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Minimum 2-Edge Connected Spanning Subgraph of Certain Interconnection Networks

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Abstract : Given an undirected graph, finding a minimum 2-edge connected spanning subgraph is NP-hard. We solve the problem for silicate network, brother cell and sierpiński gasket rhombus.

Keywords : silicate network; brother cell; sierpiński gasket rhombus.

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1 Introduction

The study of connectivity in graph theory has important applications in the areas of network reliability and network design. In fact, with the introduction of fiber optic technology

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in telecommunication, designing a minimum cost survivable network has become a major objective in telecommunication industry. Survivable networks have to satisfy some connectivity requirements, this means that they are still functional after the failure of certain links [5]. As pointed out in [5, 9], the topology that seems to be very efficient is the network that survives after the loss of $k-1$ or less edges, for some $k \geq 2$, where k depends on the level of reliability required in the network [9]. In this paper, we concentrate on the minimum 2-edge connected spanning subgraph. A connected graph $G = (V, E)$ is said to be 2-edge connected if $|V| \geq 2$ and the deletion of any set of < 2 edges leaves a connected graph. The minimum 2-edge connected spanning subgraph (2-ECSS) problem is defined as follows: Given a 2-edge connected graph G, find efficiently a spanning subgraph $S(G)$ which is also 2-edge connected and has a minimum number of edges. We denote the number of edges in a graph G by $\varepsilon(G)$ and the edges of minimum 2-edge connected spanning subgraph of G by $\varepsilon(S(G))$.

Kuller and Raghavachari [12] presented the first algorithm which, for all k, achieves a performance ratio smaller than a constant which is less than two. They proved an upper bound of 1.85 for the performance ratio of their algorithm. Cristina G. Fernandes [7] improved their analysis, proving that the performance ratio of algorithm [13] is smaller than 1.7 for large enough k, and that it is at most 1.75 for all k. Cherian et.al [6] gave an approximation algorithm for minimum size 2-ECSS problem where an ear decomposition is used to construct a feasible 2-ECSS. The depth-first search algorithm was used to present a 3/2 approximation algorithm for the minimum size 2-ECSS problem in which a notion called tree carving is used [13]. An approximation for finding a smallest 2-edge connected subgraph containing a specified spanning tree was studied by Hiroshi Nagamochi [8]. The sufficient conditions for a graph to be perfectly 2-edge connected was given by Ali Ridha Mahjoub [2]. Woonghee [15] devised an algorithm for r-regular, r-edge connected graphs. For cubic graphs, results of [11] imply a new upper bound on the integrality gap of the linear programming formulation for the 2-edge connectivity problem. Even though there are numerous results and discussions on minimum 2-edge connected spanning subgraph problem, most of them deal only with approximation results. According to the literature survey, the minimum 2-edge connected spanning subgraph problem is not solved for an interconnection network. In this paper we derive an exact number of edges of minimum 2-edge connected spanning subgraph of silicate network, brother cell and sierpiński gasket rhombus.

2 Silicate Network

Lemma 2.1. *[1] If one end of every edge of a graph G is of degree 2 then no proper spanning subgraph of G is 2-edge connected.*

Consider a honeycomb network $HC(r)$ of dimension r. Place silicon ions on all the vertices of $HC(r)$. Subdivide each edge of $HC(r)$ once. Place oxygen ions on the new vertices. Introduce $6r$ new pendant edges one each at the 2-degree silicon ions of $HC(r)$ and place oxygen ions at the pendent vertices. See Figure 1(a). With every silicon ion associate the three adjacent oxygen ions and form a tetrahedron as in Figure 1(b). The resulting network is a silicate network of dimension r, denoted $SL(r)$. The diameter of $SL(r)$ is $4r$. The graph in Figure 1(b) is a silicate network of dimension two. The 3-degree oxygen nodes of silicates are called boundary nodes. In Figure 1(b), c_1, c_2, $\cdots$, c_{12} are boundary nodes SL_2.

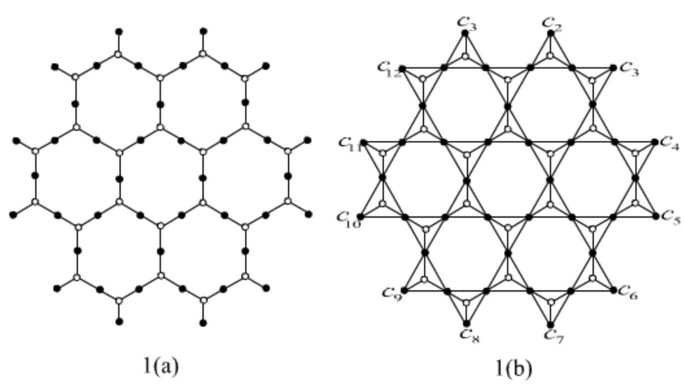


Figure 1: Silicate Network $SL(2)$

When we delete all the silicon nodes from a silicate network we obtain a new network which we shall call as an Oxide Network [14]. See Figure 2(a). An r-dimensional oxide network is denoted by $OX(r)$. By [14], there are r edge disjoint symmetric cycles in $OX(r)$ which are also vertex disjoint cycles. Let them be x_1, x_2, ..., x_r. See Figure 2(b). The number of edges in $x_i, 1 \leq i \leq r$ is $18i - 6$.

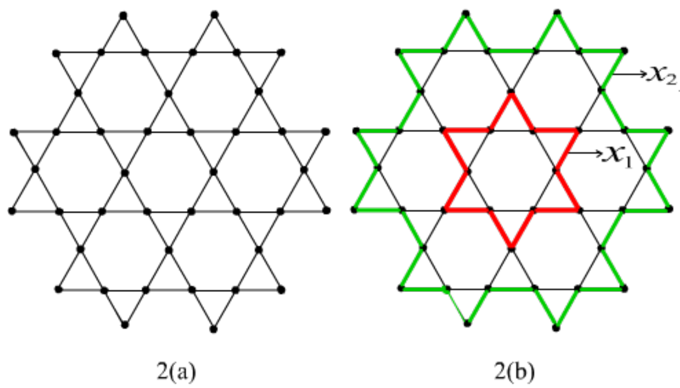


Figure 2: Oxide Network $OX(2)$

Theorem 2.2. *Let $OX(r), r \geq 2$ be an r-dimensional oxide network.Then* $\varepsilon(S(OX(r))) = \varepsilon(x_1) + \varepsilon(x_2) - 1 + \cdots + \varepsilon(x_r) - 1 + 2(r-1)$.

Proof. Let us prove the theorem by induction on r. When $r = 2$, there are r=2 edge disjoint cycles x_1 and x_2 in $OX(2)$. keeping x_1 and x_2, removing all the edges, we get a disconnected oxide network with 2-edge disjoint cycles x_1 and x_2 in $OX(2)$. See Figure 3(a).

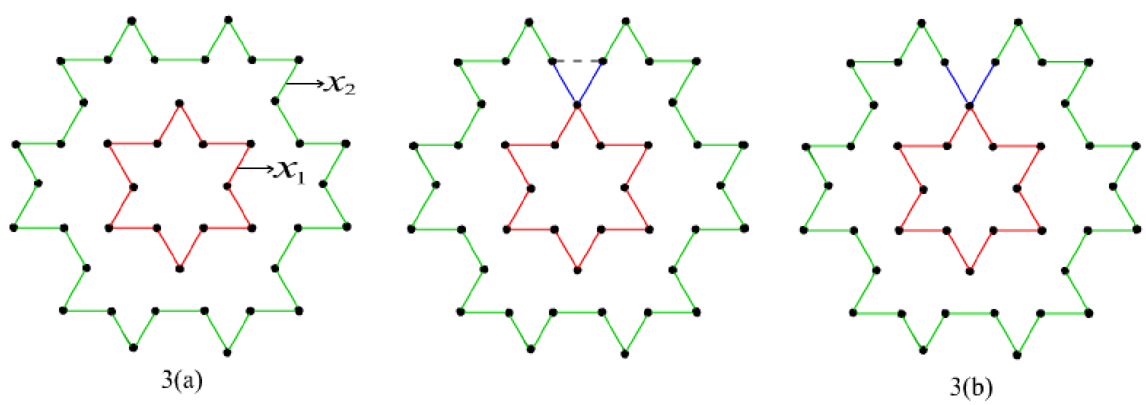


Figure 3: $\varepsilon(S(OX(r = 2))) = 12 + 29 + 2 = 43$.

Adding 2 edges from a boundary vertex of x_1 to two non boundary adjacent vertices of x_2 and deleting the edge between those non boundary vertices of x_2 [edge to be removed is shown in dashed line], we get a minimum 2-edge connected spanning subgraph. See Figure 3(b). This is minimum because by Lemma 2.1, deleting any single edge gives no 2-edge connected spanning subgraph. The number of edges in x_1 and x_2 are $18(1) - 6$ and $18(2) - 6$. Hence $\varepsilon(S(OX(r = 2))) = 12 + 29 + 2 = \varepsilon(x_1) + \varepsilon(x_2) - 1 + 2(r-1)$. Thus the result is true for $r = 2$.

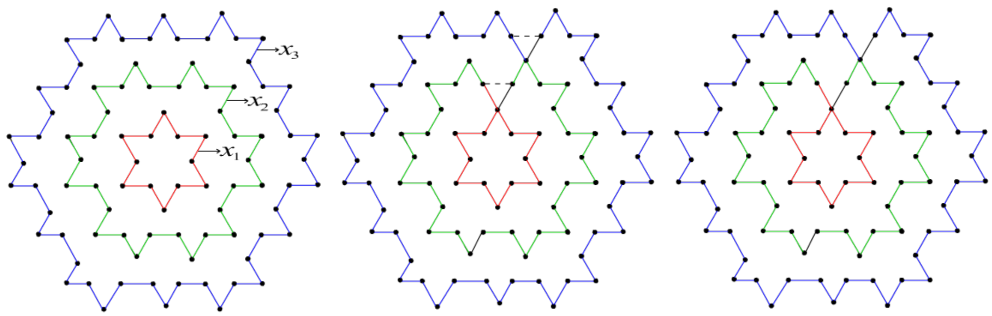


Figure 4: $\varepsilon(S(OX(r = 3))) = 12 + 29 + 47 + 4 = 92$.

We assume that the result is true for $r = k$. When $r = k + 1$, there are $r = k + 1$ edge disjoint cycles $x_1, x_2, \cdots, x_{k+1}$. Adding 2 edges from a boundary vertex of x_i, $1 \leq i \leq k$ to two non boundary adjacent vertices of x_{i+1}, $1 \leq i \leq k$ and deleting the edges between those non boundary vertices of $x_2, x_3, \cdots, x_{k+1}$, we get a minimum 2-edge connected spanning subgraph. Hence $\varepsilon(S(OX(r = k + 1))) = 18(1){-}6 - 1 + 18(2){-}6 - 1 + \cdots + 18((k{+}1)) - 6 - 1 + 2k$ $= \varepsilon(x_1) + \varepsilon(x_2) - 1 + \cdots + \varepsilon(x_{k+1}) - 1 + 2((k + 1) - 1)$. □

3 Sierpinski Gasket Rhombus

Definition 3.1. *[4] A sierpiński Gasket Rhombus of level r [denoted by SR_r] is obtained by identifying the edges in two Sierpinski Gasket S_r along one of their side. For the definition of sierpiński Gasket, refer[10].*

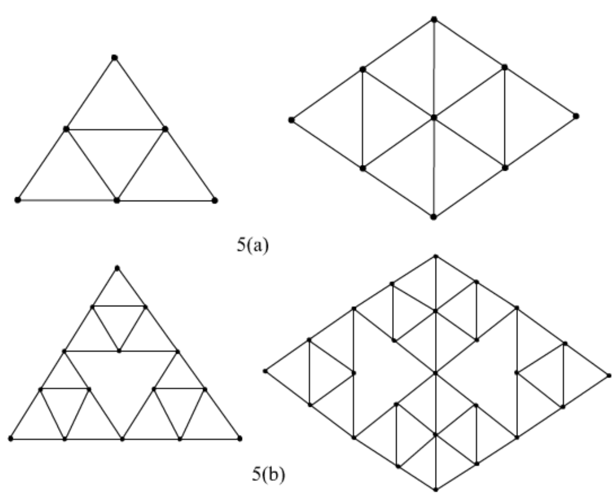
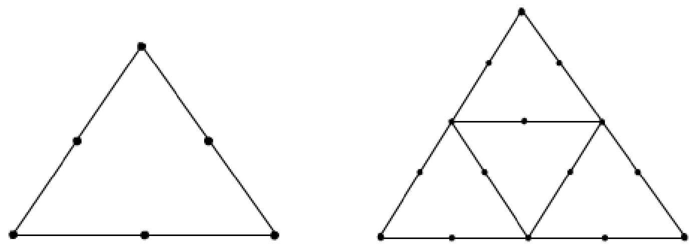


Figure 5:(a) S_2 and SR_2 and (b) S_3 and SR_3

The sierpiński Gasket graphs S_r has 3^r edges [14]. From the Definition 3.1, sierpiński Gasket

Rhombus SR_r consists two copies of sierpiński Gasket graph S_r and identifying the edges of two sierpiński Gasket graphs S_r along one of their side, 2^{r-1} edges are shared by both S_r. Therefore the number of edges in SR_r is $2{\times}3^r - 2^{r-1}$.

Theorem 3.2. *[1] Let $S_r, r \geq 3$ be the r dimensional Sierpiński gasket graph. Then $\varepsilon(S(S_r)) = 2 \times 3^{r-1}$.*

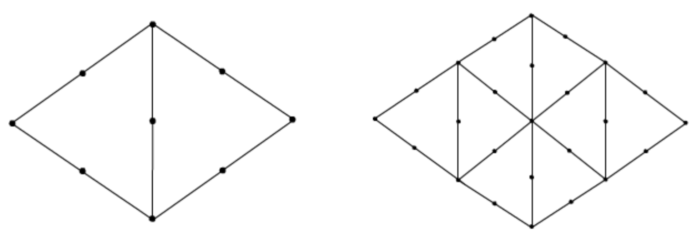


6(a) 6(b)

Figure 6: (a) $\varepsilon(S(S_2)) = 6$ and (b) $\varepsilon(S(S_3)) = 18$.

Theorem 3.3. *Let $SR_r, r \geq 2$ be the r dimensional sierpiński Gasket Rhombus. Then $\varepsilon(S(SR_r)) = 2(2 \times 3^{r-1}) - 2^{r-1}$.*

Proof. We prove this theorem by induction on r. When $r = 2$, SR_2 contains 2 copies of S_2 and has $2 \times 3^2 - 2^{2-1}$ edges. Now we construct minimum 2-edge connected spanning subgraph of SR_2 using 2 copies of minimum 2-edge connected spanning subgraph of S_2. See Figure 7(a).



7(a) 7(b)

Figure 7:(a) $\varepsilon(S(SR_2) = 10$ and (b) $\varepsilon(S(SR_3) = 32$

By Lemma 2.1, no edge can be deleted from Figure 7(b). Thus $S(SR_2) = 2\varepsilon(S(S_2))$. Since 2^{2-1} edges are shared by both S_2, $\varepsilon(S(SR_2) = 2\varepsilon(S(S_2)) - 2^{2-1} = 2(2 \times 3^{2-1}) - 2^{2-1}$
We assume that the result is true for $r = k$ (i.e.) $\varepsilon(S(SR_k)) = 2\varepsilon(S(S_k)) - 2^{k-1} = 2(2 \times 3^{k-1}) - 2^{r-1}$. Consider $r = k + 1$. SR_{k+1} contains two copies of S_k. Construct a minimum 2-edge connected spanning subgraph of SR_{k+1} using two copies of minimum 2-edge connected

spanning subgraph of S_{k+1} where $2^{(k+1)-1}$ edges are shared by two S_k. Thus $\varepsilon(S(S_{k+1})) = 2\varepsilon(S(S_{k+1})) - 2^{(k+1)-1} = 2(2 \times 3^{(k+1)-1}) - 2^{(k+1)-1}$. $\square$

4 Brother Cell

Definition 4.1. *[14] Assume that k is an integer with $k \geq 2$. The kth brother cell $BC(k)$ is the five tuple $(G_k, w_k, x_k, y_k, z_k)$, where $G_k = (V, E)$ is a bipartite graph with bipartition W (white) and B (black) and contains four distinct nodes w_k, x_k, y_k and z_k. w_k is the white terminal; x_k the white root; y_k the black terminal and z_k the black root. We can recursively define $BC(k)$ as follows:*

(1) $BC(2)$ is the 5-tuple $(G_2, w_2, x_2, y_2, z_2)$ where $V(G_2) = w_2, x_2, y_2, z_2, s, t$, and
$E(G_2 = (w_2, s), (s, x_2), (x_2, y_2), (y_2, t), (t, z_2), (w_2, z_2)(s, t)$.

(2)The kth brother cell $BC(k)$ with $k \geq 3$ is composed of two disjoint copies of $(k-1)$th brother cells

$BC^1(k-1) = (G^1_{k-1}, w^1_{k-1}, x^1_{k-1}, y^1_{k-1}, z^1_{k-1})$,

$BC^2(k-1) = (G^2_{k-1}, w^2_{k-1}, x^2_{k-1}, y^2_{k-1}, z^2_{k-1})$,

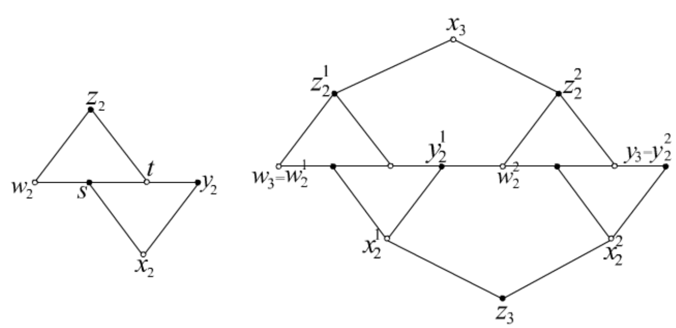
a white root x_k, and a black root z_k. To be specific,

$V(G_k) = V(G^1_{k-1}) \cup V(G^2_{k-1}) \cup \{x_k, z_k\}$,

$E(G_k) = E(G^1_{k-1}) \cup E(G^2_{k-1}) \cup$

$\{(z_k, x^1_{k-1}), (z_k, x^2_{k-1}), (x_k, z^1_{k-1}), (x_k, z^2_{k-1}), (y^1_{k-1}, w^2_{k-1})\}$,

$z_k = w^1_{k-1}$, *and* $y_k = y^2_{k-1}$.



8(a) 8(b)

Figure 8: (a) $BC(2)$ and (b) $BC(3)$

From the definition, we construct $BC(k)$ from two disjoint copies of $(k-1)$ and each time we add five more edges $(z_k, x_{k-1}^1), (z_k, x_{k-1}^2), (x_k, z_{k-1}^1), (x_k, z_{k-1}^2), (y_{k-1}^1, w_{k-1}^2)$. And each time constructing a $BC(k)$, deleting the edge (y_{k-1}^1, w_{k-1}^2) does not affect 2-edge connectivity of $BC(k)$.

Theorem 4.2. *Let $BC(r)$, $r \geq 2$ be a brother cell. Then* $\varepsilon(S(BC(r))) = 5 \times 2^{k-1} - 4$

Proof. By the Definition 4.1, BC_2 has 7 edges. Now label the vertices of $BC(2)$ as shown in the Figure 9(a). Deleting the edge(s,t), we get a cycle on 6 vertices which is a minimum 2-edge connected spanning subgraph and $\varepsilon(S(BC(2))) = 7 - 1 = 5 \times 2^{2-1} - 4 = 6$.

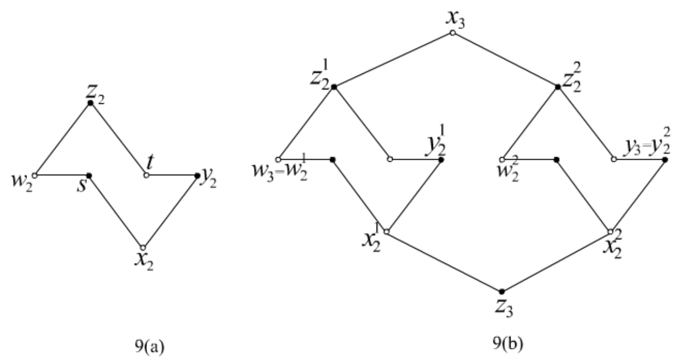


Figure 9: (a)$\varepsilon(S(BC(2))) = 6$ and (b)$\varepsilon(S(BC(3))) = 16$

We prove this theorem by induction on r. When $r = 3$, $BC(3)$ contains 2 disjoint copies of $BC(2)$ and five edges $(z_3, x_2^1), (z_3, x_2^2), (x_3, z_2^1), (x_3, z_2^2), (y_2^1, w_2^2)$ connecting theses two $BC(2)$. Now we construct minimum 2-edge connected spanning subgraph of $BC(3)$ using 2 disjoint copies of minimum 2-edge connected spanning subgraph of BC_2 and with four edges $(z_3, x_2^1), (z_3, x_2^2), (x_3, z_2^1)$, (x_3, z_2^2). See Figure 10(b). By Lemma 2.1, this is the minimum. Hence $\varepsilon(S(BC(3))) = 2\varepsilon(S(BC(2))) + 4 = 2 \times (5 \times 2^{2-1} - 4) + 4 = 16 = 5 \times 2^{3-1} - 4$.

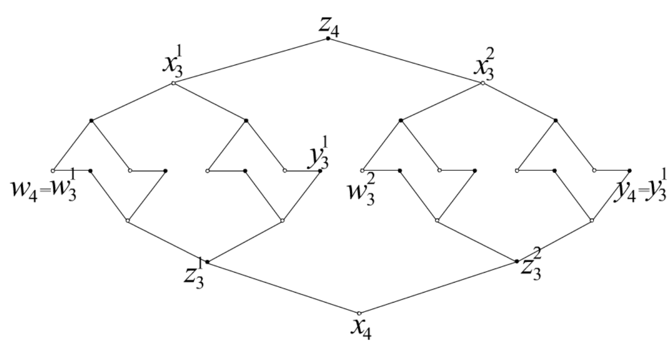


Figure 10: $\varepsilon(S(BC_4)) = 36$

We assume that the result is true for $r = k$ (i.e.) $\varepsilon(S(BC(k))) = 2\varepsilon(S(BC(k-1)))+4 = 2\times(5\times 2^{k-1}-4)+4$. Consider $r = k+1$. $BC(k+1)$ contains two copies of $BC(k)$. Construct minimum 2-edge connected spanning subgraph of $BC(k+1)$ using 2 copies of minimum 2-edge connected spanning subgraph of $BC(k)$ and with four edges $(z_k, x^1_{k-1}), (z_k, x^2_{k-1}), (x_k, z^1_{k-1}), (x_k, z^2_{k-1})$. Thus $\varepsilon(S(BC(k+1))) = 2\varepsilon(S(BC(k))) + 4 = 2 \times (5 \times 2^{k-1} - 4) + 4 = 5 \times 2^{(k+1)-1} - 4$. □

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