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On Sums of s-orthogonal Matrices

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Abstract : In this paper we discussed the sum of s-orthogonal and s-unitary matrices. Also we show that every $A \in M_n(\mathbb{Z}_{2n-1})$ can be written as a sum of s-orthogonal matrices in M_n. Moreover, we show that every $A \in M_n(\mathbb{Z}_{2k})$ can be written a sum of of s-orthogonal matrices in $M_n(\mathbb{Z}_{2k})$ if and only if the row sums and column sums of A have the same parities.

Keywords : s-orthogonal matrix, s-unitary matrix, Sum of s-orthogonal and s-unitary matrices.

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1 Introduction and Basic Definitions

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Let $\mathcal{U}_n^s(\mathbb{F})$ be the set of s-unitary matrices in $M_n(\mathbb{F})$, and let $\mathcal{O}_n^s(\mathbb{F})$ be the set of s-orthogonal matrices in $M_n(\mathbb{F})$. Suppose $n \geq 2$. In this chapter we show that every $A \in M_n(\mathbb{F})$ can be written as a sum of matrices in $\mathcal{U}_n^s(\mathbb{F})$ and of matrices in $\mathcal{O}_n^s(\mathbb{F})$. Let $A \in M_n(\mathbb{F})$ be given and let $k \geq 2$ be the least integer that is a least upper bound of the singular values of A. When $\mathbb{F} = \mathbb{C}$, we show that A can be written as a sum of k matrices from $\mathcal{U}_n^s(\mathbb{F})$. When $\mathbb{F} = \mathbb{R}$, we show that if $k = 3$, then A can

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be written as a sum of 6 s-orthogonal matrices; if k = 2, or if $k \geq 4$, we show that A can be written as a sum of $k+2$ s-orthogonal matrices.

Let $\mathbb{F} = \mathbb{C}$(the set of complex numbers) or $\mathbb{F} = \mathbb{R}$ (the set of real numbers). Let n be a given positive integer. We let $M_n(\mathbb{F})$ be the set of all $n \times n$ matrices with entries in $\mathbb{F}$. We also let $E_{ij} \in M_n(\mathbb{F})$ be the matrix whose (i,j) entry is 1 and all other entries are 0.

We study the sums of s-unitary matrices and we also study the sums of s-orthogonal matrices. We determine which matrices (if any) in $M_n(\mathbb{F})$ can be written as a sum of s-unitary or s-orthogonal matrices. We note that the sum of unitary matrices in $M_n(\mathbb{C})$) has been previously studied (see [1] and the references therein). Moreover, for $A, B \in M_n(\mathbb{C})$, sums of the form $UAU^* + VBV^*$, where $U, V \in M_n(\mathbb{C})$ are unitary, have also been studied [2]. In this paper we extend the result concerning unitary and orthogonal matrices into s-unitary and s-orthogonal matrices.

Notation 1.1. *The secondary transpose (conjugate secondary transpose) of A is defined by* $A^s = VA^TV$ ($A^\Theta = VA^*V$)*, where "V"is the fixed disjoint permutation matrix with units in its secondary diagonal.*

Definition 1.2. *[3] Let* $A \in \mathcal{M}_n(F)$

(i). The matrix A is called s-normal, if $AA^\Theta = A^\Theta A$

(ii). The matrix A is called s-orthogonal, if $AA^s = A^sA = I$*. That is* $A^TVA = V$

(iii). The matrix A is called s-unitary, if $AA^\Theta = A^\Theta A = I$*. That is* $A^*VA = V$

Lemma 1.3. *Let n be a given positive integer. Let* $\mathcal{G} \subset M_n(\mathbb{F})$ *be a group under multiplication. Then* $A \in M_n(\mathbb{F})$ *can be written as a sum of matrices in* $\mathcal{G}$ *if and only if for every* $Q, P \in \mathcal{G}$*, the matrix* QAP *can be written as a sum of matrices in* $\mathcal{G}$*.*

Notice that both $\mathcal{U}_n^s(\mathbb{F})$ and $\mathcal{O}_n^s(\mathbb{F})$ are groups under multiplication. Let $\alpha \in \mathbb{F}$ be given. Then Lemma 1.3 guarantees that for each $Q \in \mathcal{G}$, we have that αQ can be written as a sum of matrices from $\mathcal{G}$ if and only if αI can be written as a sum of matrices from $\mathcal{G}$.

Lemma 1.4. *Let* $n \geq 2$ *be a given integer. Let* $G \subset M_n(\mathbb{F})$ *be a group under multiplication. Suppose that* $\mathcal{G}$ *contains* $K \equiv diag(1, -1, ..., -1)$ *and the permutation matrices. Then every* $A \in M_n(\mathbb{F})$ *can be written as a sum of matrices in* $\mathcal{G}$ *if and only if for each* $\alpha \in \mathbb{F}, \alpha I$ *can be written as a sum of matrices in* $\mathcal{G}$*.*

Proof. The forward implication is trivial. For the other direction, suppose that for each $\alpha \in \mathbb{F}, \alpha I$ can be written as a sum of matrices in $\mathcal{G}$. Now, $K \in \mathcal{G}$ so that for each $\alpha \in \mathbb{F}$, Lemma 1.3 guarantees that αK can also be written as a sum of matrices in $\mathcal{G}$. It follows that $\alpha E_{11} = \frac{\alpha}{2}I + \frac{\alpha}{2}K$ can be written as a sum of matrices in $\mathcal{G}$. Now, for each $1 \leq i, j \leq n$, notice that $E_{ij} = PE_{11}Q$ for some permutation matrices P and Q, and that $P, Q \in \mathcal{G}$. Therefore, if $A \in M_n(\mathbb{F})$, then A can be written as a sum of matrices in $\mathcal{G}$, as $A = [a_{ij}] = \sum\limits_{i,j} a_{ij}E_{ij}$. □

Lemma 1.5. *Let* $k \geq 2$ *be a given integer. Let* $\mathcal{A}_k \equiv \{z \in \mathbb{C} : |z| \leq k\}$ *and let* $\mathcal{C}_k \equiv \{\sum\limits_{j=1}^{k} e^{i\theta_j} : \theta_j \in \mathbb{R} for j = 1, ..., k\}$*. Then* $\mathcal{A}_k = \mathcal{C}_k$*.*

2 Sums of s-orthogonal Matrices

The only matrices in $\mathcal{O}_1^s(\mathbb{F})$ are ± 1. Hence, not every element of $M_1(\mathbb{F})$ can be written as a sum of elements in $\mathcal{O}_1^s(\mathbb{F})$. In fact, only the integers can be written as a sum of elements of $\mathcal{O}_1^s(\mathbb{F})$.

2.1 The Case $\mathcal{U}_n^s(\mathbb{C})$

Let $\alpha \in \mathbb{C}$ be given. Then there exist an integer $k \geq 2$ and $\theta_1, ..., \theta_k \in \mathbb{R}$ such that $\alpha = f_k(\theta_1, ..., \theta_k)$. Now, notice that $\alpha I = f_k(\theta_1, ..., \theta_k)I = e^{i\theta_1}I + ... + e^{i\theta_k}I$ is a sum of matrices in $\mathcal{U}_n^s(\mathbb{C})$. When $n = 1$, every $\alpha \in \mathbb{C}$ can be written as a sum of elements of $\mathcal{U}_1^s(\mathbb{C})$. When $n \geq 2$, Lemma 1.4 guarantees that every $A \in M_n(\mathbb{C})$ can be written as a sum of matrices in $\mathcal{U}_n^s(\mathbb{C})$.

Lemma 2.1. *Let n be a given positive integer. Then every $A \in M_n(\mathbb{C})$ can be written as a sum of matrices in $\mathcal{U}_n^s(\mathbb{C})$.*

Let $A \in M_n(\mathbb{C})$ be given. We look at the number of matrices that make up the sum A.

Let $\alpha \in \mathbb{C}$ be given. If $|\alpha| \leq k$ for some positive integer k, then $\alpha \in \mathcal{A}_k$. Moreover, $\alpha \in \mathcal{A}_m$ for every integer $m \geq k$. For any such m, Lemma 1.5guarantees that there exist $\theta_1, ..., \theta_m \in \mathbb{R}$ such that $\alpha = e^{i\theta_1}I + ... + e^{i\theta_m}I$. However, if $|\alpha| > k$, then $\alpha \neq \mathcal{A}_k$ and α cannot be written as a sum of k elements of $\mathcal{U}_1^s(\mathbb{C})$.

Write $A = U\Sigma V$ (the singular value decomposition of A, where $U, V \in M_n(\mathbb{C})$ are s-unitary and $\Sigma = diag(\sigma_1, ..., \sigma_n)$ with $\sigma_1 \geq ... \geq \sigma_n \geq 0$. Let k be the least integer such that $\sigma_1 \leq k$. Suppose that $k \leq 2$. Then, for each l, we have $\sigma_l \in \mathcal{A}_k$. Moreover, $\sigma_1 \notin \mathcal{A}_{k-1}$. Hence, A cannot be written as a sum of $k-1$ s-unitary matrices. However, for each l, we have $\sigma_l = e^{i\theta_{l1}} + ... + e^{i\theta_{lk}}$, where each $\theta_{l1}, ..., \theta_{lk} \in \mathbb{R}$. For each $t = 1, ..., k$, set $U_t = diag(e^{i\theta_{1t}}, ..., e^{i\theta_{nt}})$. Then $U_t \in M_n(\mathbb{C})$ is s-unitary and $\sum_{t=1}^{k} U_t = \Sigma$. Hence, A can be written as a sum of k s-unitary matrices.

Suppose that $k = 1$. If $\sigma_n = 1$, then $\Sigma = I$ and A is s-unitary. If $\sigma_n \notin 1$, then for each l, we have $\sigma_l \in \mathcal{A}_2$, and A can be written as a sum of two s-unitary matrices. We summarize our results.

Theorem 2.2. *Let $A \in M_n(\mathbb{C})$ be given. Let k be the least (positive) integer so that there exist $U_1, ..., U_k \in \mathcal{U}_n^s(\mathbb{C})$ satisfying $U_1 + ... + U_k = A$.*

(1). If A is s-unitary, then $k = 1$.

(2). If A is not s-unitary and $\sigma_1(A) \in 2$, then $k = 2$.

(3). If $m \geq 2$ is an integer such that $m < \sigma_1(A) \leq m + 1$, then $k = m + 1$.

For positive integers $m \geq k$, we have $\mathcal{A}_k \subseteq \mathcal{A}_m$. Hence, every $U \in \mathcal{U}_n^s(\mathbb{C})$ can be written as a sum of two or more elements of $\mathcal{U}_n^s(\mathbb{C})$. It follows that every $A \in M_n(\mathbb{C})$ that can be written as a sum of k elements of $\mathcal{U}_n^s(\mathbb{C})$ can be written as a sum of m elements of $\mathcal{U}_n^s(\mathbb{C})$.

2.2 The Case $\mathcal{O}_n^s(\mathbb{C})$

Theorem 2.3. *Let $n \geq 2$ be a given integer. Then every $A \in M_n(\mathbb{C})$ can be written as a sum of matrices from $\mathcal{O}_n^s(\mathbb{C})$.*

Suppose that $A = Q_1 + Q_2$, where $Q_1, Q_2 \in \mathcal{O}_n^s(\mathbb{C})$. Then one checks that $AA^s = Q_1 A^s A Q_1^s$, so that AA^s and $A^s A$ are similar. Analogue to the proof of Theorem 13 of [4] ensures that $A + QS$, where Q is s-orthogonal and S is s-symmetric (or that A has a QS decomposition). Suppose now that has a QS decomposition. Is it true that A can be written as a sum of two (complex) s-orthogonal matrices? Take the case n = 1, and notice that every $A \in M_n(\mathbb{C})$ is a scalar and has a QS decomposition. However, only the integers can be written as a sum of orthogonal matrices in this case.

Lemma 2.4. *Let an integer $n \geq 2$ and $0 \neq \alpha\mathbb{C}$ be given. If $\alpha I = Q + V$ is a sum of two matrices from $\mathcal{O}_n^s(\mathbb{C})$, then there exists a s-skew symmetric $D \in M_n(\mathbb{C})$ such that $Q = \frac{\alpha}{2}I + D$, $V = \frac{\alpha}{2}I - D$, and $DD^s = (1 - \frac{\alpha^2}{4})I$. Conversely, if there exists a s-skew symmetric $D \in M_n(\mathbb{C})$ such that $DD^s = (1 - \frac{\alpha^2}{4})I$, then $Q \equiv \frac{\alpha}{2}I + D$ and $V \equiv \frac{\alpha}{2}I - D$are in $\mathcal{O}_n^s(\mathbb{C})$ and $Q + V = \alpha I$.*

Proof. Let an integer $n \geq 2$ and $0 \neq \alpha \in \mathbb{C}$ be given. Suppose that $\alpha I \in M_n(\mathbb{C})$ can be written as a sum of two s-orthogonal matrices, say, $\alpha I = Q + V$. Write $Q = [a_{ij}] = [q_1 ... q_n]$ and $V = [b_{ij}] = [v_1 ... v_n]$. Then, $b_{ij} = -a_{ij}$ for $i \neq j$. Now, for each $i = 1, ..., n$, we have $\sum\limits_{j=1}^{n} a_{ij}^2 = q_i^s q_i = 1 = v_i^s v_i = \sum\limits_{j=1}^{n} b_{ij}^2 = b_{ii}^2 + \sum\limits_{j \neq i, j=1}^{n} a_{ij}^2$.

Hence, $b_{ii} = \pm a_{ii}$. Because $Q + V = \alpha I$ and $\alpha \neq 0$, we have $b_{ii} = a_{ii} = \frac{\alpha}{2}$. Set $D = [d_{ij}]$, with $d_{ij} = a_{ij}$ if $i \neq j$, and $d_{ii} = 0$, so that $Q = \frac{\alpha}{2}I + D$ and $V = \frac{\alpha}{2}I - D$. Now, since Q and V are s-orthogonal, we have

$$QQ^s = \frac{\alpha^2}{4}I + \frac{\alpha}{2}(D + D^s) + DD^s = I \tag{2.1}$$

and

$$VV^s = \frac{\alpha^2}{4}I - \frac{\alpha}{2}(D + D^s) + DD^s = I. \tag{2.2}$$

Subtracting equation (2.2) from equation (2.1), we get $D = -D^s$, so that D is s-skew symmetric. Moreover, $DD^s = (1 - \frac{\alpha^2}{4})I$. For the converse, suppose that $D \in M_n(\mathbb{C})$ is s-skew symmetric and satisfies $DD^s = (1 - \frac{\alpha^2}{4})I$. Set $Q \equiv \frac{\alpha}{2}I + D$ and set $V \equiv \frac{\alpha}{2}I - D$. Then one checks that both Q and V are s-orthogonal matrices and $Q + V = \alpha I$. □

Theorem 2.5. *Let n be a given positive integer. For each $\alpha \in \mathbb{C}$ and each s-orthogonal $Q \in M_{2n}(\mathbb{C})$, αQ can be written as a sum of two s-orthogonal matrices.*

Let an integer $n \geq 2$ be given. If $\alpha \in \{-2, 0, 2\}$, then one checks that $\alpha I \in M_n(\mathbb{C})$ can be written as a sum of two s-orthogonal matrices.

Theorem 2.6. *Let $\alpha \in \mathbb{C}$ and let a positive integer n be given. Then $\alpha I \in M_{2n+1}(\mathbb{C})$ can be written as a sum of two matrices from $\mathcal{O}_n^s(\mathbb{C})$ if and only if $\alpha \in \{-2, 0, 2\}$.*

Proof. For the forward implication, let $\in \mathbb{C}$ and let a positive integer n be given. Suppose that $\alpha I \in M_{2n+1}(\mathbb{C})$ can be written as a sum of two s-orthogonal matrices. Then $\alpha = 0$ or $\alpha \neq 0$. If $\alpha = 0$, then $\alpha \in \{-2, 0, 2\}$. If $\alpha \neq 0$, we show that $\alpha \pm 2$. Lemma 2.4 guarantees that there exists a s-skew symmetric $D \in M_n(\mathbb{C})$ satisfying $DD^s = (1 - \frac{\alpha^2}{4})I$. Now, since n is odd, the s-skew symmetric D is singular. Hence, DD^s is singular and $\alpha \pm 2$. The backward implication can be shown by direct computation. □

2.3 The Case $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{H}$

Theorem 2.7. *Let $n \geq 2$ be a given integer. Every $A \in M_n(\mathbb{R})$ can be written as a sum of matrices from $\mathcal{O}_n^s(\mathbb{R}) = \mathcal{U}_n^s(\mathbb{R})$.*

Theorem 2.8. *Let a positive integer n and let $A \in M_{2n}(\mathbb{R})$ be given. Suppose that $k \geq 2$ is an integer such that $\sigma_1(A) \leq k$. Then A can be written as a sum of 2k matrices in $\mathcal{O}_{2n}^s(\mathbb{R})$. Moreover, for every integer $m \geq 2k$, the matrix A can be written as a sum of m matrices in $\mathcal{O}_{2n}^s(\mathbb{R})$.*

Proof. Let $A = U\Sigma V$ be a singular value decomposition of A. Then Lemma 1.3 guarantees that we only need to concern ourselves with Σ.

For $n = 1$, notice that $diag(\sigma_1, \sigma_2) = sI_2 + rK_2$, where $s = \frac{1}{2}(\sigma_1 + \sigma_2)$ and $t = \frac{1}{2}(\sigma_1 - \sigma_2)$. Now, $0 \leq t \leq s \leq k$. Hence, sI_2 and tK_2 can each be written as a sum of k s-orthogonal matrices. Moreover, for each integer $p \geq k$, notice that sI_2 can be written as a sum of p s-orthogonal matrices. Hence, $sI_2 + rK_2$ can be written as a sum of $p+k$ s-orthogonal matrices. For $n > 1$, write $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_{2n-1}, \sigma_{2n}) = diag(\sigma_1, \sigma_2) \oplus ... \oplus diag(\sigma_{2n-1}, \sigma_{2n})$. Notice now that for each $j = 1, ..., n$, $diag(\sigma_{2j-1}, \sigma_{2j})$ can be written as a sum of 2k s-orthogonal matrices, say $P_{j1}, ..., P_{j(2k)}$. For each $l = 1, ..., 2k$, set $Q_l \equiv P_{1l} \oplus ... \oplus P_{nl}$, and notice that $\Sigma = Q_1 + ... + Q_{2k}$.

Finally, notice that for each integer $m \geq 2k$ and for each each $j = 1, ..., n$, the matrix $diag(\sigma_{2j-1}, \sigma_{2j})$ can be written as a sum of m s-orthogonal matrices. □

Lemma 2.9. *Let $C \in M_2(\mathbb{R})$ be given. Suppose that $k \geq 2$ is an integer such that $\sigma_1(C) \leq k$. Then for each integer $t \geq k + 2$, C can be written as a sum of t matrices from $\mathcal{U}_2^s(\mathbb{R})$.*

Theorem 2.10. *Let n be positive integer, and let $A \in M_{2n}(\mathbb{R})$ be given. Suppose that $k \geq 2$ is an integer such that $\sigma_1(A) \leq k$. Then for each integer $t \geq k+2$, A can be written as a sum of t matrices in $\mathcal{U}_{2n}^s(\mathbb{R})$.*

Theorem 2.11. *Let $A \in M_{2n+1}(\mathbb{R})$ be given. Suppose $k \geq 2$ is an integer such that $\sigma_1(A) \leq k$. If $k \leq 3$, then A can be written as a sum of at most six matrices in $\mathcal{O}_{2n+1}^s(\mathbb{R})$. If $k \geq 4$, then A can be written as a sum of $k + 2$ matrices in $\mathcal{O}_{2n+1}^s(\mathbb{R})$.*

Theorem 2.12. *Let n be a given positive integer. Let $A \in M_n(\mathbb{H})$ be given. Then A can be written as a sum of matrices from $\mathcal{U}_n^s(\mathbb{H})$. Moreover, if $k \geq 2$ is an integer such that $\sigma_1(A) \leq k$, then A can be written as a sum of k matrices from $\mathcal{U}_n^s(\mathbb{H})$.*

Theorem 2.13. *Let $n \geq 2$ be a given integer. Let $A \in M_n(\mathbb{H})$ be given. Then A can be written as a sum of matrices from $\mathcal{O}_n^s(\mathbb{H})$.*

3 On Sums of s-orthogonal Matrices $M_n(\mathbb{Z}_k)$

Lemma 3.1. *Let k and n be given positive integers. If $E_{11} \in M_n(\mathbb{Z}_k)$ can be written as a sum of matrices from $\mathcal{O}_{n,k}^s$, then every $A \in M_n(\mathbb{Z}_k)$ can be written as a sum of matrices in $\mathcal{O}_{n,k}^s$.*

Lemma 3.2. *Let k and n be given positive integers, with $k \geq 2$. Then $2E_{11} \in M_n(\mathbb{Z}_k)$ can be written as a sum of matrices from $\mathcal{O}_{n,k}^s$.*

Theorem 3.3. *Let k and n be given positive integers, with $k \geq 2$. Then $A \in M_n(\mathbb{Z}_{2k-1})$ can be written as a sum of matrices from $\mathcal{O}^s_{n,2k-1}$.*

Lemma 3.4. *Let $a_1, ..., a_k \in \mathbb{Z}_{2m}$ be given. If $a_1^2 + ... + a_k^2 = 1$, then $a_1 + ... + a_k$ is odd.*

Proof. Notice that if there are an even number of ai that are odd, then the sum $a_1^2 + ... + a_k^2$ is even. Hence, there are an odd number of a_i that are odd, and $a_1 + ... + a_k$ is odd. □

Theorem 3.5. *Let k and n be given positive integers. If $A \in M_n(\mathbb{Z}_{2k})$ is s-orthogonal, then the row sums and the column sums of A are all odd.*

Corollary 3.6. *Let k and n be given positive integers. If $A \in M_n(\mathbb{Z}_{2k})$ is a sum of matrices in $\mathcal{O}^s_{n,2k}$, then the row sums of A have the same parities and the column sums of A have the same parities. Moreover, the row sums and the column sums also have the same parities.*

Lemma 3.7. *Let k and n be given positive integers with $n \geq 2$. Let $A \in M_n(\mathbb{Z}_{2k})$ have even entries only. Then A can be written as a sum of matrices in $\mathcal{O}^s_{n,2k}$.*

Theorem 3.8. *Let k be a given positive integer. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_n(\mathbb{Z}_{2k})$. The following are equivalent.*

(1). B can be written as a sum of matrices in $\mathcal{O}^s_{2,2k}$.

(2). a and d have the same parities, and b and c have the same parities.

(3). The row sums of B and the column sums of B have the same parities.

Theorem 3.9. *Let k and $n \geq 2$ be given positive integers. Let $B \in Mn(\mathbb{Z}_{2k})$ be given. Then B can be written as a sum of matrices in $\mathcal{O}^s_{n,2k}$ if and only if the row sums of B and the column sums of B have the same parities.*

Lemma 3.10. *Let n and k be given positive integers with $n \geq 3$. Let $B = [b_{ij}] \in M_n(\mathbb{Z}_{2k})$ have an odd parity. Suppose that $b_{11} = 1$. Suppose further that the entries in the first row are not all 1 and that the entries in the first column are not all 1. Then $B = C + D$, where $C = [1] \oplus F$,with $F \in M_{n-1}(\mathbb{Z}_{2k})$ having an odd parity and $D \in M_n(\mathbb{Z}_{2k})$ is a sum of matrices in $\mathcal{O}^s_{n,2k}$.*

Proof. It is without loss of generality to assume that $b_{1n} = 0$ and $b_{n1} = 0$. When $n = 3$, then $b_{12} = b_{21} = 0$, and we may take $D = 0$. Suppose now that $n \geq 4$. Let r_1 be the first row of B and let c_1 be the first column of B. Take s to be the vector with the same entries as r_1 but with the first entry changed to 0. Take t to be the vector which has the same entries as c_1 except in the first position. Set D as the matrix with first and last columns t, first and last rows s, and 0 elsewhere. Notice that the number of 1 in t is even, and that the number of 1 in s is also even. Hence, D can be written as a sum of matrices that are permutation equivalent to $E_2 \oplus 0$, and hence, can be written as a sum of matrices in $\mathcal{O}^s_{n,2k}$. Notice also that $C = B + D$ has the form $C = [1] \oplus F$, where $F \in M_{n-1}(\mathbb{Z}_2)$. Because B has an odd parity and because D has an even parity, F has an odd parity. □

Corollary 3.11. *Let $n \geq 2$ be a given integer. Then $B \in M_n(\mathbb{Z})$ can be written as a sum of s-orthogonal matrices in $M_n(\mathbb{Z})$ if and only if the row sums of B and the column sums of B have the same parities.*

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