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Plane wave propagation at solid-solid imperfect interface

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Abstract : In this paper, the reflection and transmission phenomenon at the imperfect interface between viscoelastic solid half space and elastic solid half space is presented. P-wave or SV-wave is considered to be incident on the interface through viscoelastic solid half space. The amplitude ratios of various reflected and transmitted waves to that of incident wave are derived and deduced for normal force stiffness, transverse force stiffness and for welded contact. After obtaining the amplitude ratios, they have been computed numerically for a particular model and results thus obtained are depicted graphically with angle of incidence of incident wave. It is found that these amplitude ratios depend on angle of incidence of the incident wave and material properties of the medium and these are affected by the stiffness also.

Keywords : Viscoelastic solid, Elastic solid, Reflection and Transmission.

1 Introduction

The state of the deep interior of the earth cannot be explained by assuming the earth to be an elastic solid. Keeping this fact in mind several problems of reflection and refraction in a linear viscoelastic solid

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have been discussed by many researchers like Cooper and Reiss [1], Cooper [2] etc.

Interface modeling has been the subject of numerous studies in material science, as applied to composite structures. Imperfect interface considered in this problem means that the stress components are continuous and small displacement field is not. The small vector difference in the displacements assumed to depend linearly on the traction vector. More precisely jumps in the displacement components are assumed to be proportional (in terms of spring-factor-type interface parameters) to their respective interface components. The infinite values of interface parameters imply vanishing of displacement jumps and therefore correspond to perfect interface conditions. On the other hand, zero values of the interface parameters imply vanishing of the corresponding interface tractions which corresponds to complete debonding. The finite values of the interface parameters define an imperfect interface. The values of the interface parameters depend on the material properties of the medium i.e. microstructure as well as the bi-material properties. Recently many authors have used the imperfect conditions at an interface to solve the various types of problems (Chen et.al. (2004), Kumar and Rupender [6] and Kumar and Chawala [7] etc.).

Using the Borcherdt (1973) theory for linear viscoelastic solid, the reflection and transmission of longitudinal wave (P-wave) or transverse wave (SV-wave) at an imperfect interface between linear isotropic elastic solid half space and linear viscoelastic solid half space is investigated. Amplitudes ratios for various reflected and transmitted waves are computed for a particular model and depicted graphically and discussed accordingly. The model considered is assumed to exist in the oceanic crust part of the earth and the propagation of wave through such a model will be of great use in the fields related to earth sciences.

2 Basic Equations

For M_1 (linear viscoelastic solid medium)

Following Borcherdt (1973), the equation governing the small motions in a linear viscoelastic solid may be written as

$$\left(\mathrm{K}'+4\mathrm{M}'/3\right)\nabla\left(\nabla.\mathbf{u}'\right)-\mathrm{M}'\nabla\times\left(\nabla\times\mathbf{u}'\right)=\rho_l\ddot{\mathbf{u}}' \tag{2.1}$$

where symbols K' is the complex bulk modulus, M' is the shear modulus, ρ_l is the density of linear viscoelastic solid and $\mathbf{u}'$ is the displacement vector. Superposed dots on right hand side of equation (2.1) stand for second partial derivative with respect to time. The stresses in the linear viscoelastic solid are given by

$$\sigma'_{kl}=\left(\mathrm{K}'-2\mathrm{M}'/3\right)\theta\delta_{kl}+2\mathrm{M}'e_{kl} \tag{2.2}$$

where

$$e_{kl}=\frac{1}{2}\left(\frac{\partial u_k{}'}{\partial x_l}+\frac{\partial u_l{}'}{\partial x_k}\right), \qquad \theta=\nabla.\mathbf{u}' \tag{2.3}$$

Using Helmholtz's theorem

$$u'=\nabla\phi'+\nabla\times\psi', \qquad \nabla.\psi'=0, \tag{2.4}$$

We can show that ϕ' and ψ' satisfy

$$\alpha^2\nabla^2\phi'=\ddot{\phi}' \text{ and } \beta^2\nabla^2\psi'=\ddot{\psi}' \tag{2.5}$$

where

$$\alpha^2=\left(K'+\frac{4M'}{3}\right)/\rho_l, \qquad \beta^2=M'/\rho_l \tag{2.6}$$

and

$$\psi'=-\left(\psi'\right)_{y'} \tag{2.7}$$

$$u'=\frac{\partial\phi'}{\partial x}+\frac{\partial\psi'}{\partial z}, \qquad w'=\frac{\partial\phi'}{\partial z}-\frac{\partial\psi'}{\partial x}. \tag{2.8}$$

For $\mathbf{M_2}$ (Homogeneous isotropic elastic solid medium)

The equation governing the small motions in a homogeneous isotropic elastic are

$$\mu^*\nabla^2\mathbf{u}^*+(\lambda^*+\mu^*)\quad\nabla(\nabla.\mathbf{u}^*)=\rho^*\ddot{\mathbf{u}}^*, \tag{2.9}$$

where symbols λ^*,μ^* are Lame's constants, ρ^* is the density and $\mathbf{u}^*$ is the displacement vector. Superposed dots on right hand side of equation (2.9) stand for second partial derivative with respect to time.

The stress strain relation in the isotropic elastic medium is given by

$$\sigma_{ij}{}^*=\lambda^*e_{kk}{}^*\ \delta_{ij}+2\mu^*\ e_{ij}{}^*, \tag{2.10}$$

where

$$e_{ij}{}^*=\frac{1}{2}\left(\frac{\partial u_i{}^*}{\partial x_j}+\frac{\partial u_j{}^*}{\partial x_i}\right), \tag{2.11}$$

are the components of the strain tensor, $e_{kk}{}^*$ is the dilatation and $\sigma_{ij}{}^*$ are the components of stress tensor in the isotropic elastic medium. For the two dimensional problem, the displacement vector $\mathbf{u}'$ is taken as

$$\mathbf{u}^*=(u^*,0,w^*), \tag{2.12}$$

The displacement components u^* and w^* are related to potential functions ϕ^* and ψ^* as

$$u^*=\frac{\partial\phi^*}{\partial x}+\frac{\partial\psi^*}{\partial z}, \quad w^*=\frac{\partial\phi^*}{\partial z}-\frac{\partial\psi^*}{\partial x}, \tag{2.13}$$

Using equations (2.12) and (2.13) in equation (2.9), we obtain as

$$\nabla^2\phi^*=\frac{1}{v_1^{*2}}\frac{\partial^2 ?^*}{\partial t^2}, \tag{2.14}$$

$$\nabla^2\psi^*=\frac{1}{v_2^{*2}}\frac{\partial^2\psi^*}{\partial t^2}, \tag{2.15}$$

where$v_1^* = \sqrt{\frac{\lambda^*+2\mu^*}{\rho^*}}$ and $v_2^*=\sqrt{\frac{\mu^*}{\rho^*}}$ are the velocities of longitudinal wave (P-wave) and transverse wave (SV-wave) in isotropic elastic medium respectively.

3 Formulation of the problem and its solution

Considering a two dimensional problem by taking the z-axis pointing into lower half-space and the imperfect interface at $z = 0$ separating the linear viscoelastic solid half space M_1 $[z > 0]$ and elastic solid half space M_2 $[z < 0]$ (see figure1). A longitudinal wave (P-wave) or transverse wave (SV-wave) propagates through linear viscoelastic solid half space medium M_1 and incident at the plane $z = 0$ and making an angle θ_0 with normal to the surface. Corresponding to each incident wave (P-wave or SV-wave), we get two reflected waves P-wave and SV-wave in the medium M_1 and two transmitted waves P-wave and SV-wave in medium M_2.

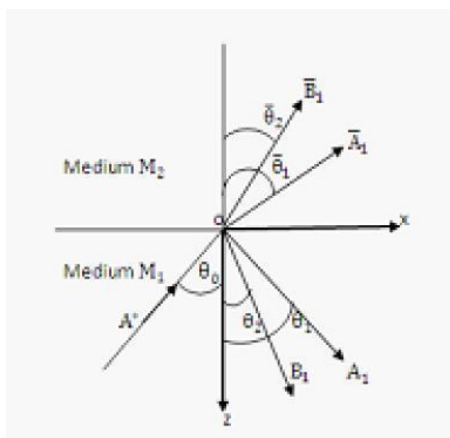

Figure 1: Geometry of the Problem

In medium M_1

The potential function satisfying the equation (2.8) can be taken as

$$\phi^{'}=A_0\ \exp\left[ik_1\left(x\ \sin\theta_0 - z\ \cos\theta_0\right)+i\omega_1 t\right]+A_1\exp\left[ik_1\left(x\ \sin\theta_1+z\ \cos\theta_1\right)+i\omega_1 t\right] \tag{3.1}$$

$$\psi^{'}=B_0\ \exp\left[ik_2\left(x\ \sin\theta_0 - z\ \cos\theta_0\right)+i\omega_2 t\right]+B_1\exp\left[ik_2\left(x\ \sin\theta_2+z\ \cos\theta_2\right)+i\omega_2 t\right] \tag{3.2}$$

where A_0 and B_0 are amplitudes of the incident P-wave and SV-wave, respectively and A_1, B_1 are amplitudes of the reflected P-wave and SV-wave respectively and to be determined from boundary conditions.

In medium M_2

The potential function satisfying the equation (2.14)–(2.15) can be taken as

$$\phi^{*}=\overline{\mathrm{A}}_{1}\exp\left[\mathrm{i}\overline{\mathrm{k}}_{1}\left(\mathrm{x}\ \sin\overline{\theta}_{1}-\mathrm{z}\ \cos\overline{\theta}_{1}\right)+\mathrm{i}\overline{\omega}_{1}\mathrm{t}\right], \tag{3.3}$$

$$\psi^{*}=\overline{\mathrm{B}}_{1}\exp\left\{\mathrm{i}\overline{\mathrm{k}}_{2}\left(\mathrm{x}\ \sin\overline{\theta}_{2}-\mathrm{z}\ \cos\overline{\theta}_{2}\right)+\mathrm{i}\overline{\omega}_{2}\mathrm{t}\right\}, \tag{3.4}$$

where $\overline{\mathrm{k}}_1$ and $\overline{\mathrm{k}}_2$ are wave numbers of transmitted P-wave and transmitted SV-wave, respectively. $\overline{\mathrm{A}}_1$ and $\overline{\mathrm{B}}_1$ are amplitudes of transmitted P-wave and transmitted SV-wave and are unknown to be determined from boundary conditions.

4 Boundary conditions

The appropriate boundary conditions at the interface $z = 0$ are the continuity of displacement and stresses. Mathematically, these boundary conditions can be expressed as:

$$\sigma_{\mathrm{zz}}{}' = \sigma_{\mathrm{zz}}{}^{*}, \qquad \sigma_{\mathrm{zx}}{}' = \sigma_{\mathrm{zx}}{}^{*}, \qquad \sigma_{\mathrm{zx}}{}^{*}=\mathrm{K}_{\mathrm{t}}\left(\mathrm{u}'-\mathrm{u}^{*}\right), \quad \sigma_{\mathrm{zz}}{}^{*}=\mathrm{K}_{\mathrm{n}}\left(\mathrm{w}'-\mathrm{w}^{*}\right). \tag{4.1}$$

In order to satisfy the boundary conditions, the extension of the Snell's law will be

$$\frac{\sin\theta_0}{\mathrm{V}_0}=\frac{\sin\theta_1}{\mathrm{V}_1}=\frac{\sin\theta_2}{\mathrm{V}_2}=\frac{\sin\overline{\theta}_1}{\overline{\mathrm{V}}_1}=\frac{\sin\overline{\theta}_2}{\overline{\mathrm{V}}_2}. \tag{4.2}$$

Also

$$\mathrm{k}_1\mathrm{V}_1=\mathrm{k}_2\mathrm{V}_2=\overline{\mathrm{k}}_1\overline{\mathrm{V}}_1=\overline{\mathrm{k}}_2\overline{\mathrm{V}}_2=\omega, \text{ at } z=0. \tag{4.3}$$

For P-wave,

$$\mathrm{V}_0=\mathrm{V}_1, \quad \theta_0=\theta_1. \tag{4.4}$$

For SV-wave,

$$\mathrm{V}_0=\mathrm{V}_2, \quad \theta_0=\theta_2. \tag{4.5}$$

For incident longitudinal wave at the interface $z = 0$, putting $\mathrm{B}_0 = 0$ in equation (3.2) and for incident transverse wave putting $\mathrm{A}_0 = 0$ in equation (3.1). Substituting the expressions of potentials given by (3.1–3.4) in equations (2.2), (2.8), (2.10) and (2.13) and using equations (4.1–4.5), we get a system of four non homogeneous which can be written as

$$\sum_{\mathrm{j}=0}^{4}\mathrm{a}_{\mathrm{ij}}\mathrm{Z}_{\mathrm{j}}=\mathrm{Y}_{\mathrm{i}}, \qquad (\mathrm{i}=1,2,3,4) \tag{4.6}$$

where

$$\mathrm{Z}_1=\frac{\mathrm{A}_1}{\mathrm{A}^*}, \quad \mathrm{Z}_2=\frac{\mathrm{A}_2}{\mathrm{A}^*}, \quad \mathrm{Z}_3=\frac{\overline{\mathrm{A}}_1}{\mathrm{A}^*}, \quad \mathrm{Z}_4=\frac{\overline{\mathrm{B}}_1}{\mathrm{A}^*} \tag{4.7}$$

Also a_{ij} in non dimensional form can be written as

$$\begin{aligned}
&a_{11} = -\frac{K'}{M'} - 2\sin^2\theta_0 + \frac{4}{3}, && a_{12} = 2\sin\theta_2\cos\theta_2\frac{k_2^2}{k_1^2},\\
&a_{13} = \frac{\lambda^*\bar{k_1}^2}{M'k_1^2}, && a_{14} = \frac{\mu^*\bar{k_2}^2}{k_1^2 M'}\sin 2\bar{\theta_2},\\
&a_{21} = -2\sin\theta_1\cos\theta_1, && a_{22} = -\frac{k_2^2}{k_1^2}\left(\cos^2\theta_2 - \sin^2\theta_2\right),\\
&a_{23} = -\frac{\mu^*\bar{k_1}^2}{M'k_1^2}\sin 2\bar{\theta_1}, && a_{24} = -\frac{\mu^*\bar{k_2}^2}{M'k_1^2}\cos 2\bar{\theta_2},\\
&a_{31} = i\sin\theta_1, && a_{32} = \frac{ik_2\cos\theta_2}{k_1},\\
&a_{33} = -\frac{i\bar{k_1}\sin\bar{\theta_1}}{k_1} - \frac{\mu^*\bar{k_1}^2\sin 2\bar{\theta_1}}{K_t k_1}, && a_{34} = \frac{i\bar{k_2}\cos\bar{\theta_2}}{k_1} - \frac{\mu^*\bar{k_2}^2\cos 2\bar{\theta_2}}{K_t k_1},\\
&a_{41} = i\cos\theta_1, && a_{42} = -\frac{ik_2\sin\theta_2}{k_1},\\
&a_{43} = \frac{\overline{ik_1}\cos\bar{\theta_1}}{k_1} + \frac{\lambda^*\bar{k_1}^2}{K_n k_1}, && a_{44} = \frac{i\bar{k_2}\sin\bar{\theta_2}}{k_1} + \frac{\mu^*\bar{k_2}^2\sin 2\bar{\theta_2}}{K_n k_1}
\end{aligned} \tag{4.8}$$

For incident longitudinal wave:

$$A^* = A_0,\ B_0 = 0, Y_1 = -a_{11}, Y_2 = a_{21}, Y_3 = -a_{31}, Y_4 = a_{41}, \tag{4.9}$$

For incident transverse wave:

$$A^* = B_0,\ A_0 = 0, Y_1 = a_{12}, Y_2 = -a_{22}, Y_3 = a_{32},\ Y_4 = -a_{42}, \tag{4.10}$$

5 Particular case

Case I: Normal force stiffness ($K_n \neq 0, K_t \to \infty$)

In this case, we obtain a system of four non homogeneous equations as those given by equation (4.6) with the changed a_{ij} as

$$a_{33} = -\frac{i\,\bar{k}_1}{k_1}\sin\bar{\theta}_1, \qquad a_{34} = \frac{i\,\bar{k}_2\cos\bar{\theta}_2}{k_1}, \tag{5.1}$$

Case II: Transverse force stiffness ($K_t \neq 0, K_n \to \infty$)

In this case also, a system of four non homogeneous equations as those given by equation (4.6) is obtained with the changed a_{ij} as given below

$$a_{43} = \frac{\overline{i\ k_1}\cos\bar{\theta}_1}{k_1}, \qquad a_{44} = \frac{i\,\bar{k}_2\sin\bar{\theta}_2}{k_1}, \tag{5.2}$$

Case III: Welded contact ($K_n \to \infty, K_t \to \infty$)

Again in this case, a system of four non homogeneous equations is obtained as those given by equation (4.6) with some a_{ij} changed as

$$a_{33} = -\frac{i\,\bar{k}_1}{k_1}\sin\bar{\theta}_1, \qquad a_{34} = \frac{i\,\bar{k}_2\cos\bar{\theta}_2}{k_1}, \qquad a_{43} = \frac{\overline{i\ k_1}\cos\bar{\theta}_1}{k_1}, \qquad a_{44} = \frac{i\,\bar{k}_2\sin\bar{\theta}_2}{k_1} \tag{5.3}$$

6 Numerical results and discussion

The theoretical results obtained above indicate that the amplitude ratios Z_i $(i = 1, 2, 3)$ depend on the angle of incidence of incident wave. In order to study in more detail the behaviour of various amplitude ratios on the angle of incidence, we have computed them numerically by taking the following values relevant elastic parameters.

In medium M_1, Following Silva [4], the physical parameters representing the crust as a linear viscoelastic solid are as follows

$$Q_P = 100, \quad Q_S = 45, \quad \rho_l = 2.6 \text{ gm/cm}^3, \quad V_P = 6.1 \text{ km/s}, \quad V_S = 3.5 \text{ km/s} \tag{6.1}$$

In medium M_2, the physical parameters for isotropic elastic solid are as follows

$$\rho' = 2.65 \frac{\text{Mg}}{\text{m}^3}, \quad \mu' = 2.238 \frac{\text{MN}}{\text{m}^2}, \quad \lambda' = 2.238 \frac{\text{MN}}{\text{m}^2} \tag{6.2}$$

Using MATLAB, a computer programme has been developed and modulus of amplitude ratios $|Z_i|, (i = 1, 2, 3, 4,)$ for various reflected and transmitted waves have been computed. $|Z_1|$ and $|Z_2|$ represent the modulus of amplitude ratios for reflected P and reflected SV-wave respectively. Also, $|Z_3|$ and $|Z_4|$ represent the modulus of amplitude ratios for transmitted P and transmitted SV-wave respectively. The variations in all the figures are shown for the range $0^0 = \theta = 90^0$.

Figures (2)-(5) represent the variations of the amplitude ratios of reflected P-wave, reflected SV-wave, transmitted P-wave and transmitted SV-wave with angle of incidence of incident P-wave whereas figures Figures (6)-(9) show the variations of the amplitude ratios for reflected P-wave, reflected SV-wave, transmitted P-wave and transmitted SV-wave with angle of incidence of the incident SV-wave. In all the figures Figures (2)-(9), dashed line represent the general case (GEN) of imperfect boundary, whereas dashed line represent the normal force stiffness case(NFS).Also, bold dashed line represent the transverse force stiffness case(TFS) and solid line depicts the welded contact case(WD).

In figure (2), the values are same for NFS and WD cases. Also, for GEN and TFS cases the values are same. In figure (3), the values are almost same in all the four cases. In figure (4), the values are same for TFS and WD cases. Also, for GEN and NFS cases the values of amplitude ratio $|Z_1|$ are same. In figure (5), it is clear that the values are same for TFS and GEN cases but are different for other two cases. In figures (3), (5), the curves first attain their maximum values and then start to decrease and reach to their minimum value i.e. zero.

In figure (6), the curves first attain their maximum values and then suddenly start to decrease and reach to zero value. In figure (7), it is clear that the values are same for TFS and GEN cases but are different for other two cases and in all cases the values of amplitude ratio approach to one. In figure (8), the values are almost same in all the cases except for welded case .Also, all curves approach to zero value. The behaviour of all curves is almost same in figure (9). The curve for welded contact oscillates and approaches to zero.

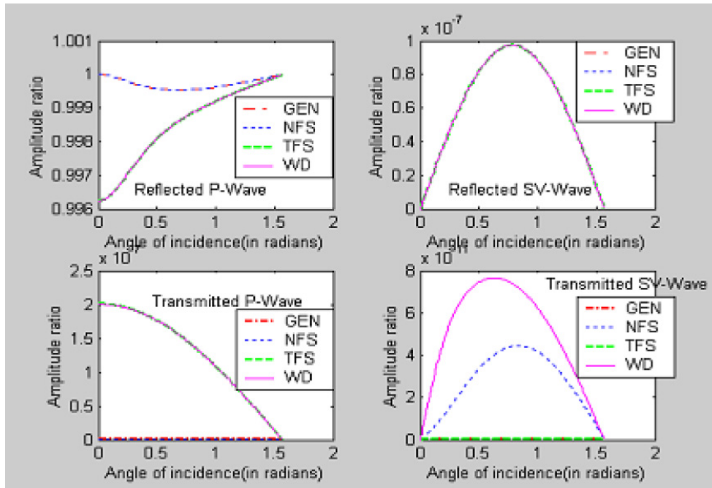

Figure 2-5: Variation of the amplitude ratios of reflected P-wave, reflected SV-wave, transmitted P-wave and transmitted SV-wave with angle of incidence of P-wave.

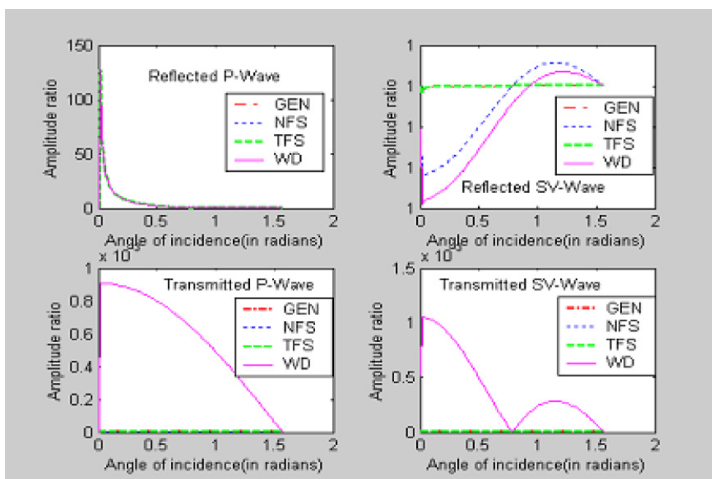

Figure 6-9: Variation of the amplitude ratios of reflected P-wave, reflected SV-wave, transmitted P-wave and transmitted SV-wave with angle of incidence of SV-wave.

7 Conclusion

Reflection and transmission phenomenon of incident elastic waves at an imperfect interface between linear elastic solid half space and linear viscoelastic solid half space has been studied when P-wave or SV-wave is incident. It is observed that the amplitudes ratios of various reflected and transmitted waves depend on the angle of incidence of the incident wave and material properties. Effect of stiffness is observed on amplitude ratios. The research work is supposed to be useful in further studies; both theoretical and observational of wave propagation in more realistic models of linear viscoelastic solid present in the earth's interior. The problems may be of use in engineering, seismology and geophysics etc.

References

[1] Henery F. Cooper and Edward F. Reiss, *Propagation and Reflection of Viscoelastic Waves*, J.Acoust. Soc. Am.,38(24)(1965).

[2] Henery F. Cooper, *Reflection and Transmission of Oblique Plane Waves at a Plane Interface between Viscoelastic Media*, AJ.Acoust. Soc. Am., 42(1064)(1967).

[3] R. Kumar and B. Singh,B, *Reflection of plane waves at a planar viscoelastic micropolar interface*, Indian J. Pure and Applied Mathematics, 31(3)(2000), 287–303.

[4] W. Silva, *Body waves in a layered anelastic solid*, Bulletin of the Seismological Society of America, 66(5)(1976), 1539–1554.

[5] R. Kumar, A. Miglani and S. Kumar, *Reflection and Transmission of plane waves between two different fluid saturated porous half spaces*, Bull. Pol. Ac., Tech., 59(2)(2011), 227–234.

[6] R. Kumar and Rupender, *Propagation of plane waves at the imperfect boundary of elastic and electro-microstretch generalized thermoelastic solid*, Applied Mathematics and Mechanics, 30(11)(2009), 1445–1454.

[7] R. Kumar and V. Chawla, *Effect of rotation and stiffness on surface wave propagation in elastic layer lying over a gereralized thermodiffusive elastic half-space with imperfect boundary*, Journal of solid mechanics, 2(1)(2010), 28–42.