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EOQ model for a deteriorating item with lead-time under inflation over a finite time horizon

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Abstract : This paper presents the optimal inventory replenishment policy of a non-instantaneous deteriorating item under inflationary conditions using a discounted cash flow(DCF) approach,over a finite planning horizon.For proper recognition of financial implication of opportunity cost and inventory carrying cost in inventory control analysis, DCF approaches are used here.The unit cost depends on inflation as well as lead lime.The demand rate is assumed to be a function of unit cost under inflation. Shortages are allowed and partially backlogged according to an negative exponential waiting time function.Learning effect and influence of lead time are introduced on ordering cost.Maximum quantity of lost sale is prescribed.A general approach for finding the minimum cost is presented.Numerical examples are used to illustrate the approach and the model.Some sensitivity analyses to observe the effect of deterioration and inflation on the optimal inventory policy are performed.

Keywords : Inflation, Lead Time, Non-instantaneous Deterioration, Partial Backlogging, Discounted Cash Flow.

1 Introduction

Deterioration of an item occurs in the form of decay, evaporation, obsolescence, loss of utility etc. and such commodities are gasoline, fertilizers, different type of oils, milks, medicine etc. Inventory models for different deteriorating items with different types of deterioration have been developed by several researchers in past and recent years. Ghare and Schrader [1] have developed an EOQ model for exponentially decaying inventory. Later Covert and Philip[2] introduced two parameter weibull distribution for deterioration. Philip[3] developed a three parameter weibull distribution for deterioration with no shortages. Shah[4] extended Philip's[3] model with shortages.

Goyal and Giri[5] provided an excellent and detailed review of deteriorating inventory literature.All researchers assumed that the deterioration of the items in inventory starts from the instant of their arrival in stock.Infact, most goods would have a span of maintaining quality or original condition, namely, during a period, there is no deterioration. Wu, Ouyang and Yang[6] defined the phenomenon as "non-instantaneous deterioration". In the real world, this type of phenomenon exists for many items such as firsthand vegetable and fruits will have a short span of maintaining fresh quality, in which there is almost

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no spoilage. Afterwards, some of these items start to decay. Inventory management of this type inventory problems for non-instantaneous deteriorating items is considered here.

All the above articles are based on the assumption that the cost involved in inventory system remains constant over the planning horizon.This assumption may not be true in real life as many countries experience high annual inflation rate. So demand of certain products is also changed due to influences of inflation. As inflation increases, the value of money goes down which erodes the future worth of saving and force one for more current spending. Usually, these spending are on peripherals and luxury items that give rise to demand of these items. So effect of inflation and time value of money can not be ignored.

Buzacott[7] discussed EOQ model with inflation subject to different types of pricing policies. Bose et al[8] presented a paper on deteriorating items with linear time dependent demand rate and shortages under inflation. Wee and Law[9] addressed the problem with finite replenishment rate of deteriorating items taking account of time value of money. Chang[10] proposed an inventory model for deteriorating items under inflation and such a situation in which supplier provides the purchaser a permissible delay of payment if the purchase order is a large quantity.

Reproduction in labour time a cost, as a result of worker proficiency in performing a repetitive task due to the gain in previous work experience,is well known as the 'learning effect'. This phenomenon was first disclosed by Wright[11]in the aircraft industry. Replogle[12] presented an EOQ model in which setup cost is reduce overtime due to the 'learning curve effect'. He uses a numerical example to compare his model with the classical EOQ model. Cheng[13]further modifies the model of Replogle[12]. Recently Nanda and Nam[14] developed joint buyer-manufacturer quantity discount lot-sizing models for single buyer and multiple buyer cases under a range of learning rates and different levels of learning retention due to production breaks.

Abad[15] considered the problem of deteriorating items with partial backlogging. Here, when a stock-out situation occurs only a fraction of demand is backordered. This fraction is a strictly decreasing function of the waiting time. Teng et al[16] developed an inventory model for deteriorating items with time-varying demand in which unsatisfied demand is partially backordered.

Skouri and Papachristos[17] presented a production inventory model with production rate,demand rate and deteriorating rate, all considered as function of time. Their model allowed shortages and the partial rate is a hyperbolic function of the time up to the order point. They proposed an algorithm for finding the solution of the problem.However, they could not ensure that the optimal solution is a global optimum. San Jose et al[18] have studied an inventory system over an infinite planning horizon considering partial backlogging.

In this paper an attempt has been made to develop an inventory model over a finite time horizon with shortages which are partially backlogged. Here deterioration rate is constant and demand rate increases exponentially due to inflation and decreases due to unit price.

The unit price increases exponentially due to inflation and decreases due to lead time over a finite planning horizon and the discounted cash flow (DCF) approach is used for the analysis of optimal inventory replenishment policy. Here maximum quantity of lost sale has been prescribed. Optimal solution for the proposed model is derived minimizing the total cost, and a comprehensive sensitivity analysis has also been performed to observe the effects of deterioration and inflation on the optimal inventory replenishment policies. Numerical examples are used to illustrate how the procedure works.

2 Mathematical formulation of proposed inventory model:

The following assumptions and notations are adopted for the finite time horizon inventory model.

2.1 Assumptions and notations

(i) A_0 is a constant and c_0, c_{10}, and c_{20} are the unit purchase cost of an item, the out-of-pocket inventory carrying cost per unit time and shortage cost per unit per unit time respectively, at the beginning of the time horizon.

(ii) Assuming continuous compounding of inflation, the ordering cost, unit cost of the item, out-of-pocket inventory carrying cost and shortage cost at any time t and i-th cycle are respectively
$A(t) = A_0 e^{\alpha t} e^{\frac{-ib}{n}} \left\{1 - \mu\left(\frac{k_1 H}{n}\right)^{\epsilon}\right\}$,
$C(t) = \frac{C_0}{L^{\gamma_1}} e^{\alpha t}$, where γ_1 $(0 \leq \gamma_1 \leq 1)$ is a constant
$C_1(t) = C_{10} e^{\alpha t}$, and
$C_2(t) = C_{20} e^{\alpha t}$ (buzacott(1975))
where L represent the lead time and $0 \leq \alpha \leq 1$ is a constant.

(iii) The demand rate exponentially increases with inflation, decreases with unit price.It is represented by $\lambda(t) = \frac{\lambda_0 e^{\alpha t}}{[c(t)]^{\gamma}}$, where $0 \leq \alpha, \gamma, \gamma_1 \leq 1$ are constants, α is inflation rate.

(iv) The rate of replenishment is infinite.

(v) The constant fraction $\theta(0 \leq \theta \leq 1)$ of on-hand inventory deteriorates per unit time.

(vi) Shortages are allowed, which are partially backlogged. The fraction of backlogged demand is $B(\tau)$, where τ is the amount of time the customers wait before receiving the good i.e., $B(\tau)$ represents the proportion of patient customers.

(vii) There is no repair or replenishment of the deteriorated items during the inventory cycle.

(viii) A Discounted Cash Flow (DCF) approach is used to consider the various costs at various times, r $(r \geq \alpha)$ is the discount rate.

(ix) The item does not deteriorate up to time t_d.

(x) H is the length of the finite planning horizon.

(xi) $q_i(t)$ is the inventory level at time of i-th cycle.

(xii) q_{0i} denote initial inventory level of the i-th cycle.

(xiii) s_i denote the shortage level at time t_i.

(xiv) t_d is the time which no units are deteriorated.

(xv) Lead time L is a fraction of cycle length i.e, $L = k_1 T$, $(0 < k_1 < 1)$.

(xvi) l_r is the learning rate , $(0 < l_r < 1)$.

(xvii) b learning coefficient.$b = -\frac{\log(l_r)}{\log(2)}$.

(xviii) maximum quantity of lost sale is M_{LS}.

2.2 Mathematical formulation of crisp model.

The planning horizon (H) has been divided into n equal cycles, each of length T (i.e.T=$\frac{H}{n}$). Let us consider the i-th cycle, i.e. $t_{i-1} \leq t \leq t_{i-1}+t_d$, where $t_0 = 0$,$t_n = H, t_i - t_{i-1} = T$ and $t_i = iT(i = 1,2,3.,.,.,.,n)$. At the beginning of the i-th cycle, a batch of q_i units enters the inventory system from which s_i units are delivered towards back orders leaving a balance of q_{0i} units as the initial inventory level for the i-th cycle(t_{i-1}, t_i), i.e. $q_i = q_{0i} + s_i$. Thereafter, as time passes, the inventory level gradually decreases due to demand in time $t_{i-1} \leq t \leq t_{i-1} + t_d$, where t_d is the time upto which units are non-deteriorated. And in time $t_{i-1} + t_d \leq t \leq t_i$ the inventory level gradually decreases due to demand and partly due to deterioration and reaches zero at time $t_{i1}(Figure - 1)$. Further, demands during the remaining period of the cycle, i.e, from t_{i1} to t_i, are partially backlogged and are fulfilled by a new procurement. Now $t_{i1} = t_i - kT = (i-k)\frac{H}{n}, (i = 1,2,3,...,n), (0 \leq k \leq 1)$, where kT is the fraction of the cycle T having shortages. Let $q_i(t)$ be the inventory level of the i-th cycle at time t $(t_{i-1} \leq t \leq t_i, i = 1,2,....,n)$. The differential equations describing the instantaneous states of $q_i(t)$ over $(t_{i-1}, t_i), i = 1,2,...,n$, are

$$\frac{dq_i}{dt} = \begin{cases} -\frac{\lambda_0 e^{\alpha t}}{[c(t)]^{\gamma}}, & t_{i-1} \leq t \leq t_{i-1}+t_d, \\ -\frac{\lambda_0 e^{\alpha t}}{[c(t)]^{\gamma}} - \theta q_i(t), & t_{i-1}+t_d \leq t \leq t_{i1} \\ -\frac{\lambda_0 e^{\alpha t}}{[c(t)]^{\gamma}} B(t_i - t), & t_{i1} \leq t \leq \ t_i \end{cases} \tag{2.1}$$

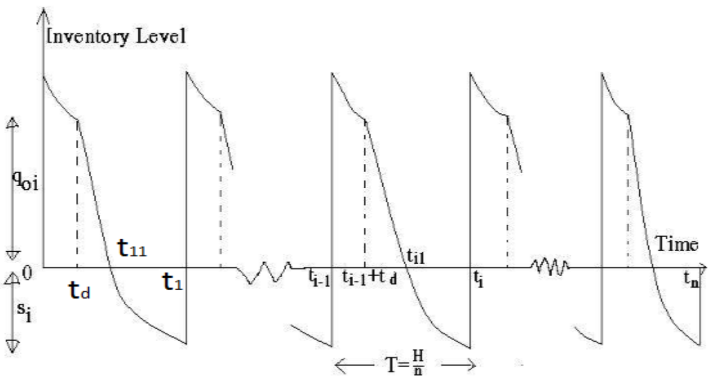

Figure 1: Graphical representation of inventory system

The solution of the above differential equation along with the boundary conditions $q_i(t_{i-1}) = q_{0i}$ and $q_i(t_{i1}) = 0$ are

$$q_i(t) = \begin{cases} q_{0i} - \frac{\lambda_0 c_0^{-\gamma}\left(\frac{k_1 H}{n}\right)^{\gamma\gamma_1}}{\alpha(1-\gamma)}\left\{e^{\alpha(1-\gamma)t} - e^{\alpha(1-\gamma)t_{i-1}}\right\}, & t_{i-1} \leq t \leq t_{i-1}+t_d \\ \frac{\lambda_0 c_0^{-\gamma}\left(\frac{k_1 H}{n}\right)^{\gamma\gamma_1}}{\alpha(1-\gamma)+\theta}\left\{e^{[\alpha(1-\gamma)+\theta]t_{i1}-\theta t} - e^{\alpha(1-\gamma)t}\right\}, & t_{i-1}+t_d \leq t \leq t_{i1} \\ -\frac{\lambda_0 \rho_0 c_0^{-\gamma}\left(\frac{k_1 H}{n}\right)^{\gamma\gamma_1} e^{-at_i}}{\alpha(1-\gamma)+a}\left\{e^{[\alpha(1-\gamma)+a]t} - e^{[\alpha(1-\gamma)+a]t_{i1}}\right\}, & t_{i1} \leq t \leq t_i \end{cases} \tag{2.2}$$

Since $q_i(t_{i1}) = 0$ and $q_i(t_i) = -s_i$ then we have

$$q_{0i} = \frac{\lambda_0(\frac{k_1H}{n})^{\gamma\gamma_1}}{\alpha(1-\gamma)c_0^\gamma}\left\{e^{\alpha(1-\gamma)(t_{i-1}+t_d)} - e^{\alpha(1-\gamma)t_{i-1}}\right\}$$
$$+ \frac{\lambda_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}}{\alpha(1-\gamma)+\theta}\left\{e^{[\alpha(1-\gamma)+\theta]t_{i1}-\theta(t_{i-1}+t_d)} - e^{\alpha(1-\gamma)(t_{i-1}+t_d)}\right\}, i = 1,2,3,.....,n$$

and

$$s_i = \frac{\lambda_0\rho_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1} e^{-at_i}}{\alpha(1-\gamma)+a}\left\{e^{[\alpha(1-\gamma)+a]t_i} - e^{[\alpha(1-\gamma)+a]t_{i1}}\right\}, i = 1,2,3,.....,n$$

Further,batch size $q_i(= q_{0i} + s_i)$ for the i-th cycle is

$$q_i = \frac{\lambda_0\left(\frac{k_1H}{n}\right)^{\gamma}\gamma_1}{\alpha(1-\gamma)c_0^\gamma}\left\{e^{\alpha(1-\gamma)(t_{i-1}+t_d)} - e^{\alpha(1-\gamma)t_{i-1}}\right\}$$
$$+ \frac{\lambda_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}}{\alpha(1-\gamma)+\theta}\left\{e^{[\alpha(1-\gamma)+\theta]t_{i1}-\theta(t_{i-1}+t_d)} - e^{\alpha(1-\gamma)(t_{i-1}+t_d)}\right\}$$
$$+ \frac{\lambda_0\rho_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1} e^{-at_i}}{\alpha(1-\gamma)+a}\left\{e^{[\alpha(1-\gamma)+a]t_i} - e^{[\alpha(1-\gamma)+a]t_{i1}}\right\}, i = 1,2,3,.....,n \quad (2.3)$$

Now at the beginning of each cycle there will be a cash out flow of ordering cost and purchase cost.Further, since the inventory carrying cost is proportional to the value of the inventory, the out-of-pocket (physical shortage) inventory carrying cost per unit time at time t is $q_i(t)c_1(t)$. Similarly, the shortage cost can also be obtained.By using DCF approach, the present worth of the various cost for the i-th cycle are as follows:

(1). Present worth of ordering cost for i-th cycle, A_i is

$$A_i = A(t_{i-1})e^{rt_{i-1}}e^{\frac{-ib}{n}}\left\{1-\mu\left(\frac{k_1H}{n}\right)^{\epsilon}\right\}, i = 1,2,3,.....,n \quad (2.4)$$

(2). Present worth of the purchase cost for i-th cycle, P_i is

$$P_i = q_iC(t_{i-1})e^{-rt_{i-1}} = \frac{q_ic_0}{\left(\frac{k_1H}{n}\right)^{\gamma_1}}e^{(\alpha-r)t_{i-1}}, i = 1,2,...,n \quad (2.5)$$

(3). Present worth of the inventory carrying cost for i-th cycle, H_i is

$$H_i = C_1(t_{i-1})e^{-rt_{i-1}}\int_{t_{i-1}}^{t_{i1}} q_i(t)e^{-rt}dt, i = 1,2,...,n \quad (2.6)$$

(4). Present worth of the shortage cost for i-th cycle, Π_i is

$$\Pi_i = C_2(t_{i-1})e^{-rt_{i-1}}\int_{t_{i1}}^{t_i} q_i(t)e^{-rt}dt, i = 1,2,...,n \quad (2.7)$$

Hence, the present worth of the total variable cost for the i-th cycle, PW_i, is the sum of the ordering cost (A_i), purchase cost (P_i), inventory carrying cost (H_i), and shortage cost(Π_i), i.e.

$$PW_i = A_i + P_i + H_i + \Pi_i, i = 1,2,3....,n. \quad (2.8)$$

The present worth of the total variable cost of the system during the entire time horizon H is given by

$$PW(k,k_1,n)_H = \sum_{i=1}^{n} PW_i = \sum_{i=1}^{n}(A_i + P_i + H_i + \Pi_i). \quad (2.9)$$

Substituting the values of A_i,P_i,H_i and Π_i from Equations.(2.3),(2.4),(2.5) and (2.6) respectively in Equation (2.8) and after simplification, we get

$$\begin{aligned}
PW_H(k,k_1,n) &= A_0 e^{\frac{-b}{n}} \frac{1-e^{(\alpha-r)H-b}}{1-e^{\frac{(\alpha-r)H-b}{n}}}\{1-\mu(\frac{k_1H}{n})^{\epsilon}\} + \lambda_0 c_0^{1-\gamma}\left(\frac{k_1H}{n}\right)^{(\gamma-1)\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-r]H}}{1-e^{[\alpha(2-\gamma)-r]\frac{H}{n}}}\\
&\left\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)} + \frac{e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}} - e^{\alpha(1-\gamma)t_d}}{\alpha(1-\gamma)+\theta} + \frac{\rho_0 e^{\alpha(1-\gamma)\frac{H}{n}}}{\alpha(1-\gamma)+a}\{1-e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}\}\right\}\\
+&\ \lambda_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}\left[c_{10}\left\{\frac{1-e^{-rt_d}}{r}\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)}\right.\right.\\
+&\ \frac{e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}} - e^{\alpha(1-\gamma)t_d}}{\alpha(1-\gamma)+\theta}\} + \frac{1}{\alpha(1-\gamma)}\{\frac{1-e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r} + \frac{1-e^{-rt_d}}{r}\}\\
+&\ \frac{1}{\alpha(1-\gamma)+\theta}\{\frac{e^{-(r+\theta)t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}} - e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}}{r+\theta}\\
-&\ \left.\frac{e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}} - e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r}\}\right\}\\
+&\ \frac{c_{20}\rho_0}{\alpha(1-\gamma)+a}e^{[(1-\gamma)\alpha-r]\frac{H}{n}}\left\{\frac{1}{\alpha(1-\gamma)+a-r}\right.\\
-&\ \left.\left.\frac{\{\alpha(1-\gamma)+a\}e^{-[\alpha(1-\gamma)+a-r]\frac{kH}{n}}}{r\{\alpha(1-\gamma)+a-r\}} + \frac{e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}}{r}\right\}\right]. \qquad (2.10)
\end{aligned}$$

Now the problem is to determine the optimal values of k, k_1 and n which minimize $PW_H(k,k_1,n)$.

The Model:

Now the problem is to determine the optimal values of k, k_1 and n which minimize $PW_H(k,k_1,n)$ subject to the restriction of lost sale. Therefore, the model is

$$\begin{aligned}
&\text{Minimize} && PW_H(k,k_1,n)\\
&\text{subject to} && \sum_{i=1}^{n} LQ_i \le M_{LS}.
\end{aligned} \qquad (2.11)$$

3 Mathematical Analysis

Since the cost function,$PW_H(k,k_1,n)$, is a function of two continuous and one discrete variables-k, k_1 and n respectively, therefore, for any given value of $n=n_0$ (say), the necessary condition for $PW_H(k,k_1,n_0)$ to be minimum is

$$\begin{aligned}
\frac{\partial PW_H(k,k_1,n_0)}{\partial k} &= 0\\
\frac{\partial PW_H(k,k_1,n_0)}{\partial k_1} &= 0
\end{aligned} \qquad (3.1)$$

$$\begin{aligned}
i.e, \quad & \lambda_0 c_0^{1-\gamma}\left(\frac{k_1H}{n}\right)^{(\gamma-1)\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-r]H}}{1-e^{[\alpha(2-\gamma)-r]\frac{H}{n}}}\left\{\frac{\rho_0 H}{n}e^{\alpha(1-\gamma)\frac{H}{n}}e^{[\alpha(1-\gamma)+\theta]\frac{kH}{n}}\right.\\
-&\ \left.\frac{H}{n}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}e^{-\theta t_d}\right\} + \lambda_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}\left[c_{10}\left\{-\frac{H(1-e^{-rt_d})}{nr}\right.\right.\\
&\left. e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}} - \frac{He^{-(r+\theta)t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}}{n(r+\theta)} + \frac{H\{\alpha(1-\gamma)+\theta\}}{n(r+\theta)}e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}\right\}\\
&+\frac{c_{20}\rho_0}{\alpha(1-\gamma)+a}e^{[\alpha(1-\gamma)-r]\frac{H}{n}}\left\{\frac{H\{\alpha(1-\gamma)+a\}}{nr}e^{[\alpha(1-\gamma)+a-r]\frac{kH}{n}}\right.\\
&\left.\left.-\frac{kH}{nr}e^{[\alpha(1-\gamma)+a]\frac{kH}{n}}\right\}\right] = 0
\end{aligned} \qquad (3.2)$$

and

$$\begin{aligned}
&-A_0 e^{\frac{-b}{n}} \frac{1-e^{(\alpha-r)H-b}}{1-e^{\frac{(\alpha-r)H-b}{n}}}\Big\{-\mu\left(\frac{H}{n}\right)^{\epsilon}\epsilon(k_1)^{\epsilon-1}\Big\}+\lambda_0 c_0^{1-\gamma}(\gamma-1)\gamma_1(k_1)^{(\gamma-1)\gamma_1-1}\left(\frac{H}{n}\right)^{(\gamma-1)\gamma_1} \\
&\frac{1-e^{[\alpha(2-\gamma)-r]H}}{1-e^{[\alpha(2-\gamma)-r]\frac{H}{n}}}\Bigg\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)} \\
+\;&\frac{e^{-\theta t_d}e^{\{\alpha(1-\gamma)+\theta\}(1-k)\frac{H}{n}}-e^{\alpha(1-\gamma)t_d}}{\alpha(1-\gamma)+\theta}+\frac{\rho_0 e^{\alpha(1-\gamma)\frac{H}{n}}}{\alpha(1-\gamma)+a}\Big\{1-e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}\Big\}\Bigg\} \\
+\;&c_{10}\gamma\gamma_1(k_1)^{\gamma\gamma_1-1}\lambda_0 c_0^{-\gamma}\left(\frac{H}{n}\right)^{\gamma\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}\Bigg\{\frac{1-e^{-rt_d}}{r}\Big\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)} \\
+\;&\frac{e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}-e^{\alpha(1-\gamma)}}{\alpha(1-\gamma)+\theta}\Big\}+\frac{1}{\alpha(1-\gamma)}\Big\{\frac{1-e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r}+\frac{1-e^{-rt_d}}{r}\Big\} \\
+\;&\frac{1}{\alpha(1-\gamma)-\theta}\Big\{\frac{e^{-(r+\theta)t_d}e^{[\alpha(1-\gamma)-\theta](1-k)\frac{H}{n}}-e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}}{r+\theta} \\
-\;&\frac{e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}-e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r}\Big\}\Bigg\} \\
+\;&\frac{c_{20}\lambda_0\rho_0\gamma\gamma_1(k_1)^{\gamma\gamma_1-1}c_0^{-\gamma}(\frac{H}{n})^{\gamma\gamma_1}}{\alpha(1-\gamma)+a}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}e^{[\alpha(1-\gamma)-r]\frac{H}{n}}\Bigg\{\frac{1}{\alpha(1-\gamma)+a-r} \\
-\;&\frac{\{\alpha(1-\gamma)+a\}e^{-[\alpha(1-\gamma)+a-r]\frac{kH}{n}}}{r\{\alpha(1-\gamma)+a-r\}}+\frac{e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}}{r}\Bigg\}=0 \qquad (3.3)
\end{aligned}$$

Further, $PW_H(k, k_1, n_0)$ is a convex function in k for some fixed k_1 (see Appendix A). Therefore, Equation (2.9) will provide the optimal values of k , k_1 for $n = n_0$.

4 Solution Procedure

In order to obtain the values of k, k_1 and n which minimize $PW_H(k, k_1, n)$,the following procedure is adopted.

Step 1: Solve Equation (2.11) for k and k_1 by substituting $n = n_p$ and $n = n_p + 1$,the corresponding values of k are k_{n_p} and $k_{n_p} + 1$, k_1 are k_{1n_p} and k_{1n_p+1}respectively, $(n_p = 1, 2.....)$.

Step 2: Compute $PW_H(k_{n_p}, k_{1n_p}, n_p)$ and $PW_H(k_{n_p+1}, k_{1n_p+1}, n_p + 1)$.

Step 3: If $PW_H(k_{n_p}, k_{1n_p}, n_p) \leq PW_H(k_{n_p+1}, k_{1n_p+1}, n_p+1)$, then optimal values of $k^* = k_{n_p}, k_1^* = k_{1n_p}$ and $n^* = n_p$. The optimal value of T can be obtained using the relation $T = \frac{H}{n^*}$ while the optimal value of $PW(k, k_1, n)_H$ can be obtained by substituting k^*, k_1^* and n^* in Equation (2.9) and lot sizes (q_i) for $i = 1, 2, ..., n^*$. can be obtained from Equation (2.3) and Stop. Else go to Step 4.

Step 4: Replace n_p by $n_p + 1$ and go to Step 1.

5 Special cases:

Case-I: When $k = 0$ (1.e, when no shortages are allowed)

Equation (2.9) reduces to

$$
\begin{aligned}
& PW_H(k_1,n) = A_0 e^{\frac{-b}{n}} \frac{1-e^{(\alpha-r)H-b}}{1-e^{\frac{(\alpha-r)H-b}{n}}}\{1-\mu\Big(\frac{k1H}{n}\Big)^{\epsilon}\} \\
+ \;& \lambda_0 c_0^{1-\gamma}\Big(\frac{k_1H}{n}\Big)^{(\gamma-1)\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-r]H}}{1-e^{[\alpha(2-\gamma)-r]\frac{H}{n}}}\Big\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)} \\
+ \;& \frac{e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta]\frac{H}{n}}-e^{\alpha(1-\gamma)t_d}}{\alpha(1-\gamma)+\theta}\Big\} \\
+ \;& c_{10}\lambda_0 c_0^{-\gamma}\Big(\frac{k_1H}{n}\Big)^{\gamma\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}\Big\{\frac{1-e^{-rt_d}}{r}\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)} \\
+ \;& \frac{e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta]\frac{H}{n}}-e^{\alpha(1-\gamma)t_d}}{\alpha(1-\gamma)+\theta}\}+\frac{1}{\alpha(1-\gamma)}\{\frac{1-e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r}+\frac{1-e^{-rt_d}}{r}\} \\
+ \;& \frac{1}{\alpha(1-\gamma)+\theta}\{\frac{e^{-(r+\theta)t_d}e^{[\alpha(1-\gamma)+\theta]\frac{H}{n}}-e^{[\alpha(1-\gamma)-r]\frac{H}{n}}}{r+\theta} \\
- \;& \frac{e^{[\alpha(1-\gamma)-r]\frac{H}{n}}-e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r}\}\Big\}
\end{aligned}
\tag{5.1}
$$

Case-II: When $\alpha = 0$ (i.e, without inflation)

Equation (2.9) reduces to

$$
\begin{aligned}
& PW_H(k,k_1,n) = A_0 e^{\frac{-b}{n}}\frac{1-e^{-(rH+b)}}{1-e^{\frac{-(rH+b)}{n}}}\{1-\mu\Big(\frac{k1H}{n}\Big)^{\epsilon}\} \\
& +\lambda_0 c_0^{1-\gamma}\Big(\frac{k_1H}{n}\Big)^{(\gamma-1)\gamma_1}\frac{1-e^{-rH}}{1-e^{-r\frac{H}{n}}}\Big\{t_d+\frac{e^{-\theta t_d}e^{\theta(1-k)\frac{H}{n}}-1}{\theta} \\
+ \;& \frac{\rho_0}{a}\{1-e^{-a\frac{kH}{n}}\}\Big\}+c_{10}\lambda_0 c_0^{-\gamma}\Big(\frac{k_1H}{n}\Big)^{\gamma\gamma_1}\frac{1-e^{-2rH}}{1-e^{-2r\frac{H}{n}}}\Big\{\frac{1-e^{-rt_d}}{r}\{\frac{e^{-\theta t_d}e^{\theta(1-k)\frac{H}{n}}-1}{\theta}\} \\
+ \;& \frac{(rt_d-1)(1-e^{-rt_d})+rt_de^{-rt_d}}{r^2}+\frac{1}{(r+\theta)\theta}e^{-(r+\theta)t_d}e^{\theta(1-k)\frac{H}{n}}-\frac{1}{\theta r}e^{-rt_d} \\
& +\frac{1}{r(r+\theta)}e^{r(1-k)\frac{H}{n}}\Big\} \\
+ \;& \frac{c_{20}\lambda_0\rho_0 c_0^{-\gamma}(\frac{k_1H}{n})^{\gamma\gamma_1}}{a}\frac{1-e^{-2rH}}{1-e^{-2r\frac{H}{n}}}e^{-r\frac{H}{n}}\Big\{\frac{1}{a-r}-\frac{ae^{-(a-r)\frac{kH}{n}}}{r(a-r)}+\frac{e^{\frac{-akH}{n}}}{r}\Big\}
\end{aligned}
\tag{5.2}
$$

Case-III: When $b = 0$ (i.e,model without learning effect)

$$
\begin{aligned}
& PW_H(k,k_1,n) = A_0\frac{1-e^{(\alpha-r)H}}{1-e^{\frac{(\alpha-r)H}{n}}}\{1-\mu(\frac{k_1H}{n})^{\epsilon}\}+\lambda_0 c_0^{1-\gamma}\Big(\frac{k_1H}{n}\Big)^{(\gamma-1)\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-r]H}}{1-e^{[\alpha(2-\gamma)-r]\frac{H}{n}}} \\
& \Big\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)}+\frac{e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}-e^{\alpha(1-\gamma)t_d}}{\alpha(1-\gamma)+\theta} \\
+ \;& \frac{\rho_0 e^{\alpha(1-\gamma)\frac{H}{n}}}{\alpha(1-\gamma)+a}\{1-e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}\}\Big\} \\
+ \;& \lambda_0 c_0^{-\gamma}\Big(\frac{k_1H}{n}\Big)^{\gamma\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}\Big[c_{10}\Big\{\frac{1-e^{-rt_d}}{r}\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)} \\
+ \;& \frac{e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}-e^{\alpha(1-\gamma)t_d}}{\alpha(1-\gamma)+\theta}\}+\frac{1}{\alpha(1-\gamma)}\{\frac{1-e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r}+\frac{1-e^{-rt_d}}{r}\} \\
+ \;& \frac{1}{\alpha(1-\gamma)+\theta}\{\frac{e^{-(r+\theta)t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}-e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}}{r+\theta} \\
- \;& \frac{e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}-e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r}\}\Big\} \\
+ \;& \frac{c_{20}\rho_0}{\alpha(1-\gamma)+a}e^{[(1-\gamma)\alpha-r]\frac{H}{n}}\Big\{\frac{1}{\alpha(1-\gamma)+a-r} \\
- \;& \frac{\{\alpha(1-\gamma)+a\}e^{-[\alpha(1-\gamma)+a-r]\frac{kH}{n}}}{r\{\alpha(1-\gamma)+a-r\}}+\frac{e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}}{r}\Big\}\Big].
\end{aligned}
\tag{5.3}
$$

6 Numerical illustration:

Let $A_0 = 3500$, $\lambda_0 = 1100$ units per year, $r = 0.12$ per years, $C_0 = 15$, $c_{10} = 3$, $c_{20} = 8$, $H = 18$ years, $\alpha = .05$ per years and $\theta = .05$ per years $\gamma = .8$, $\gamma_1 = .9$,$t_d = .25$, $a = .06$, $\rho = .95$, $b = .5$, $\mu = .01$ and $\epsilon = .5$ and $M_{LS} = 1800$.

Using the solution procedure as mentioned above, we get $n^* = 5$, $k^* = .4761$, $k_1^* = .5375$, $T^* = 3.6$ years, $PW_H(k^*, k_1^*, n^*) = 34205.64$ (cf. Table-1) and the lot sizes q_i^* for $i = 1, 2, ..., 5.$ have been shown in the Table 2. Further sensitivity analysis has been performed to study the impact of inflation (α) and deterioration (θ) on optimal number of cycles (n^*), optimal cycles length (T^*), present worth of total variable cost ($PW_H(k^*, k_1^*, n^*)$) and lot size (q_i^*) (Table 2.).The optimum results for the particular models- case-I,II and III are also presented in Tables-1 and 2.

Table-1:Optimum Results for different models

Type of Model	No: of cycle	k	$k1$	Ordering cost	Holding cost	Shortage cost	Purchase cost	PW
Primary Model	3	0.6486	0.1485	5474.76	1087.35	4913.11	23856.94	35332.17
	4	0.5344	0.3168	7098.48	1792.74	3919.45	21810.48	34421.16
	5	**0.4761**	**0.5375**	**8713.95**	2113.13	3036.27	20342.29	34205.64
	6	0.4236	0.7604	10328.88	2290.83	2300.10	19545.35	34465.17
	7	0.3825	1.0241	11929.54	2416.76	1806.74	18878.54	35041.59
Case-I	6	0	0.5903	10347.72	5649.40	0	22209.95	38207.07
Case-II	6	0.4413	0.7092	8241.19	1691.22	1939.86	14185.45	26057.72
Case-III	4	0.5338	t 0.3183	9167.67	1803.11	3723.16	21796.63	36490.56

Table-2:Ordered quantity for different cycle

Type of Models	cycle no(i)	1	2	3	4	5	6	Total
Primary Model	q_i(in units)	723	749	777	805	834	–	$\sum q_i = 3888$
Case-I	q_i(in units)	618	636	656	676	696	718	$\sum q_i = 4000$
Case-II	q_i(in units)	638	638	638	638	638	638	$\sum q_i = 3190$
Case-III	q_i(in units)	719	752	787	823	–	–	$\sum q_i = 3081$

7 Discussion:

The present paper proposes a solution procedure for EOQ model of a non-instantaneous deteriorating item with stock dependent demand rate with inflation.Here, shortages are allowed and backlogged partially is a function of waiting time.The study has been conducted under the Discounted Cash Flow (DCF) approach as it permits a proper recognition of the financial implication of the opportunity cost inventory analysis.

8 Sensitivity Analyses:

Sensitivity of cycle number, total cost and ordered quantities at each cycle are obtained due to the variation in the values of inflation and deterioration and presented in Table-3.Figure -2 and Figure-3 show the changes in total cost PW_H against the changes in k and k_1 respectively. A pictorial representation of PW_H is given in Figure-4.

Table-3: Sensitivity analysis for inflation and deterioration rates

$\alpha\downarrow$	$\theta\rightarrow$	0	0.05	0.10	0.15	0.20	0.25
nearly	n^*	5	6	6	6	6	6
0	PW	25727.46	26057.73	26335.46	26565.69	26754.96	26907.84
	q^*	3884.49	3825.49	3693.09	3513.16	3253.99	3027.92
	n^*	5	5	5	6	5	5
0.02	PW	28408.63	28851.54	29202.33	29473.18	29678.15	29830.56
	q^*	3568.09	3456.78	3263.63	3571.42	t2792.38	2577.82
	n^*	5	5	5	5	5	5
0.04	PW	31726.14	32224.14	32619.31	32929.08	33160.75	33329.80
	q^*	3851.65	3740.64	3476.68	3132.78	2857.56	2644.96
	n^*	5	5	5	5	4	4
0.06	PW	35853.69	36426.38	36902.56	37271.61	37504.25	37633.58
	q^*	4197.69	3953.14	3549.90	3161.60	2310.26	2116.11

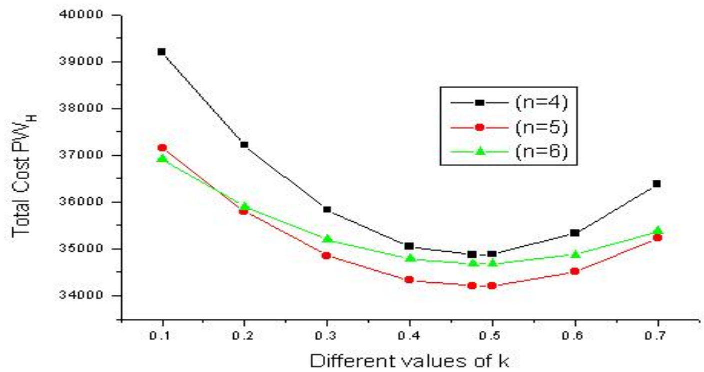

Figure-2 : PW_H curve for fixed value of k_1

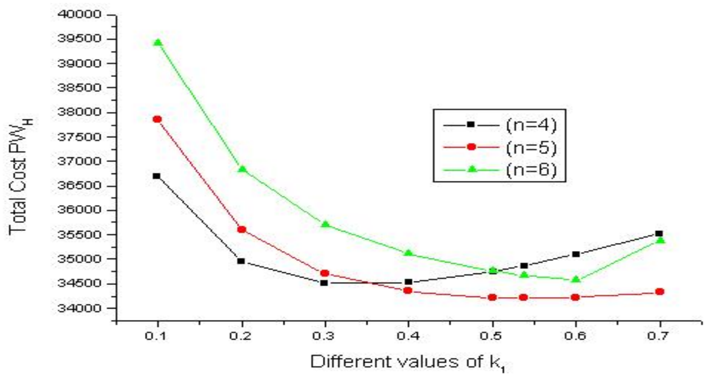

Figure-3 : PW_H curve for fixed value of k

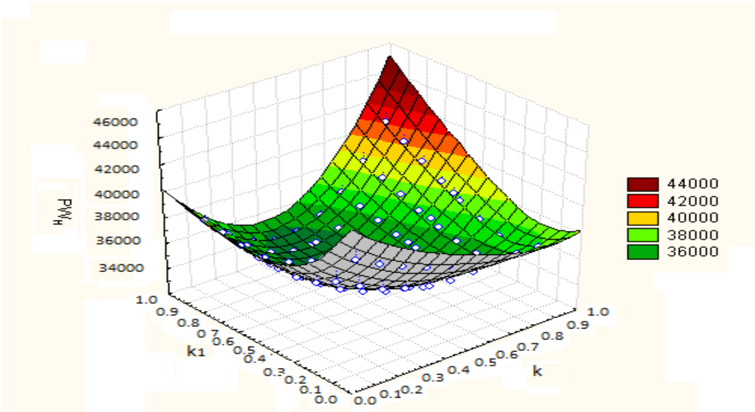

Figure-4 : Pictorial representation of $PW_H(k,k_1)$

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Appendix-A

In this section,convexity of the cost function, $PW_H(k, k_1, n)$,on appropriate domain is shown.

Theorem 1. $PW_H(k, k_1, n_0)$is a convex function of k for any positive value of $n = n_0$ and a fixed value k_1 (say).

Proof. We treat $PW_H(k, k_1, n_0)$ to be defined on $0 \leq k \leq 1$ for any positive value of n_0.

$$
\begin{aligned}
PW_H(k,k_1,n) = {} & A_0 e^{\frac{-b}{n}} \frac{1-e^{(\alpha-r)H-b}}{1-e^{\frac{(\alpha-r)H-b}{n}}} \left\{1-\mu\left(\frac{k_1H}{n}\right)^{\epsilon}\right\} + \lambda_0 c_0^{1-\gamma}\left(\frac{k_1H}{n}\right)^{(\gamma-1)\gamma_1} \\
& \frac{1-e^{[\alpha(2-\gamma)-r]H}}{1-e^{\left\{\alpha(2-\gamma)-r\right\}\frac{H}{n}}} \left\{ \frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)} \right. \\
+ & \left. \frac{e^{-\theta t_d} e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}} - e^{\alpha(1-\gamma)t_d}}{\alpha(1-\gamma)+\theta} + \frac{\rho_0 e^{\alpha(1-\gamma)\frac{H}{n}}}{\alpha(1-\gamma)+a}\left\{1-e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}\right\}\right\} \\
+ & c_{10}\lambda_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1} \frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}} \left\{ \frac{1-e^{-rt_d}}{r}\left\{\frac{e^{\alpha(1-\gamma)t_d}-1}{\alpha(1-\gamma)} \right.\right. \\
+ & \left. \frac{e^{-\theta t_d} e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}} - e^{\alpha(1-\gamma)}}{\alpha(1-\gamma)+\theta}\right\} + \frac{1}{\alpha(1-\gamma)}\left\{\frac{1-e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r} + \frac{1-e^{-rt_d}}{r}\right\} \\
+ & \frac{1}{\alpha(1-\gamma)-\theta}\left\{\frac{e^{-(r+\theta)t_d} e^{[\alpha(1-\gamma)-\theta](1-k)\frac{H}{n}} - e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}}{r+\theta} \right. \\
- & \left.\left. \frac{e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}} - e^{[\alpha(1-\gamma)-r]t_d}}{\alpha(1-\gamma)-r}\right\}\right\} \\
+ & \frac{c_{20}\lambda_0\rho_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}}{\alpha(1-\gamma)+a} \frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}} e^{[\alpha(1-\gamma)-r]\frac{H}{n}} \left\{ \frac{1}{\alpha(1-\gamma)+a-r} \right. \\
- & \left. \frac{\left\{\alpha(1-\gamma)+a\right\} e^{-[\alpha(1-\gamma)+a-r]\frac{kH}{n}}}{r\left\{\alpha(1-\gamma)+a-r\right\}} + \frac{e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}}{r}\right\},
\end{aligned}
\tag{9.1}
$$

$$\frac{d}{dk}PW_H(k,k_1,n_0)=\lambda_0 c_0^{1-\gamma}\left(\frac{k_1H}{n}\right)^{(\gamma-1)\gamma_1}\frac{H}{n}\frac{1-e^{[\alpha(2-\gamma)-r]H}}{1-e^{[\alpha(2-\gamma)-r]\frac{H}{n}}}\Bigg\{\frac{\rho_0 H}{n}e^{\alpha(1-\gamma)}e^{[\alpha(1-\gamma)+\theta]\frac{kH}{n}}$$

$$-\quad \frac{H}{n}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}e^{-\theta t_d}\Bigg\}+c_{10}\lambda_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}$$

$$\Bigg\{\frac{H(1-e^{-rt_d})}{nr}\{-e^{-\theta t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}-\frac{He^{(r+\theta)t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}}{n(r+\theta)}$$

$$+\quad \frac{H\{\alpha(1-\gamma)+\theta\}}{n(r+\theta)}e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}\Bigg\}+\frac{c_{20}\lambda_0\rho_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}}{\alpha(1-\gamma)+a}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}$$

$$e^{[\alpha(1-\gamma)-r]\frac{H}{n}}\Bigg\{\frac{H\{\alpha(1-\gamma)+a\}}{nr}e^{[\alpha(1-\gamma)+a-r]\frac{kH}{n}}-\frac{kH}{nr}e^{[\alpha(1-\gamma)+a]\frac{kH}{n}}\Bigg\}=0$$

$$\frac{d^2}{dk^2}PW_H(k,k_1,n_0)=\lambda_0 c_0^{1-\gamma}\left(\frac{k_1H}{n}\right)^{(\gamma-1)\gamma_1}\frac{H^2}{n^2}\frac{1-e^{[\alpha(2-\gamma)-r]H}}{1-e^{[\alpha(2-\gamma)-r]\frac{H}{n}}}e^{\alpha(1-\gamma)(1-k)\frac{H}{n}}$$

$$\Bigg\{[\alpha(1-\gamma)+\theta]e^{\theta[(1-k)\frac{H}{n}-t_d]}-\rho_0[\alpha(1-k)+a]e^{\frac{-akH}{n}}\Bigg\}$$

$$+\quad c_{10}\lambda_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}\frac{H^2}{n^2}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}$$

$$\Bigg\{\frac{e^{-\theta t_d}-e^{-(r+\theta)t_d}}{r}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}[\alpha(1-\gamma)+\theta]\frac{\alpha(1-\gamma)+\theta}{\alpha+\theta}e^{-(\alpha+\theta)t_d}e^{[\alpha(1-\gamma)+\theta](1-k)\frac{H}{n}}$$

$$+\quad \frac{r-\alpha(1-\gamma)}{r+\theta}e^{[\alpha(1-\gamma)-r](1-k)\frac{H}{n}}\Bigg\}+\frac{c_{20}\lambda_0\rho_0 c_0^{-\gamma}\left(\frac{k_1H}{n}\right)^{\gamma\gamma_1}}{\alpha(1-\gamma)+a}\frac{H^2}{n^2}\frac{1-e^{[\alpha(2-\gamma)-2r]H}}{1-e^{[\alpha(2-\gamma)-2r]\frac{H}{n}}}$$

$$e^{[\alpha(1-\gamma)-r]\frac{H}{n}}e^{-[\alpha(1-\gamma)+a]\frac{kH}{n}}\Bigg\{[\alpha(1-\gamma)+a]-[\alpha(1-\gamma)+a-r]e^{\frac{rkH}{n}}\Bigg\} \tag{9.2}$$

Since $r>(1-\gamma)\alpha$ and $0\le k\le 1\Rightarrow \frac{d^2}{dk^2}PW_H(k,k_1,n_0)>0$ for any integer $n_0>0$. Hence,$PW_H(k,k_1,n_0)$ is a convex function of k for any fixed value of k_1.