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Complete Generators in 3-Valued Logic and Wrong Wheeler's Results

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Abstract : One of central problems of k-valued logic is identification and construction of complete generators (Sheffer functions). This problem is solved in 3-valued logic but some important results getting by Wheeler are wrong. We discuss Martin's, Foxley's Wheeler's and Rousseau's results in 3-valued logic. We construct classes of functions with the same ranges and complete generators for these classes in 3-valued logic.

Keywords : multiple-valued logics, Sheffer functions.

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1 Introduction

Multiple-valued logics attract the intense attention for connection with computer technology. But the most fruitful of the logics is Post's [1, 2]. We use this logic in our paper. In Post's k-valued logic [2] the negation, disjunction, and conjunction are presented by computable functions: $\o x = x + 1 \pmod k$, $x_1 \vee x_2 = \max(x_1, x_2)$, and $x_1 \wedge x_2 = \min(x_1, x_2)$. One of central problems of k-valued logic is identification and construction of complete generators (Sheffer functions). This problem is very complex since the number of objects (functions) of k-valued logic is very large and the number of complete generators is very large, too. These numbers increase quickly with growth of k. Thus investigation of complete generators of 3-valued logic is more simple than for greater k. More detailed investigation of complete generators for $k = 3$ was given by R.F. Wheeler [6]. But some his results are wrong. In particular, he gave the number of complete generators for any number of variables and the number was used in some papers (for example, [3]). But the number is wrong. The paper contains 3 sections. This introduction is the first section. The second section discusses results getting by N.M. Martin [4], E. Foxley [5], R.F. Wheeler [6], and G. Rousseau [7]. The last section contains all contemporary results, in particular, the numbers of complete generators of functions taking 1 and 2 values. Further complete generators are called just generators.

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2 Some Results in 3-Valued Logic

2.1 Martin's Results ([4], 1954)

Martin formulated four conditions that are fulfilled by non-generators: *substitution*, *co-substitution*, *t-closing*, and *closing*. We will give more precise definitions of the conditions. A function $f(x_1, x_2)$ satisfies *substitution*, if

$$\exists\, D\; \forall x_1, x_2, x_3, x_4 \quad x_1 \sim x_3 \wedge x_2 \sim x_4(D) \rightarrow f(x_1, x_2) \sim f(x_3, x_4)\; (D)$$

where D is a decomposition of $\{0, 1, 2\}$ into two or three disjoint subsets, $\sim$ means to belong to the same subset. There are 4 decompositions: $\{\{0\}, \{1, 2\}\}$, $\{\{1\}, \{0, 2\}\}$, $\{\{2\}\{0, 1\}\}$, $\{\{0\}, \{1\}.\{2\}\}$. A function $f(x_1, x_2)$ satisfies *co-substitution*, if

$$\exists\, D\; \forall x_1, x_2, x_3, x_4\; f(x_1, x_2) \sim f(x_3, x_4)(D) \rightarrow x_1 \sim x_3 \vee x_2 \sim x_4\; (D)$$

A function $f(x_1, x_2)$ satisfies *t-closing*, if

$$\exists\, t, k\; \forall x, i, j \sim f(t^i(x), t^j(x)) = t^k(x)$$

where $t(x) \in \{\bar{x}, \bar{\bar{x}}\}$, $t^0(x) = t(x)$, $t^{n+1} = t^n(t(x))$ and $i, j, k \in \{0, 1, 2\}$. The functions $t(x)$ are cyclic: $t^3(x) = t(x)$. Martin used any cyclic functions as $t(x)$ but only the functions $\o x$ and $\o\o x$ are cyclic. A function $f(x_1, x_2)$ satisfies *closing*, if

$$\exists\, X \sim X \subset \{0, 1, 2\} \wedge X \neq \varnothing \wedge \forall x_1, x_2 \quad x_1, x_2 \in X \rightarrow f(x_1, x_2) \in X$$

Martin proved that a function which does not satisfy these four conditions is a generator.

2.2 Foxley's Results ([5], 1962)

Foxley gave a simple rule of *t-closing*: a function $f(x_1, x_2)$ satisfies *t-closing*, if

$$\exists\, m\; \forall x, i, j \quad i \neq 2 \wedge j \neq 2 \rightarrow f(t^i(x), t^j(x)) = t^m(x)$$

where $t^0(x) = x$, $t^1(x) = \bar{x}$, $t^2(x) = \bar{\bar{x}}$ and $i, j, k \in \{0, 1, 2\}$. He proved also that the condition *co-substitution* is superfluous.

2.3 Wheeler's Results ([6], 1964)

Further reduction of the number of conditions was pointed by Wheeler. We will introduce his results in more simple way. After Post we call a function δ if $f(x, ..., x) \neq x$. Further we use only 2-ary δ functions taking all three values. We will denote by δ_2 a δ functions for which $f(x, x)$ takes only two values and denote by δ_3 a δ function for which $f(x, x)$ takes all three values. Wheeler found by calculation that a function δ_3 is a generator iff *t-closing* condition is not fulfilled, and the function δ_2 is a generator iff two conditions of *closing* and *substitution* are not fulfilled. Wheeler replaced *t-closing* by *conjuction* : a function $f(x_1, x_2)$ satisfies *conjuction*, if

$$|\{\varphi(x_1, x_2) : \forall t\; \varphi(x_1, x_2) = t(f(t(x_1), t(x_2))\; \}| \neq 6$$

where $|X|$ is a cardinal of a set X, t is an element of the symmetric group G_3 (top row has values of x, bottom row has values of $t(x)$):

$$t \in \left\{ \begin{pmatrix} 0,1,2 \\ 0,1,2 \end{pmatrix}, \begin{pmatrix} 0,1,2 \\ 0,2,1 \end{pmatrix}, \begin{pmatrix} 0,1,2 \\ 1,0,2 \end{pmatrix}, \begin{pmatrix} 0,1,2 \\ 1,2,0 \end{pmatrix}, \begin{pmatrix} 0,1,2 \\ 2,0,1 \end{pmatrix}, \begin{pmatrix} 0,1,2 \\ 2,1,0 \end{pmatrix} \right\}$$

The condition is more simple for computations than *t-closing*. Wheeler found that the number of δ_3 functions satisfying *t-closing* equals 18. He found also that the number of δ_2 functions satisfying *closing* equals 1944. In the next subsection we will show that all the other Wheeler's results are wrong. In particular, the number of δ_2 functions satisfying *substitution* is wrong.

2.4 Rousseau's Results (1968, [7])

Rousseau replaced *t-closing* by *automorphism*: a function $f(x_1, x_2)$ satisfies *automorphism* if

$$f\big(t(x_1), t(x_2)\big) = t\big(f(x_1, x_2)\big)$$

where $t(x) \in \{\bar{x}, \bar{\bar{x}}\}$. The condition is more simple for computations than *t-closing* and *conjuction*.

3 All results

We use the next equivalent relation: two functions are equivalent if they have the same range. Classes of equivalences are isomorphic if they have the same cardinal. So we will use only classes with ranges $\{0\}$, $\{0,1\}$, and $\{0,1,2\}$ (but there is a class of constants with empty range, too). The class with range $\{0\}$ has the unique generator $f(x_1, x_2) = 0$. The class with range $\{0,1\}$ has 60 generators from of 512 two-ary functions and from of 128 δ functions. The least generator has values $(1,0,0,\quad 0,0,0,\quad 0,0,1)$ whenever values of variables are $((0,0),(0,1),(0,2),\quad (1,0),(1,1),(1,2),\quad (2,0),(2,1),(2,2))$. The greatest generator has values $(1,1,1,\quad 1,0,1,\quad 1,1,0)$.

Further we use the class with range $\{0,1,2\}$. The class has 3774 generators from of 19683 two-place functions, this is 19% of the functions and 86% of δ functions (their number is 4374). The least generator has values (1,0,0, 0,2,0, 0,0,0), the greatest generator has values (2,2,2, 2,2,2, 2,0,1).

Co-substition condition is superfluous. *T-closing* condition was simplified by Rousseau. *Substitution* condition was not changed. Now we will give the properties of functions δ_2 and δ_3. The functions δ_2 are generators iff they do not satisfy *substitution* and *clousing* conditions. All δ_2 functions do not satisfy *t-closing*. The functions δ_2 have 6 options of $f(x,x)$ values (for values of $x = (0,1,2)) : (1,0,0),\ (1,0,1),\ (1,2,1),\ (2,0,0),\ (2,2,0),\ (2,2,1)$. For each option there are 389 δ_2 functions that are generators. So the number of the function for all options equals 2334 and this is 53% of all δ_2 functions.

The number of δ_2 functions is equal to 4374, of which 1944 functions satisfy *closing*, 726 functions satisfy *substitution*, and 630 functions satisfy both conditions of *closing* and *substitution*. Wheeler [6] found the number of δ_2 functions satisfying *closing* but could not find the well number of functions satisfying *substitution* (this number is 726, not 150) and satisfying both conditions of *closing* and *substitution* (this number is 630, not 54, but 726-630 = 150-54, this explains the coincidence with Martin's results).

In particular, Wheeler stated that the number of δ_2 functions satisfying both conditions of *closing* and *substitution* equals 9 (for one option), but there are 10 (out of 105) δ_2 functions satisfying these conditions. These functions $f(x_1, x_2)$ have values:

$$(1,0,0,\ 0,0,0,\ 2,2,0),\ (1,0,0,\ 0,0,1,\ 2,2,0),\ (1,0,0,\ 1,0,0,\ 2,0,0),$$
$$(1,0,0,\ 1,0,0,\ 2,2,0),\ (1,0,0,\ 1,0,1,\ 2,2,0),\ (1,0,1,\ 0,0,0,\ 2,2,0),$$
$$(1,0,1,\ 0,0,1,\ 2,2,0),\ (1,0,1,\ 1,0,0,\ 2,2,0),\ (1,0,1,\ 1,0,1,\ 2,2,0),$$
$$(1,0,2,\ 0,0,2,\ 0,0,0)$$

The functions δ_3 have the next properties. These functions are generators, iff they do not satisfy *t-clousing*. All δ_3 functions (generators and non-generators) do not satisfy *closing* and do not satisfy *substitution*. The functions δ_3 have two options for values of $f(x,x)$: (1,2,0) and (2,0,1). For each option there are 720 δ_3 functions which are generators and 9 functions which are non-generators. The number of generators for all options equals 1440. This is 99% of all δ_3 functions.

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