While the second section is devoted to the study of automatic continuity of Homomorphisms in *-simple complete p-normed algebras (respectively-*semi-simple). We show that if A is a *-semi-simple complete p-normed algebra, then any surjective homomorphism (or dense range) of an algebra complete p-normed B in A is continuous (Theorem 2.2).

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2. Characterization of *-simples Algebras

2.1. *-ideals

Definition 2.1. *An Involution on a algebra is an application* $^*: A \to A$ *satisfies the following properties:*

(i) $(x^*)^* = x$, $\forall\, x \in A$

(ii) $(x+y)* = x * + y*$, $\forall\, x, y \in A$

(iii) $(\lambda x)^* = \bar{\lambda} x^*$, $\forall\, x \in A$ *and* $\forall\, \lambda \in C$

(iv) $(xy)^* = y^* x^*$, $\forall\, x, y \in A$

Equipped with involution * A *is said to be an* **-algebra. Let* A *be a unital associative algebra not necessarily commutative. Along this work,* * *is an involution on* A.

Definition 2.2. *It is said that an ideal* I *of* A *is a* **-ideal if* $I^* \subseteq I$*, where* $I^* = \{a^*/a \in I\}$. *We verify that if* I *is a* **-ideal of* A*, then* I *is two-sided ideal of* A.

Example 2.3.

(1) If I *is two-sided ideal, then* $I + I^*$ *and* $I \cap I^*$ *are both* **-ideals of* A.

(2) If I *is a left ideal (respectively Right ideal), then* $I.I^*$ *(respectively* $I.I^*$*) is a* **-ideal of* A.

Remark 2.4. *Let* I *be a non-zero* **-ideal of* A*. Then,* * *induces an involution on the quotient algebra* A/I*, denoted as* **, defined by:* $(a+I)^* = (a)^* + I$*, for all* $a \in A$.

2.2. *-simple Algebra

Definition 2.5. *A* **-ideal* I *is said* **-minimal (respectively* **-maximal) of* A*, if the only* **-ideal of* A *contained in* I *(respectively Containing* I*) are* $\{0\}$ *and* I *(respectively* I *and* A*). If* I *is a minimum* **-ideal (maximum respectively), Then* I *is an ideal* **-minimal (respectively* **-maximum)* A*. Let* I *be a two-sided ideal of* A*. Then,* I *was especially an* A*-module.* D *denote the set defined by:*

$$D = \{T \in \ End_A(I)/\ T(ai) = aT(i),\ \forall\ a \in A\ et\ \forall\ i \in I\}$$

Where $End_A(I)$ *is the set of endomorphism of* A*-modules* I *in* I.

Definition 2.6. *Let* I *be a* **-ideal of* A *and* $T \in End_A(I)$*. We say that* T *is a* **-endomorphism and noted let* $T \in End^*_A(I)$ *if* $T(a^*) = (T(a))^*$*, for all* $a \in A$*.* D^* *denote the set defined by:*

$$D^* = \{T \in End^*_A(I)/T(ai) = aT(i),\ \forall\ a \in A\ et\ \forall\ i \in I\}.$$

Proposition 2.7. *Let* I *be an* **-minimal ideal of* A*. Then, the set* D^* *is a division unital sub-algebra of the algebra* $End^*_A(I)$.

Proof. The set D is obviously an sub-algebra of $End_A^*(I)$, containing the identity operator. Let $T \in D^*$ is not zero. T is bijective: indeed, $\forall a \in A$ and $\forall i \in I$, $aT(i) = T(ai) \subseteq T(I)$, following $T(I)$ is ideal of A. Since T is a *-endomorphism (T(i))* = T(i*) T(I),, for all i I. So, T (I) is a * -ideal of A contained in I. Or, $T(I) \neq \{0\}$, $T(I) = I$. Let $N = \{i \in I/T(i) = (0)\}$. Then $\forall a \in A$ and $\forall i \in N$, $T(ai) = aT(i) = (0)$ $(\forall i \in N)$, following N is an ideal of A. Since T^* is a *-endomorphism, it follows that N is *-ideal of A. However, $T \neq \{0\}$, from where $N \neq I$ as a result, $N = \{0\}$. It was therefore, T invertible the reverse T^{-1}. In addition,

$$aT^{-1}(i) = T^{-1}T(aT^{-1}(i)) = T^{-1}(aTT^{-1}(i)) = T^{-1}(ai).$$

Hence $T^{-1} \in D^*$ □

Definition 2.8. *It is said that algebra with involution $(A,^*)$ is *-simple if the only *-ideals of A are (0) and A.*

Note that if A is a simple algebra equipped with an involution *, then A *-simple. But the converse is not true in general, as shown in the following example:

Let A be a simple algebra, A° the opposite algebra A. Consider the algebra $B = A \times A^\circ$. Provided with the exchange involution defined by: $^*(x, y) = (y, x)$, B is *-simple algebra is not simple.

It is therefore natural to ask under what conditions the converse is true. It is subject to the following proposition:

Proposition 2.9. *Let A *-simple algebra. If the involution is anisotropic, then A is simple Recall that involution is called anisotropic if $\forall a \in A$, it $a^*a = 0 \Rightarrow a = 0$.*

Proof. Let I be an ideal of A, $I \cap I^* = \{0\}$ or A. If $I \cap I^* = \{0\}$, then $x.x^* = 0$. With * is anisotropic, then $x = 0 \forall x \in I$, a result $I = \{0\}$. If $I \cap I^* = A$, then $I = A$. □

Proposition 2.10. *Let A l is *-algebra. Then A is a *-simple if, and only if, there a maximal ideal M such that $M \cap M^* = \{0\}$.*

Proof.

$\Rightarrow$ Suppose A is *-simple. Let M be a maximal ideal of A. $M \cap M^*$ is a *-ideal of A, $M \cap M^* = \{0\}$ or A. If $M \cap M^* = A$, then $M = A$, which contradicts the fact that M is proper. Hence, $M \cap M^* = \{0\}$.

$\Leftarrow$ Suppose that there exists a maximal ideal M such that $M \cap M^* = \{0\}$. Let I a *-ideal of A. If $I \subseteq M$, then $I^* = I \subseteq M^*$, where $I \subseteq M \cap M^* = \{0\}$. If I, $I \subseteq M$, then $M = I + A$, and

$$M^* + I = (M^* + I)A = (M^* + I)(M + I) \subseteq M^*M + I = I.$$

Which implies that M^*I, as a result, $M \subseteq I$. Since M is maximum *-ideal A, so it follows that $A = I$. □

Proposition 2.11. *Let A an *-simple algebra which is not simple. Then, there exists a sub-algebra simple unit I of A such that $A = I \oplus I^*$.*

Proof. Let I a proper ideal of A. $I \cap I^*$ is a *-ideal, therefore $I \cap I^* = \{0\}$ or $I \cap I^* = A$. If $I \cap I^* = A$ then $I = A$, which is absurd. From where $I \cap I^* = \{0\}$. There is also $I + I^*$ is a *ideal, then $I + I^* = \{0\}$, or $I + I^* = A$. If $I + I^* = \{0\}$, then $I = \{0\}$, which contradicts the fact that I is proper. Therefore, $A = I \oplus I^*$. Let J a ideal of A such as $J \subseteq I$. According to what precedes, $A = J \oplus J^*$. Let $i \in I$, then there exists $j, j' \in J$ such that $i = j + j'^*$. However $i - j = j'^* \in I \cap I^* = \{0\}$, from where $i = j$, therefore $I = J$. Consequently, I is a minimal ideal of A. Let J an ideal of I, then J is an ideal of A. Indeed, let $a \in A$ and $j \in J$, then it exists i, i'^* such that $a = i + i'^*$. From where $aj = (i + i'^*)j = ij + i'^*j$. However, $i'^*j \in I^*I \subseteq II^* = \{0\}$, consequently $aj = ij \in J$. As I is a minimal ideal, then $J = \{0\}$ or $I = J$. Thus, I a simple sub-algebra. On other hand, I a unital and if 1 indicates the unit of A, then there exists $e, e' \in I$ such that $1 = e + e'^*$. Let $x \in I$, we are $x = x1 = xe + xe'^*$, but $x - xe = xe'^* \in I \cap I^* = \{0\}$, from where $x = xe$. In the same way, we checked that $x = xe$. Consequently, I an unital of unit e. □

Proposition 2.12. *Let A be a *-algebra and M a maximal ideal of A. Then, $M \cap M^*$ is a *-maximal ideal of A.*

Proof. Suppose that $M \neq M^*$. Then $A/M \cap M^* = M/M \cap M^* \oplus M^*/M \cap M^*$. Indeed, M is a maximal ideal of A, then $M/M \cap M^*$ is a * is maximal ideal of $A/M \cap M^*$. So, $A/M \cap M^* = M/M \cap M^* + M * /M \cap M^*$. Let $\bar{x} \in M/M \cap M^* \cap M^*/M \cap M^*$, then there exists $m, m' \in M$ such that $\bar{x} = m + M \cap M^*$ and $\bar{x} = m'^* + M \cap M^*$, *, which implies that $m - m'^* \in M \cap M^* \subset M$. Since, $m \in M$, so $m'^* \in M$. On the other hand, $m'^* \in M^*$. Thus, $m'^*M \cap M^*$. Then $\bar{x} = \bar{0}$. So by Proposition $A/M \cap M^*$ is *-simple algebra, following $M \cap M^*$ is *-maximal. □

Proposition 2.13. *Let A be a *-algebra and M *-maximal ideal which is not maximal. Then there exists a maximal ideal N of A such that $M = N \cap N^*$.*

Proof. As M is not maximum, there is a maximal ideal N of A such that $M \subset N$. Since $M^* = M \subset N^*$, where $M \subset N \cap N^*$. Since $N \cap N^*$ is a *-ideal of A, it follows therefore that $M = N \cap N^*$. □

Definition 2.14. *Let A be a *-algebra. Called *-radical of A, denoted $Rad_*(A)$, the intersection of all ideals *- maximum of A. A is called *-semi-simple if $Rad_*(A) = \{0\}$.*

Example 2.15. *Let A be a simple algebra, A° the opposite algebra of A. Consider the algebra $B = A \times A^\circ$. Provided with the exchange involution defined by $^*(x, y) = (y, x)$. So B is an algebra *-semi-simple.*

Proposition 2.16. *Let I be a *-ideal a *-algebra A such that $I \subseteq Rad_*(A)$. So $Rad_*(A/I) = Rad_*(A)/I$. In particular, $A/Rad_*(A)$ is a *-semi simple.*

Proof. M is a *-maximum ideal of A. We put $\bar{A} = A/I$ and $\bar{M} = M/I$. We have $I \subseteq Rad_*(A) \subseteq M$. So from the following canonical isomorphism $\bar{A}/\bar{M} \approx A/M$ is *-simple, it follows that $\bar{A}/\bar{M}$ is a *-simple algebra. Consequently, M/I is a *-ideal *-maximal of A/I. From where

$$\begin{aligned} Rad_*(A/I) &= \cap\{\bar{M} : \text{M is}^* - \text{maximum ideal of A}\} \\ &= \cap\{M : \text{is}^* - \text{maximum ideal of A}\} \\ &= \overline{Rad}_*(A) = Rad_*(A)/I \end{aligned}$$

□

3. Automatic continuity

In this section, we are interested in the study of automatic continuity in *-simples complete p-normed *-algebras (respectively *-semi-Simple). To do this, we first useful preliminary, then we extend the results known in the case of Banach spaces to complete p-normed spaces.

Definition 3.1. *Let T a linear application of a complete p-normed space X in a complete p-normed space Y. Then, the separating space $\sigma(T)$ of Y is the subset of Y defined by*

$$\sigma(T) = \{y \in Y / \exists (x_n)_n \ X : x_n \xrightarrow{\| \|_p} 0 \ et \ T(xn) \xrightarrow{\| \|_p} y\}.$$

Proposition 3.2. *Let X and Y two complete p-normed space, then the separating space (T) of any a linear application $T : X \to Y$ is a closed subspace of Y.*

Proof. Evidently $\sigma(T)$ is a subspace vector of Y. Let $(y_k)_k$ a sequence in $\sigma(T)$ converging to y of Y. we prove that $y \in \sigma(T)$ for every $k \in IN^*$, $y_k \in \sigma(T)$. Then there is a sequence $(y_{k,n})_n \subset X$ such that $x_{n,k} \xrightarrow{\| \|_p} 0$ and $T(x_{n,k}) \xrightarrow{\| \|_p} y_k$. Let us choose two sequence $(z_k)_k(y_k)_k$ in X such that, for every $k \in IN^*$, $\|z_k\|_p K < 1/k$ and $\|T(z_k) - y\|_p < 1/k$. Then, we are $z_k \xrightarrow{\| \|_p} 0$ and $T(z_k) - y_k \xrightarrow{\| \|_p} 0$. On other hand, for every $k \in IN^*$, we are $T(z_k) - y = (T(z_k) - y_k) + (y_k - y)$. Then, $z_k \xrightarrow{\| \|_p} 0 \ et \ T(z_k) - y \xrightarrow{\| \|_p} 0$. Consequently, $y \in \sigma(T)$. □

Proposition 3.3. *Let X and Y be two complete p-normed spaces and $T : X \to Y$ linear application. So we have:*

(i) $\sigma(T) = \{0\}$, if and only if, the graph of T is closed.

(ii) Where R and S are two continuous operators X and Y, respectively, and if, $TR = ST$, then $S(\sigma(T) \subset \sigma(T)$.

Proof.

(i) Suppose $\sigma(T) = \{0\}$, then the graph of T is closed. Indeed let $(x_n)_n$ be a sequence of elements of X such $x_n \xrightarrow{\| \|_p} x$ and $T(x_n) \xrightarrow{\| \|_p} y$. Then we have $x_n - x \xrightarrow{\| \|_p} 0$ and $T(x_n - x) \xrightarrow{\| \|_p} y - T(x)$, which implies that $y - T(x) \in \sigma(T) = \{0\}$, therefore, $y = T(x)$. The converse is obvious.

(ii) Let $y \in \sigma(T)$, then there exists a sequence $(x_n)_n$ be a sequence of elements of X such $x_n \xrightarrow{\| \|_p} 0$ and $T(x_n) \xrightarrow{\| \|_p} y$. So we have $R(x_n) \xrightarrow{\| \|_p} 0$ and $TR(x_n) = ST(x_n) \xrightarrow{\| \|_p} S(y)$. This then results that $S(y) \in \sigma(T)$.

□

Proposition 3.4. *Let X, Y and Z complete p-normed spaces. Let $S : X \to Y$ a linear application and $R : Y \to Z$ a continuous linear application. So we have*

(i) RS is continuous if and only if, $R\sigma(T) = \{0\}$.

(ii) $\overline{R(\sigma(S))} = \sigma(RS)$.

Proof. Suppose that RS is continuous. Let $y \in \sigma(S)$, then there exists a sequence $(x_n)_n$ of elements of X such $x_n \xrightarrow{\|\ \|_p} 0$ *et* $S(x_n) \xrightarrow{\|\ \|_p} y$. Then, $RS(x_n) \xrightarrow{\|\ \|_p} R(y)$ and $RS(x_n) \xrightarrow{\|\ \|_p} 0$. As S is separated, the result $R(y) = 0$. Conversely, suppose $R(\sigma(S)) = \{0\}$. Let $Q : Y \to Y/\sigma(S)$ the canonical surjective map from Y to $Y/\sigma(S)$ and the linear application R_0 defined to $Y/\sigma(S)$ in Z by $R_0(y + \sigma(S)) = R(y)$. Since R_0 is continuous, it suffices to show that QS is continuous. This amounts to show that $\sigma(QS) = \{0\}$ (Proposition 3.2 (i)). Let $y + \sigma(S) \in \sigma(QS)$, there exists a sequence $(x_n)_n$ be a sequence of elements of X as $x_n \xrightarrow{\|\ \|_p} 0$ in X, and $QS(x_n) \xrightarrow{\|\ \|_p} y + \sigma(S)$ in $Y/\sigma(S)$. Then there exists a sequence $(y_n)_n$ elements of $\sigma(S)$ such that $S(x_n) - y - y_n \xrightarrow{\|\ \|_p} 0$. We choose a sequence $(w_n)_n$ of elements of X such that $\|w_n\|_p < 1/n$ and $\|S(w_n) - y_n\|_p < 1/n$. So $x_n - w_n \xrightarrow{\|\ \|_p} 0$ and $S(x_n - w_n) - y \xrightarrow{\|\ \|_p} 0$ for n large enough, as a result $y \in \sigma(S)$, which implies that $R(\sigma(S)) = \{0\}$, according to the previous proposal, QS is continuous. We have $R\sigma(S) \subset \sigma(RS)$, in effect: Let $y \in \sigma(S)$, then there exists a sequence $(x_n)_n$ be a sequence of elements of X such $x_n \xrightarrow{\|\ \|_p} 0$ and $S(x_n) \xrightarrow{\|\ \|_p} y$. R is continuous, therefore $RS(x_n) \xrightarrow{\|\ \|_p} R(y)$.
As $\sigma(RS)$ is closed (Proposition 1.1), $\overline{R(\sigma(S))} \subseteq \sigma(RS)$. To show the other inclusion, consider the canonical map $Q_0 : Z \to Z/\overline{R(\sigma(S))}$ where, $Q_0(z) = z + \overline{R(\sigma(S))}$. Then Q_0 is continuous. So Q_0R is continuous, on the other hand, $Q_0(\sigma(RS)) = (\bar{0})$. From where, $\sigma(RS) \subseteq \overline{R(\sigma(S))}$. □

Remark 3.5. *Under the same assumptions of the proposition (3.3), the subspace $S^{-1}[\sigma(S)]$ is closed in X.*

Proof. $S^{-1}[\sigma(S)] = Ker(QS) = (QS)^{-1}\{\bar{0}\}$. As $\sigma(S)$ is closed, $Y/\sigma(S)$ is complete p-normed. Therefore, $S^{-1}[\sigma(S)]$ is in a closed X. □

Proposition 3.6. *Let T a linear application of a complete p-normed unital algebra A in a complete p-normed unital algebra B. Then, if T is surjective, the separating space $\sigma(T)$ is proper ideal of Y.*

Proof. Let $b \in B$ and $y \in \sigma(T)$. $y \in \sigma(T)$, there then there exists a sequence $(a_n)_n \subset A$ such that: $a_n \xrightarrow{\|\ \|_p} 0$ *et* $T(a_n) \xrightarrow{\|\ \|_p} y$. Suppose that T is surjective, then there exists a A such that $T(a) = B$; Then, $a_na \xrightarrow{\|\ \|_p} 0$ *et* $T(a_na) = T(a_n)(T(a) = T(a_n)b \xrightarrow{\|\ \|_p} y_b$, Consequently $yb \in \sigma(T)$. Let us show that $\sigma(T)$ is a proper ideal of B. As A and B are unital, $T(e_A) = e_B$. For all, $a \in A$, $Sp(T(a)) \subset Sp(a)$. Then $rB(T(a)) \leq rA(a)$, $a \in A$. Let c an element of center of B, then $r_B(T(a)) \leq r_B(c - T(a)) + r_B(T(a)) \leq \|c - T(a)\|_p + \|a\|_p$. Suppose that $e_B \in \sigma(T)$. Then, there exists $a(a_n)_n \subset A$ such that: $a_n \xrightarrow{\|\ \|_p} 0$ *et* $T(an) \xrightarrow{\|\ \|_p} eB$. As e_B is an element of center of B, $r_B(T(e_A)) = r_B(e_B) \leq \|e_B - T(a_n)\|_p + \|a_n\|_p \xrightarrow{\|\ \|_p} 0$. What contradicts the fact that $r_B(e_B) = 1$. □

Proposition 3.7. *Let A be a complete p-normed algebra, then any modular maximal left ideal M of A is closed.*

Proof. Let M be a modular maximal left ideal unital the right unit e. Suppose that there is a element x of M such that $\|e - x\|_p < 1$. Let $u = \sum_1^{\infty}(e - x)^n$. So $u - u(e - x) = e - x$, consequently $e = x = ux = u - ue \in M$. So $M = A$, which contradicts the fact that M is proper ideal. Therefore $M \cap \{x \in A : \|e - x\|_p < 1\} = \phi$. It follows that, $\bar{M}$ is a modular left proper ideal, closed and contains M. Since M is maximum, therefore $M = \bar{M}$. □

Proposition 3.8. *Let A be a complete p-normed *-algebra A, then all *-maximal *-ideal M of A is closed.*

Proof. If M is a maximal ideal of A, then M is closed. Otherwise, if M not Maximal, there is a maximal ideal N of A such that $M = N \cap N^*$ (proposition 2.6). Since N (respectively N^*) is closed, it is deduced that M is closed in A. □

Proposition 3.9. *Let T a homomorphism of a complete p-normed algebra A on a complete p-normed algebra B then, if B is simple and if T is surjective (or with dense range), T is continuous.*

Proof. Let$\sigma(T)$ the separator ideal of Tin B is simple, so $\sigma(T) = \{0\}$ or $\sigma(T) = B$. If $e_B \in \sigma(T)$. which contradicts the proposition (3.4). From where $\sigma(T) = \{0\}$. As a result, T is continuous. □

Theorem 3.10. *Let T a homomorphism of a complete p-normed algebra A on an complete p-normed *algebra B then, if B is *simple and if T is surjective (or with dense range), T is continuous.*

Proof. Let B is an algebra *simple, there exists simple unital subalgebra I of B such that $B = I \oplus I^*$ (Proposition 2.4); following algebraic isomorphism $B \approx B/I^*$, one deduces that I am a maximum ideal of B. From where I (respectively I^*) is closed in B. Consequently, I (respectively I^*) is a complete p-normed subalgebra. Let us consider $Pr1 : B \to I$ (respectively $Pr2 : B \to I^*$) the canonical projection of B on I (respectively of B on I^*). Since Pr1 (respectively Pr2) is a continuous epimorphism, then according to the proposition (3.3) $Pr1\ o\ T$ (respectively $Pr2\ o\ T$) is continuous. Consequently, $T = (Pr1 + Pr2)\ o\ T = Pr1\ o\ T + Pr2\ o\ T$ is continuous. □

Theorem 3.11. *Let T is a homomorphism of a complete p-normed algebra A on a complete p-normed algebra B. If B is *-semi-simple and if T is surjective (or dense range), then T is continuous.*

Proof. Let M^*-maximal ideal of B and $Q : B \to B/M$ the canonical surjection. Since Q is continuous, it results that $Q\ o\ T$ is a dense range. In addition, B/M is a *-simple complete p-normed *-algebra. By Theorem (3.1), $Q\ o\ T$ is continuous. As a result that, $\sigma(Q\ o\ T) = \{0\}$. Since $(Q\ o\ T) = \overline{Q(\sigma(Y))}$ [3], we deduce that $\sigma(Q(T)) = \{0\}$. Hence, $\sigma(T) \subset M$. As M is arbitrary, then $\sigma(T) \subset \cap M$. However $\cap M = Rad_*(B) = \{0\}$, where $\sigma(T) = \{0\}$. Consequently, T is continuous. □

Corollary 3.12. *Let $(A, \|.\|_p)$ an complete p-normed * semi-simple algebra. Then, we have*

(i) All the complete complete p-normed on A are equivalent.

(ii) The involution * is automatically continuous.

Proof.

(i) It is enough to apply the previous theorem to the Identity of A.

(ii) That is to say q the linear application of A in IR^+ defined by $q(x) = \|x^*\|_p$ $(x \in A)$. We checks easily that Q is a complete p-normed on A. And according i), Q is equivalent to $\|.\|_p$.

□

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