International Journal of Mathematics And Its Applications
Vol.2 No.4 (2014), pp.13-18.
ISSN: 2347-1557(online)

Bilinear and Bilateral Generating Relations Involving Restricted Jacobi and Laguerre Polynomials

M.I.Qureshi[†], **Deepak Kumar Kabra**[‡,1] **and Yasmeen**[*]

[†] *Department of Applied Sciences and Humanities, Jamia Millia Islamia, New Delhi, India.*

[‡,*] *Department of Mathematics, Government College Kota, Kota, Rajasthan, India.*

Abstract : In this paper we obtain a bilinear generating relation for the restricted Jacobi polynomials, using the fractional derivative technique. By the process of confluence, a number of interesting linear, bilinear and bilateral generating relations for the restricted Jacobi and Laguerre polynomials, are derived as special cases. Known generating relations of Khan [7, 8] are also deduced.

Keywords : Restricted Jacobi polynomials, Generating relations, Bilinear and Bilateral Generating function, Restricted Laguerre Polynomials.

AMS Subject Classification: 33C45, 33C65, 33C70, 33C05, 33C15.

1 Introduction and Preliminaries

The fractional derivative operator $D_z^{(b)}$ is an extension of the familiar derivative operator $D_z^{(n)}$ (n being a positive integer), to arbitrary (integer, rational, irrational and complex) values of b. The development of the fractional derivative operators is receiving keen attention from many researchers presently. In particular, see for example, the work of Lavoie, etal. [9], Manocha [11], Manocha-Sharma [12, 13, 14], Oldham-Spanier [15], Sharma-Abiodun [17] and Deshpande [2]. In 1731, Euler extended the derivative formula in the following form.
Let $D_z^{(b)}$ denotes the operator of fractional derivative having the arbitrary order b, as usually defined

$$D_z^{(b)}[z^{a-1}] = \frac{\Gamma(a)}{\Gamma(a-b)} z^{a-b-1}, \tag{1.1}$$

which holds for all values of b, except $b = a$ and a is neither zero nor a negative integer.
Throughout in present paper, we use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \backslash \{0\}$.
Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.
The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma

[1] Corresponding author E-Mail: dkabra20@gmail.com (Deepak Kumar Kabra)

function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\dots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases}$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

The object of the present paper is to establish a generating relation for the product of two restricted Jacobi polynomials, using the fractional derivative operator (1.1). A number of interesting generating formulae for Jacobi and Laguerre polynomials are obtained as special cases.

A unification of Lauricella's fourteen triple hypergeometric functions $F_1, F_2, \dots, F_{14}$ and three additional triple hypergeometric functions H_A, H_B, H_C, was introduced by Srivastava [18], who defined a general triple hypergeometric series $F^{(3)}[x,y,z]$ in the form

$$F^{(3)}\left[\begin{array}{l} (a) :: (b); (b'); (b'') : (c); (c'); (c''); \\ (e) :: (g); (g'); (g'') : (h); (h'); (h''); \end{array} x, y, z\right] = \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p) \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

where, for convenience,

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b_j^{'})_{n+p}\prod_{j=1}^{B''}(b_j^{''})_{p+m}\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c_j^{'})_n\prod_{j=1}^{C''}(c_j^{''})_p}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g_j^{'})_{n+p}\prod_{j=1}^{G''}(g_j^{''})_{p+m}\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h_j^{'})_n\prod_{j=1}^{H''}(h_j^{''})_p}$$

where (a) abbreviates the array of A parameters given by $a_1, a_2, \dots, a_A$ with similar interpretations for (b), (b'), (b''), et cetera. The above triple hypergeometric series converges absolutely when

$$\begin{cases} 1 + E + G + G'' + H - A - B - B'' - C \geqslant o, \\ 1 + E + G + G' + H' - A - B - B' - C' \geqslant o, \\ 1 + E + G' + G'' + H'' - A - B' - B'' - C'' \geqslant o, \end{cases}$$

where the equalities hold true for suitably constrained values of $|x|, |y|$ and $|z|$.

The Jacobi's polynomials $P_n^{(\alpha,\beta)}(x)$ [16] are given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left[\begin{array}{ll} -n, 1+\alpha+\beta+n; & \frac{1-x}{2} \\ 1+\alpha & ; \end{array}\right] \tag{1.2}$$

$$= (-1)^n P_n^{(\beta,\alpha)}(-x)$$

where $Re(\alpha) > -1$ and $Re(\beta) > -1$.

The Laguerre's polynomials $L_n^{(\alpha)}(x)$ [16] are given by

$$\lim_{|\beta| \longrightarrow \infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right) = L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1\left[\begin{array}{ll} -n & ; \\ 1+\alpha; & \end{array} x\right] \tag{1.3}$$

where $Re(\alpha) > -1$

The Appell's double hypergeometric function of first kind F_1 [4] is given by

$$F_1(a; b, b'; \ c \ ; x, y) = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_r(b')_s}{(c)_{r+s}} \frac{x^r}{r!}\frac{y^s}{s!} \tag{1.4}$$

where $\max\{|x|, |y|\} < 1$.

The Humbert's double hypergeometric functions are defined by [4]

$$\Phi_1[a, b; c; x, y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_r}{(c)_{r+s}} \frac{x^r}{r!}\frac{y^s}{s!} \tag{1.5}$$

where $|x| < 1, \quad |y| < \infty$

$$\Phi_2[b, c; d; x, y] = \sum_{r,s=0}^{\infty} \frac{(b)_r (c)_s}{(d)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!} \tag{1.6}$$

where $|x| < \infty, \quad |y| < \infty$

$$\Phi_3[b; d; x, y] = \sum_{r,s=0}^{\infty} \frac{(b)_r}{(d)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!} \tag{1.7}$$

where $|x| < \infty, \quad |y| < \infty.$

The triple hypergeometric function ${}_3\Phi_D^{(1)}$ of Jain [6] is the generalization of Humbert's double hypergeometric function Φ_1 and is defined by

$${}_3\Phi_D^{(1)}[a, b, c; d; x, y, z] = \sum_{r,s,k=0}^{\infty} \frac{(a)_{r+s+k} (b)_r (c)_s}{(d)_{r+s+k}} \frac{x^r}{r!} \frac{y^s}{s!} \frac{z^k}{k!} \tag{1.8}$$

Other notations of ${}_3\Phi_D^{(1)}$ are $\Phi_D^{(3)}$ of Srivastava and Exton [19, 4] and F_{D_1} of Exton [3].

$$\Phi_D^{(3)}[a, b, c, -; d; x, y, z] = \sum_{r,s,k=0}^{\infty} \frac{(a)_{r+s+k} (b)_r (c)_s}{(d)_{r+s+k}} \frac{x^r}{r!} \frac{y^s}{s!} \frac{z^k}{k!} \tag{1.9}$$

$$F_{D_1}[a, a, a; b, c, -; d, d, d; x, y, z] = \sum_{r,s,k=0}^{\infty} \frac{(a)_{r+s+k} (b)_r (c)_s}{(d)_{r+s+k}} \frac{x^r}{r!} \frac{y^s}{s!} \frac{z^k}{k!} \tag{1.10}$$

The triple hypergeometric function $\Phi_3^{(3)}$ of Exton [4] is the generalization of Humbert's double hypergeometric functions Φ_2 and Φ_3 and is defined by

$$\Phi_3^{(3)}[a, b; c; x, y, z] = \sum_{r,s,k=0}^{\infty} \frac{(a)_r (b)_s}{(c)_{r+s+k}} \frac{x^r}{r!} \frac{y^s}{s!} \frac{z^k}{k!} \tag{1.11}$$

Any values of parameters and variables leading to the results given in sections 2 and 3 which do not make sense, are tacitly excluded.

2 Main Generating Relations

Consider the generating relation of Feldheim [5] in the form

$$\sum_{n=0}^{\infty} \frac{1}{(1+c)_n} P_n^{(c,a-n)}(x) t^n = exp\left\{\frac{(1+x)t}{2}\right\} {}_1F_1\left[\begin{matrix} -a; \\ 1+c; \end{matrix} \frac{(1-x)t}{2}\right] \tag{2.1}$$

where $P_n^{(c,a-n)}(x)$ and ${}_1F_1$ are restricted Jacobi's polynomials and Kummer's confluent hypergeometric function [16], respectively.

In equation (2.1), replacing t by bt, multiplying both the sides by $(1-by)^m b^{d-1}$,(m being a positive integer), using the operator $D_b^{(d-e)}$ on both the sides and interpreting the result with the help of the definition (1.1), we get the bilateral generating relation in the form

$$\sum_{n=0}^{\infty} \frac{(d)_n P_n^{(c,a-n)}(x)}{(e)_n (1+c)_n} {}_2F_1\left[\begin{matrix} -m, d+n; \\ e+n \ \ ; \end{matrix} by\right] (bt)^n$$
$$= F^{(3)}\left[\begin{matrix} d :: -; -; - : -; -a \ \ ; -m; \\ e :: -; -; - : -; 1+c; \ \ -; \end{matrix} \frac{b(1+x)t}{2}, \frac{b(1-x)t}{2}, by\right] \tag{2.2}$$

where ${}_2F_1$ and $F^{(3)}$ are Gauss's ordinary hypergeometric polynomial [16] and Srivastava's triple hypergeometric function respectively.

In (2.2), replacing y, d, e and b by $\frac{1-y}{2}$, $1+d+e+m$, $1+e$ and 1, respectively and using the definition (1.2) of Jacobi's polynomial, we get a bilinear generating relation for Jacobi's polynomials in the following form

$$\sum_{n=0}^{\infty}\frac{m!(1+d+e+m)_n P_n^{(c,a-n)}(x)P_m^{(e+n,d)}(y)}{(1+e)_{m+n}(1+c)_n}t^n$$
$$= F^{(3)}\left[\begin{array}{l} 1+d+e+m :: -;-;- : -;-a \quad ;-m; \\ 1+e \quad\quad :: -;-;- : -;1+c; \quad -; \end{array} \frac{(1+x)t}{2}, \frac{(1-x)t}{2}, \frac{1-y}{2}\right] \tag{2.3}$$

3 Special Cases

In (2.2) setting $b=1$, replacing x and t by $\left(\frac{2x}{c}-1\right)$ and $-(1+c)t$, respectively, taking $|c|\longrightarrow\infty$, using the confluence principle [10, 1, 19], we get

$$\sum_{n=0}^{\infty}\frac{(d)_n L_n^{(a-n)}(x)}{(e)_n}{}_2F_1\left[\begin{array}{c} -m, d+n; \\ e+n; \end{array} y\right]t^n$$
$$= {}_3\Phi_D^{(1)}\left[d,-a,-m;e;-t,y,-xt\right] \tag{3.1}$$

$$= \Phi_D^{(3)}\left[d;-a,-m,-;e;-t,y,-xt\right] \tag{3.2}$$

$$= F_{D_1}\left[d,d,d;-a,-m,-;e,e,e;-t,y,-xt\right] \tag{3.3}$$

Here $L_n^{(a-n)}(x)$ are the restricted Laguerre's polynomials (1.3).
In (3.1) or (3.2) or (3.3), replacing e, t and y by $1+e$, $\frac{t}{d}$ and $\frac{y}{d}$, respectively and taking $|d|\longrightarrow\infty$, we get a bilinear generating function for restricted Laguerre's polynomials

$$\sum_{n=0}^{\infty}\frac{m!L_n^{(a-n)}(x)L_m^{(e+n)}(y)}{(1+e)_{m+n}}t^n = \Phi_3^{(3)}\left[-a,-m;1+e;-t,y,-xt\right] \tag{3.4}$$

Setting $t=-y$ in (3.3) and using a transformation of Exton [3], we get

$$\sum_{n=0}^{\infty}\frac{(d)_n L_n^{(a-n)}(x)}{(e)_n}{}_2F_1\left[\begin{array}{c} -m, d+n; \\ e+n \quad ; \end{array} y\right](-y)^n = \Phi_1\left[d;-(a+m);e;y,xy\right] \tag{3.5}$$

Replacing y by $-y$ and taking $m=0$, (3.5) reduces to a known generating function of Khan [8]

$$\sum_{n=0}^{\infty}\frac{(d)_n L_n^{(a-n)}(x)}{(e)_n}y^n = \Phi_1\left[d;-a;e;-y,-xy\right] \tag{3.6}$$

When y is replaced by $\frac{y}{d}$, taking $|d|\longrightarrow\infty$, (3.6) reduces to another known generating function of Khan [7]

$$\sum_{n=0}^{\infty}\frac{L_n^{(a-n)}(x)}{(e)_n}y^n = \Phi_3\left[-a;e;-y,-xy\right] \tag{3.7}$$

When $y=0$ or $m=0$, (3.3) reduces to (3.6).
When $x=0$, (3.3) reduces to

$$\sum_{n=0}^{\infty}\frac{(a)_n(d)_n}{(e)_n}{}_2F_1\left[\begin{array}{c} -m, d+n; \\ e+n \quad ; \end{array} y\right]\frac{t^n}{n!} = F_1\left[d;a;-m;e;t,y\right] \tag{3.8}$$

Replacing t by $\frac{t}{a}$ in (3.8) and taking $|a|\longrightarrow\infty$, we get

$$\sum_{n=0}^{\infty}\frac{(d)_n}{(e)_n}{}_2F_1\left[\begin{array}{c} -m, d+n; \\ e+n \quad ; \end{array} y\right]\frac{t^n}{n!} = \Phi_1\left[d;-m;e;y,t\right] \tag{3.9}$$

On replacing y, t and e by $\frac{y}{d}$, $\frac{t}{d}$ and $1+e$, respectively and taking $|d| \longrightarrow \infty$, (3.8) reduces to

$$\sum_{n=0}^{\infty} \frac{m!(a)_n}{(1+e)_{m+n}} L_m^{(e+n)}(y) \frac{t^n}{n!} = \Phi_2\left[a, -m; 1+e; t, y\right] \tag{3.10}$$

Similarly by the process of confluence, (3.10) gives

$$\sum_{n=0}^{\infty} \frac{m!}{(1+e)_{m+n}} L_m^{(e+n)}(y) \frac{t^n}{n!} = \Phi_3\left[-m; 1+e; y, t\right] \tag{3.11}$$

When $y = 0$ or $m = 0$ in (3.4), we again get (3.7) and when $x = 0$, (3.4) reduces to (3.10). Alternatively (3.1), (3.8) and (3.9) can also be written in the following bilateral and linear generating relations

$$\sum_{n=0}^{\infty} \frac{(1+e+d+m)_n}{(1+e+m)_n} L_n^{(a-n)}(x) P_m^{(e+n,d)}(y) t^n$$

$$= \frac{(1+e)_m}{m!} {}_3\Phi_D^{(1)}\left[1+e+d+m, -a, -m; 1+e; -t, \frac{1-y}{2}, -xt\right] \tag{3.12}$$

$$\sum_{n=0}^{\infty} \frac{(a)_n(1+e+d+m)_n}{(1+e+m)_n} P_m^{(e+n,d)}(y) \frac{t^n}{n!} = \frac{(1+e)_m}{m!} F_1\left[1+e+d+m, a, -m; 1+e; t, \frac{1-y}{2}\right] \tag{3.13}$$

and

$$\sum_{n=0}^{\infty} \frac{(1+e+d+m)_n}{(1+e+m)_n} P_m^{(e+n,d)}(y) \frac{t^n}{n!} = \frac{(1+e)_m}{m!} \Phi_1\left[1+e+d+m; -m; 1+e; \frac{1-y}{2}, t\right] \tag{3.14}$$

Acknowledgment

Authors are indebted to Prof. M. A. Pathan, Aligarh Muslim University, Aligarh, U.P. (India) for the valuable discussions during the preparation of this paper.

References

[1] R. Askey and G. Gasper, *Certain rational functions whose power series have positive coefficients*, Amer. Math. Monthly, 79(1972), 327–341.

[2] V.L. Deshpande, *Generating relations involving hypergeometric functions of three variables*, Jñānābha, 4(1974), 47–57.

[3] H. Exton, *On certain confluent hypergeometric functions of three variables*, Ganita, 21(2)(1970), 79–92.

[4] H. Exton, *Multiple hypergeometric functions and applications*, John Wiley and Sons (Halsted Press), New York, Ellis Horwood, Chichester, U.K, (1976).

[5] E. Feldheim, *Relations entre les polynômes de Jacobi, Laguerre et Hermite*, Acta Math., 75(1941), 117–138.

[6] R.N. Jain, *The confluent hypergeometric functions of three variables*, Proc. Nat. Acad. Sci., 36(1966), 395–408.

[7] I.A. Khan, *Generating functions for Jacobi, Laguerre and Bessel polynomials*, Indian J. Pure Appl. Math., 3(1972), 437–442.

[8] I.A. Khan, *Generating functions for Jacobi and related polynomials*, Proc. Amer. Math. Soc., 32(1)(1972), 179–186.

[9] J.L. Lavoie, *Fractional derivatives and special functions*, SIAM Rev., 18(1976), 240–268.

[10] Y.L.Luke, *The special functions and their approximations-I*, Academic Press, New York and London,(1969).

[11] H.L. Manocha, *Some formulae involving Appell's function F_2*, Mathematica(Cluj), 9(32)(1967), 85–89.

[12] H.L. Manocha and B.L. Sharma, *Summation of infinite series*, J.Austral. Math.Soc., 6(1966), 470–476.

[13] H.L. Manocha and B.L. Sharma, *Some formulae by means of fractional derivatives*, Compositio Math., 18(1967), 229–234.

[14] H.L. Manocha and B.L. Sharma, *Fractional derivatives and summation*, J. Indian Math. Soc.(N.S.), 38(1974), 371–382.

[15] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York,(1974).

[16] E.D. Rainville,; *Special Functions*, The Macmillan , New York, Reprinted by Chelsea Publishing Company, Bronx, New York, (1960).

[17] B.L. Sharma and R.F.A. Abiodun, *Some generating functions for Laguerre and related polynomials*, Proc. Nat. Acad. Sci., 49(1979), 1–6.

[18] H.M. Srivastava, *Generalized Neumann expansions involving hypergeometric functions*, Proc. Cambridge Philos.Soc., 63(1967), 425–429.

[19] H.M. Srivastava and H. Exton, *On Laplace's linear differential equations of general order*, Nederl. Akad. Wetensch. Proc.,Ser.A,76=Indag.Math., 35(1973), 371–374.