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On Edge-Distance and Edge-Eccentric Graph of a Graph

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Abstract : An *elementary circuit* (or *tie*) is a subgraph of a graph and the set of edges in this subgraph is called an *elementary tieset*. The *distance* $d(e_i, e_j)$ between two edges in an undirected graph is defined as the minimum number of edges in a tieset containing e_i and e_j. The *eccentricity* $\varepsilon_\tau(e_i)$ of an edge e_i is $\varepsilon_\tau(e_i) = \max\limits_{e_j \in E} d(e_i, e_j)$. In this paper, we have introduced the edge - self centered and edge - eccentric graph of a graph and have obtained results on these concepts.

Keywords : Cycle; eccentric edge; edge - self centered graph; edge - distance degree regular graph; edge - eccentric graph.

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1 Introduction

Unless mentioned otherwise, for terminology and notation the reader may refer Buckley and Harary [3], new ones will be introduced as and when found necessary.

Let $G = (V(G), E(G))$ be an undirected, connected graph, without multiple edges and self-loops, where $V(G)$ and $E(G)$ are the vertex set and edge set, respectively. Their respective cardinalities are p and q. Usually, in graph theory literature we see the distance concept is defined between vertices, as the length of a shortest path between any two vertices. Also, it is well known fact that the distance (between two vertices) is a "metric" in case of undirected graphs. As the study of distance, serves as an underlying tool in understanding many concepts, parameters in graphs, a tremendous amount of work is seen on these concepts. A monograph in the form of book on distance in graphs by Buckley and Harary [3] emphasizes the importance of the subject.

Often a parameter defined on vertices in graphs is (generalized) extended to edges, viz., the chromatic number to edge - chromatic number; vertex - connectivity to edge - connectivity. But in case of distance(vertex), the edge - distance is not that prolific nor it was the first one to be defined. Instead, the generalization was on distance between graphs by Zelinka [19], on the rotation distance of graphs by Faudree and Schelp [7], distances between graphs under edge operations by Goddard and Swart [8], comparison of various distances between isomorphism classes of graphs by Zelinka [20], etc. Finally, the edge - distance was introduced by Sengoku et al. [16]. They defined distance between two edges based on the cycle they lie in. Formally, the definitions are given as follows:

A *path* in G is a subgraph and the set $P_i \subseteq E$ of edges in the subgraph is called *pathset*. An *elementary circuit* (or *tie*) is a subgraph and the set of edges in the subgraph is called an *elementary tieset*, which is called simply a *tieset*. The number of edges in a path or the pathset is called its *length* and the number of edges in a tie or the tieset is called its *length*. That is, the length of a pathset P_i is $|P_i|$ and the length of a tieset τ_e is $|\tau_e|$.

Let $\tau_k(e_i, e_j)$ be a tieset containing both e_i and e_j of G and let $R(e_i, e_j)$ be the set of these tiesets. The *distance* $d(e_i, e_j)$ between two edges e_i and e_j as follows:

(a) $d(e_i, e_j) = \min\limits_{\tau_k(e_i,e_j)\in R(e_i,e_j)} |\tau_k(e_i, e_j)|$, if $R(e_i, e_j) \neq \emptyset$, for $e_i, e_j \in E$

(b) $d(e_i, e_j) = \infty$, if $R(e_i, e_j) = \emptyset$, for $e_i, e_j \in E$

For an edge $e_i \in E$, let

(c) $d(e_i, e_i) = 0$.

The *eccentricity* $\varepsilon_\tau(e_i)$ of an edge e_i is defined as $\varepsilon_\tau(e_i) = \max\limits_{e_j\in E} d(e_i, e_j)$. The *radius* $rad_\tau(G)$ concerning the edges of G is defined as $\min\limits_{e_i\in E} \varepsilon_\tau(e_i)$ while the *diameter* $diam_\tau(G)$ is $\max\limits_{e_i\in E} \varepsilon_\tau(e_i)$. An edge e is a *central edge* if $\varepsilon_\tau(e) = rad_\tau(G)$.

In [2], Akiyama et al. have defined *eccentric graph* of a graph G, denoted by G_e, has the same set of

vertices as G with two vertices u and v being adjacent in G_e if and only if either v is an eccentric vertex of u in G or u is an eccentric vertex of v in G, that is $dist_G(u, v) = min\{e_G(u), e_G(v)\}$.
Given graphs G and H, the lexicographic product $G[H]$ has vertex set $\{(g, h) : g \in V(G), h \in V(H)\}$ and two vertices (g, h), (g', h') are adjacent if and only if either $[g, g']$ is an edge of G or $g = g'$ and $[h, h']$ is an edge of H.

A *cutpoint* of a graph is one whose removal increases the number of components. Thus if v is a cutpoint of a connected graph G, then $G - v$ is disconnected. A *nonseparable* graph is connected, nontrivial, and has no cutpoints. A *block* of a graph is a maximal nonseparable subgraph. If G is nonseparable, then G itself is often called a block. If $x = uv$ is an edge of G, and w is not a vertex of G, then x is subdivided when it is replaced by the edges uw and wv. If every edge of G is subdivided, the resulting graph is the *subdivision* graph $S(G)$.

2 Main Results

2.1 Edge - self centered graphs

Motivated by the results proved by Sengoku et al. [17], in this section we introduce some concepts based on edge - distance, namely, edge - self centered, edge-distance degree sequence and edge - distance degree regular. The first one being the natural extension of the self - centered graph studied for vertex - distance.

Definition 2.1. *A graph G is said to be an edge - self centered graph if every edge in G is a central edge.*

Next we define the distance degree sequence in terms of edge - distance as follows:

Definition 2.2. *The edge - distance degree sequence of an edge e in a graph G is defined as the sequence* $(d_0, d_1, d_2, d_3, \ldots)$, *where each* d_i, $i \geq 0$ *is the number of edges at distance i from e.*

But compared to the distance degree sequence for vertex - distance the notion of edge - distance degree sequence slightly changes as we see from the following note.
Note: In the edge - distance degree sequence $(d_0, d_1, d_2, d_3, \ldots)$, always
(i) $d_0 = 1$, since $d(e_i, e_i) = 0$ for all $e_i \in E(G)$.
(ii) $d_1 = 0$ and $d_2 = 0$, since the graph considered is loopless and without multiple edges.

Next, we introduce the concept of edge - DDR graph. The DDR graphs (based on vertex - distance degree sequence) were defined by Bloom et al. [4] with applications in Chemistry. Further they were studied by Quintas et al. [5], Medha Itagi Huilgol et al. [11], Medha Itagi Huilgol et al. [12], [13], etc.

As a generalization of this prolific concept we deal here the edge - distance degree regular graph.

Definition 2.3. *A graph G is edge - distance degree regular (edge - DDR) graph if all edges of G have the same edge - distance degree sequence.*

Note: The distance between any two edges is always greater than or equal to the girth of a graph.

Theorem 2.1. *A graph G has finite diameter if and only if G is nonseparable.*

Proof. Suppose G has finite diameter, then the eccentricity of every edge is finite, that is, every pair of edges is contained in at least one cycle. Therefore, every pair of edges lies on a common cycle. Hence G is nonseparable. Conversely, suppose G is nonseparable. Every pair of edges lies on a common cycle. Hence, every edge has finite eccentricity, and G has finite diameter. ☐

Theorem 2.2. *Every cycle is edge - self centered.*

Proof. Let C_p be any cycle. Since every pair of edges lies on the cycle C_p and there exists no other cycle whose length is less than the length of C_p, eccentricity of every edge is p. Hence C_p is edge - self centered. ☐

Remark 2.3. *Every cycle is edge - DDR having* $dds(e) = (\underbrace{1, 0, 0, \ldots, 0, p-1}_{p+1 \text{ entries}})$.

Theorem 2.4. *Every complete graph K_p, $p \geq 4$ is edge - self centered with radius 4.*

Proof. Let K_p be the complete graph on $p \geq 4$ vertices and $e = uv$ be any edge in K_p. Let S be the set of edges which are adjacent to e and T be the set of edges which are nonadjacent to e. Since $p \geq 4$, both S and T are nonempty sets. Now, we prove the set of edges in S are at distance 3 and the set of edges in T are at distance 4. Let $e_1 = u_1v_1 \in S$ and $v = u_1$ then there exists an edge $e_1' = uv_1$ in K_p. Hence, the edges in S are at distance 3 from e. Let $e_2 = u_2v_2 \in T$ then there exist edges $e_2' = uu_2$ and $e_2'' = vv_2$. So, the edges in T are at distance 4 from e. Hence, the edges in T are the eccentric edges of an edge e. Hence, the eccentricity of e is 4. Since e is arbitrarily chosen, K_p, $p \geq 4$ is edge - self centered with radius 4. ☐

Remark 2.5. *K_3 is edge - self centered with radius 3.*

Remark 2.6. *Every complete graph K_p is edge-DDR having* $dds(e) = (\underbrace{1, 0, 0, 2p-4, \frac{(p-3)(p-2)}{2}}_{5 \text{ entries}})$.

The next result deals with the lexicographic product of graphs.

Theorem 2.7. *For any graph G with at least one edge, the lexicographic product $C_p[G]$ is edge - self centered graph if and only if p is odd.*

Proof. Let C_p: 1, 2, 3, …, p be an odd cycle. Let G be any graph with at least one edge and G_1, G_2, …, G_p be the copies of G in the lexicographic product $C_p[G]$, replaced in the places of 1, 2, 3, …, p, respectively in C_p. Let $E(G_i)$ be the set of edges in G_i and $E(G_i\text{-}G_{i+1(mod\ p)})$ be the set of edges between G_i and $G_{i+1(mod\ p)}$ in the lexicographic product $C_p[G]$. Now, we prove that the eccentricity of every edge is the same.

For an $e \in G_i$, $1 \leq i \leq p$, there exists a shortest cycle of length $p+1$ containing an edge e and $e' \in G_{(i+\frac{p-1}{2})(mod\ p)}$, that is, $d(e,e') = p+1$ and there exists no other edge farther than e' from e. Hence, e' is an eccentric edge of e and the eccentricity of e is equal to $p+1$.

For an $e \in E(G_i\text{-}G_{i+1(mod\ p)})$ for $1 \leq i \leq p$, there exists a shortest cycle of length $p+1$ containing an edge e and $e' \in G_{(i+\frac{p-1}{2}+1)(mod\ p)}$, that is, $d(e,e') = p+1$ and there exists no other edge farther than e' from e. Hence, e' is an eccentric edge of e and the eccentricity of e is equal to $p+1$.

Hence, the eccentricity remains the same for all edges in $C_p[G]$, making it an edge - self centered graph.

Conversely, suppose $C_p[G]$ is edge - self centered. On the contrary, suppose p is even, then the eccentricity of an edge $e \in E(G_i)$ is $p+2$, since there exists a shortest cycle of length $p+2$ containing both e and $e' \in E(G_{i+\frac{p}{2}(mod\ p)})$ and there exists no other edge farther than e' from e, but, the eccentricity of an edge $e \in E(G_i\text{-}G_{i+1(mod\ p)})$ is $p+1$, since there exists a shortest cycle of length $p+1$ containing both e and $e' \in E(G_{i+\frac{p}{2}+1(mod\ p)})$ and there exists no other edge farther than e' from e, a contradiction to the fact that $C_p[G]$ is edge - self centered. Hence, p is odd. □

For example, let us consider the lexicographic product of C_7 with K_2 as shown in Figure 1. Let the vertices of C_7 be labeled as 1, 2, …, 7, so that there are seven copies of K_2, named as $G_1, \ldots, G_7$. For an edge $e_1 \in E(G_1)$, there exists a shortest cycle $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ of length 8 containing both e_1 and $e_5 \in E(G_4)$, that is, $d(e_1, e_5) = 8$ and it is clear from the Figure 1 that there exists no other edge farther than e_5 from e_1. Hence, e_5 is an eccentric edge of e_1 and eccentricity of e_1 is equal to 8. Similarly, we can prove eccentricity of every edge $e \in E(G_i)$, $1 \leq i \leq 7$ is equal to 8.

For an edge $e_2 \in E(G_1 - G_2)$, there exists a shortest cycle $e_2, e_3, e_4, e_9, e_{10}$, e_{11}, e_{12}, e_{13} of length 8 containing both e_2 and $e_{10} \in E(G_5)$, that is, $d(e_2, e_{10}) = 8$ and it is clear from the Figure 1 that there exists no other edge farther than e_{10} from e_2. Hence, e_{10} is an eccentric edge of e_2 and eccentricity of e_2 is equal to 8. Similarly, we can prove eccentricity of every edge $e \in E(G_i - G_{i+1(mod\ 7)})$, $1 \leq i \leq 7$ is equal to 8.

Hence, the eccentricity of every edge in $C_7[K_2]$ remains the same making it an edge - self centered graph.

Now, if we consider p to be even, then we show that we arrive at a contradiction. Let us consider

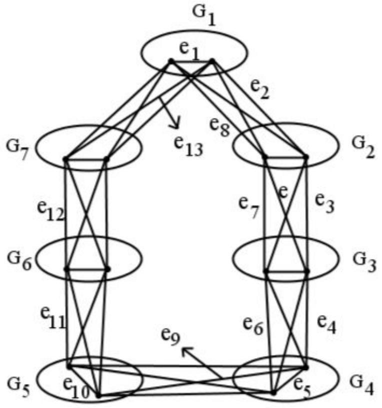

Figure 1: $C_7[K_2]$

$p = 8$, so that the lexicographic product of C_8 with K_2 is taken. This graph is shown as in Figure 2. The eccentricity of an edge $e_1 \in E(G_1)$ is 10, since there exists a shortest cycle $e_1, e_2, e_3, e_4, e_5, e_6,$ e_7, e_9, e_{10}, e_{11}, of length 10 containing both e_1 and $e_6 \in E(G_5)$ and it is clear from the Figure 2 that there exists no other edge farther than e_6 from e_1, but the eccentricity of an edge $e_2 \in E(G_1 - G_2)$ is 9, since there exists a shortest cycle $e_2, e_3, e_4, e_5, e_{12}, e_8, e_9, e_{10}, e_{13}$ of length 9, containing both e_2 and $e_8 \in E(G_6)$ and there exists no other edge farther than e_8 from e_2. Hence, $C_8[K_2]$ is not edge - self centered.

Remark 2.8. *For any positive even integer $p \geq 4$, the lexicographic product $C_p[\overline{K}_m]$ is an edge - self centered graph.*

Proposition 2.4. *Let G be a nonseparable graph having girth k. If every pair of edges lies on a cycle of length k, then G is edge - DDR.*

Proof. Since the girth of G is k and every pair of edges lie on a cycle of length k, the distance $d(e_i, e_j) = k$ for every pair (e_i, e_j) in $E(G)$. Hence, $dds(e) = (\underbrace{1, 0, 0, \ldots, q-1}_{k+1\ entries})$ for all $e \in E(G)$. Hence G is edge - DDR. □

Example 2.5. *Petersen graph is a nonseparable graph having girth 5 in which every pair of edges lie on a cycle of length 5 and hence it is edge - DDR.*

Proposition 2.6. *If G is a edge - self centered graph with radius r then subdivision graph $S(G)$ is also edge - self centered with radius $2r$.*

Proof. In the subdivision graph $S(G)$ the length of every cycle becomes twice the length of corresponding cycle in G. Hence the proof. □

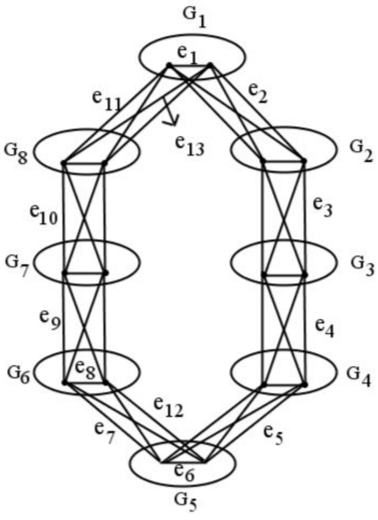

Figure 2: $C_8[K_2]$

Remark 2.9. *Subdivision graph of a Petersen graph is edge - self centered with radius 10.*

2.2 Edge - eccentric graph of a graph

In this section we introduce a new class of graphs viz., the edge - eccentric graph of a graph and study related results. This concept is defined on similar lines as defined by Akiyama et al. [2] about eccentric graph of a graph, with respect to the eccentricity defined on vertex distance.

Definition 2.7. *Edge - eccentric graph $ED_e(G)$ of a graph G is a graph with vertex set consisting of set $E(G)$ of edges of G and two vertices e_i and e_j are adjacent in $ED_e(G)$ if and only if they are at eccentric distance from each other.*

Theorem 2.10. *Edge - eccentric graph of any forest is isomorphic to a complete graph K_q, where q is the number of edges in the forest.*

Proof. Let F be a forest containing q edges. Since F is acyclic, eccentricity of every edge is infinity and for every edge all other edges are eccentric. Hence, the edge - eccentric graph of any forest is isomorphic to a complete graph K_q. □

Corollary 2.11. *For any tree T of order $p \geq 2$, $ED_e(T) \cong K_{p-1}$.*

Corollary 2.12. *$ED_e(K_p)$ is a regular graph with regularity $\frac{(p-2)(p-3)}{2}$.*

Proof. For every edge e, there exist $\frac{(p-3)(p-2)}{2}$ eccentric edges. Hence, edge - eccentric graph of complete graph K_p is a regular graph with regularity $\frac{(p-2)(p-3)}{2}$. Hence the proof. □

Proposition 2.8. *For any cycle C_p, $ED_e(C_p) = K_p$.*

Proof. Let C_p be any cycle on $p \geq 3$ vertices. Since, the shortest cycle containing every pair of edges is C_p itself, each edge is eccentric to every other edge and vice - versa, with eccentricity of each edge equal to p. Hence $ED_e(C_p)$ is isomorphic to the complete graph K_p. □

Remark 2.13. *If $G(p, q)$ is a graph in which for every pair of edges, there exist a shortest cycle with length equal to girth of G then $ED_e(G) \cong K_q$.*

Example 2.9. *Edge - eccentric graph of Petersen graph is isomorphic to K_{10}.*

Remark 2.14. *Edge - eccentric graph of a disconnected graph is connected.*

Remark 2.15. *Edge - eccentric graph need not always be connected. Edge eccentric graph of K_4 is the union of 3 disjoint $K_2's$.*

Theorem 2.16. *Edge - eccentric graph of a disconnected graph having k components is a complete k - partite graph if each component is nonseparable.*

Proof. Let G be a disconnected graph all of whose k components are nonseparable. Since G is disconnected, eccentricity of every edge is ∞. Let C be any component and $e \in C$, then every edge not in C is eccentric to e and vice versa. Also, since each component is nonseparable, there exist no two edges in C which are eccentric to each other. Hence, edge - eccentric graph of G is complete k - partite graph. □

Theorem 2.17. *If G is a graph with $r \geq 1$ cut vertices and $ED_e(G)$ is isomorphic to complete $r + 1$ - partite graph.*

Proof. Let G be any connected graph having $r \geq 1$ cut vertices, then G has $r + 1$ blocks. No two edges from different blocks lie on same cycle, hence any two edges belonging to different blocks are eccentric to each other. Hence $ED_e(G)$ is isomorphic to complete $r + 1$ - partite graph. □

Corollary 2.18. *If G is a connected graph containing unique cut vertex then $ED_e(G)$ is isomorphic to complete bipartite graph.*

Remark 2.19. *If b_i is the number of vertices in a block B_i, $i = 1, 2$ which are adjacent to a cut vertex v and v is the only cut vertex in the connected graph G having 2 blocks then $ED_e(G)$ is isomorphic to $K_{q_1+b_1, q_2+b_2}$, where q_i, $i = 1, 2$, is the number of edges in each block B_i.*

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