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Rough Anti-Fuzzy Ideals in Rings and Their Properties

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Abstract : In this paper, we shall introduce the concept of rough anti-fuzzy ideals in a ring and give some properties of homomorphisms and anti-homomorphisms on a rough anti-fuzzy ideal.

Keywords : rough ring, rough ideal,rough fuzzy ideal, anti-fuzzy ideal, rough anti-fuzzy ideal.

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1 Introduction

The fuzzy set introduced by L A Zadeh in 1965 and the rough set introduced by Z Pawlak in 1982 are generalisations of the classical set theory. Both these set theories are new mathematical tool to deal the uncertain, vague and imprecise data. In Zadeh's fuzzy set theory, the degree of membership of elements of a set plays the key role, whereas in Pawlak's rough set theory, equivalence classes of a set are the building blocks for the upper and lower approximations of the set, in which a subset of universe is approximated by the pair of ordinary sets, called upper and lower approximations. Combining the theory of rough set with abstract algebra is one of the trends in the theory of rough set. Some authors studied the concept of rough algebraic structures. On the other hand, some authors substituted an algebraic structure for the universal set and studied the roughness in algebraic structure. The algebraic approach to rough sets have been given and studied by Iwinski, Bonikowaski, Biswas & Nanda [2], Kuroki etc.And then B. Davvaz studied relationship between rough sets and ring theory and considered ring as a universal set and introduced the notion of rough ideals of a ring in [3]. A further study of this work is done by Osman Kazanci and B. Davaaz in [8]. Extensive researches has also been carried out to compare the theory of rough sets with other theories of uncertainty such as fuzzy sets and conditional events. Dubois and Prade [4] were one of the first who combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets.

This paper concerns a relationship between rough sets, fuzzy sets and ring theory. In section 2, we review some basic definitions. Section 3 deals with some properties of rough anti-fuzzy ideal. In section 4, we give some homomorphic and anti-homomorphic properties of rough anti-fuzzy ideal.

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2 Preliminaries

In this section we give some basic definitions which are used in this article.

Definition 2.1. *Let θ be an equivalence relation on R, then θ is called a congruence relation if $(a, b) \in \theta$ implies$(a + x, b + x)$, (ax, bx), $(xa, xb) \in \theta$ for all $x \in R$.*

A congruence relation θ on R is called complete if $[ab]_\theta = [a]_\theta[b]_\theta$.

As it is well known in the fuzzy set theory established by Zadeh, a fuzzy subset μ of a set R is defined as a map from R to the unit interval $[0, 1]$.

Definition 2.2 ([8])**.** *Let θ be an equivalence relation on R and μ a fuzzy subset of R. Then we define the fuzzy sets $\theta_-(\mu)$, $\theta^-(\mu)$ as follows:*

$$\theta_-(\mu)(x) = \bigwedge_{z\in[x]_\theta} \mu(z) \quad \text{and} \quad \theta^-(\mu)(x) = \bigvee_{z\in[x]_\theta} \mu(z).$$

The fuzzy sets $\theta_-(\mu)$ and $\theta^-(\mu)$ are called , respectively the θ- lower and θ-upper approximations of the fuzzy set μ.
$\theta(\mu) = (\theta_-(\mu)\ ,\ \theta^-(\mu))$ is called a rough fuzzy set with respect to θ if $\theta_-(\mu) \neq \theta^-(\mu)$.

Definition 2.3 ([11])**.** *Let X and Y be two non-empty sets, $f : X \to Y$, μ and σ be fuzzy subsets of X and Y respectively. Then*
$f(\mu)$, the image of μ under f is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x);\ f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$f_-(\mu)$, the anti-image of μ under f is a fuzzy subset of Y defined by

$$f_-(\mu)(y) = \begin{cases} \inf\{\mu(x);\ f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$f^{-1}(\sigma)$, the pre-image of σ under f is a fuzzy subset of X defined by

$$f^{-1}(\sigma)(x) = \sigma(f(x)) \ \forall x \in X.$$

Definition 2.4 ([11])**.** *For a function $f : R_1 \to R_2$, a fuzzy subset μ of a ring R_1 is called f-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, x, $y \in R_1$.*

We say that a fuzzy subset μ of a ring R_1 has the sup property if for any subset T of R_1 , there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t\in T} \mu(t)$.

Definition 2.5 ([10])**.** *A fuzzy subset μ of a ring R is called upper rough f-invariant if $\theta^-(\mu)$ is f-invariant and a lower rough f-invariant if $\theta_-(\mu)$ is f-invariant.*

Let μ be a fuzzy subset of R and $\theta(\mu) = (\theta_-(\mu)\ ,\ \theta^-(\mu))$ a rough fuzzy set. If $\theta_-(\mu)$ and $\theta^-(\mu)$ are f-invariant, then $(\theta_-(\mu)\ ,\ \theta^-(\mu))$ is called rough f-invariant.

3 Rough Anti-Fuzzy Ideal

In this section we define rough anti-fuzzy ideals in a ring R and prove some theorems regarding them.

Definition 3.1 ([1])**.** *A fuzzy subset μ of a ring R is called an anti-fuzzy left (right) ideal of R if*

1. $\mu(x-y) \leq \mu(x) \vee \mu(y)$

2. $\mu(xy) \leq \mu(x) \vee \mu(y)$

3. $\mu(xy) \leq \mu(y)$ ($\mu(xy) \leq \mu(x)$)

for all $x, y \in R$.

Definition 3.2 (). *A fuzzy subset* μ *of a ring* R *is called an upper rough anti-fuzzy left (right) ideal of* R *if* $\theta^-(\mu)$ *is an anti-fuzzy left (right) ideal of* R *and a lower rough anti-fuzzy left (right) ideal of* R *if* $\theta_-(\mu)$ *is an anti-fuzzy left (right) ideal of* R.

Let μ *be a fuzzy subset of* R *and* $\theta(\mu) = (\theta_-(\mu)\ ,\ \theta^-(\mu))$ *a rough fuzzy set. If* $\theta_-(\mu)$ *and* $\theta^-(\mu)$ *are anti-fuzzy left (right) ideals of* R*, then* $(\theta_-(\mu)\ ,\ \theta^-(\mu))$ *is called a rough anti-fuzzy left(right) ideal.*

Example. *Consider the ring* $R = (Z_4, +, .)$ *and subring* $(S, +, .)$*, where* $S = \{0, 2\}$*. Define a congruence on* Z_4 *as* $a \equiv b \bmod S$ *iff* $a - b \in S$*. Define a fuzzy subset* $\mu : R \to [0, 1]$ *as*

$$\mu(x) = \begin{cases} 0.1 & \text{if } x=0 \\ 0.5 & \text{if } x=1,\ 2,\ 3 \end{cases}$$

$x - y$

x:y	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

$\mu(x-y)$

$\mu(x):\mu(y)$	0.1	0.5	0.5	0.5
0.1	0.1	0.5	0.5	0.5
0.5	0.5	0.1	0.5	0.5
0.5	0.5	0.5	0.1	0.5
0.5	0.5	0.5	0.5	0.1

Clearly $\mu(x-y) \leq \mu(x) \vee \mu(y)$

xy

x:y	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mu(xy)$

$\mu(x):\mu(y)$	0.1	0.5	0.5	0.5
0.1	0.1	0.1	0.1	0.1
0.5	0.1	0.5	0.5	0.5
0.5	0.1	0.5	0.1	0.5
0.5	0.1	0.5	0.5	0.5

Clearly $\mu(xy) \leq \mu(x) \vee \mu(y)$ *and* $\mu(xy) \leq \mu(x) \wedge \mu(y)$.
Therefore μ *is an anti-fuzzy ideal.*

$$\theta^-(\mu)(x) = \begin{cases} 0.5 & \text{if } x=\ 0,\ 1,\ 2,\ 3 \end{cases}$$

and

$$\theta_-(\mu)(x) = \begin{cases} 0.1 & \text{if } x=\ 0,\ 2 \\ 0.5 & \text{if } x=\ 1,\ 3 \end{cases}$$

Clearly μ *is a rough set.* $\theta^-(\mu)$ *is an anti-fuzzy ideal, is obvious.*

$\theta_-(\mu)(x-y)$

$\theta_-(\mu)(x):\theta_-(\mu)(y)$	0.1	0.5	0.1	0.5
0.1	0.1	0.5	0.1	0.5
0.5	0.5	0.1	0.5	0.1
0.1	0.1	0.5	0.1	0.5
0.5	0.5	0.1	0.5	0.1

	$\theta_-(\mu)(x):\theta_-(\mu)(y)$	0.1	0.5	0.1	0.5
	0.1	0.1	0.1	0.1	0.1
$\theta_-(\mu)(xy)$	0.5	0.1	0.5	0.1	0.5
	0.1	0.1	0.1	0.1	0.1
	0.5	0.1	0.5	0.1	0.5

Clearly $\theta_-(\mu)(x-y) \leq \theta_-(\mu)(x) \vee \theta_-\mu)(y)$

$\theta_-(\mu)(xy) \leq \theta_-(\mu)(x) \vee \theta_-(\mu)(y)$ *and* $\theta_-(\mu)(xy) \leq \theta_-(\mu)(x) \wedge \theta_-(\mu)(y)$.

Therefore $\theta_-(\mu)$ *is an anti-fuzzy ideal. Therefore,* μ *is a rough anti-fuzzy ideal.*

Theorem 3.3. *Let* θ *be a complete congruence relation on* R*. If* μ *is an anti-fuzzy left (right) ideal of* R*, then* $\theta^-(\mu)$ *is an anti-fuzzy left (right) ideal of* R*.*

Proof. For $x,y \in R$,

$$\begin{aligned}
\theta^-(\mu)(x-y) &= \bigvee_{z\in[x-y]_\theta} \mu(z) \\
&= \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a-b) \\
&\leq \bigvee_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a) \vee \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy left ideal }) \\
&= \bigvee_{a\in[x]_\theta} \mu(a) \vee \bigvee_{b\in[y]_\theta} \mu(b) \\
&= \theta^-(\mu)(x) \vee \theta^-(\mu)(y)
\end{aligned}$$

Hence $\theta^-(\mu)(x-y) \leq \theta^-(\mu)(x) \vee \theta^-(\mu)(y)$.

Also we have,

$$\begin{aligned}
\theta^-(\mu)(xy) &= \bigvee_{z\in[xy]_\theta} \mu(z) \\
&= \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(ab) \\
&\leq \bigvee_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a) \vee \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy left ideal }) \\
&= \bigvee_{a\in[x]_\theta} \mu(a) \vee \bigvee_{b\in[y]_\theta} \mu(b) \\
&= \theta^-(\mu)(x) \vee \theta^-(\mu)(y)
\end{aligned}$$

Hence $\theta^-(\mu)(xy) \leq \theta^-(\mu)(x) \vee \theta^-(\mu)(y)$.

Again we have,

$$\begin{aligned}
\theta^-(\mu)(xy) &= \bigvee_{z\in[xy]_\theta} \mu(z) \\
&= \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(ab) \\
&\leq \bigvee_{b\in[y]_\theta} (\mu(b)) \quad (\because \mu \text{ is an anti-fuzzy left ideal }) \\
&= \theta^-(\mu)(y)
\end{aligned}$$

Hence $\theta^-(\mu)(xy) \leq \theta^-(\mu)(y)$. Therefore, $\theta^-(\mu)$ is an anti-fuzzy left ideal of R. Similarly we can prove the other part also. □

Remark. *The converse of the theorem (3.3) does not hold in general.*

Example. *Consider the ring* $(R = Z_4, +, .)$ *and subring* $(S, +, .)$, *where* $S = \{0, 2\}$. *Define a congruence on* Z_4 *as* $a \equiv b \bmod S$ *iff* $a - b \in S$. *Define a fuzzy subset* $\mu : R \to [0, 1]$ *as*

$$\mu(x) = \begin{cases} 0.5 & \textit{if } x{=}0,\ 3 \\ 0.1 & \textit{if } x = 1,\ 2 \end{cases}$$

$\mu(x-y)$

$\mu(x):\mu(y)$	0.1	0.5	0.5	0.5
0.1	0.1	0.5	0.5	0.5
0.5	0.5	0.1	0.5	0.5
0.5	0.5	0.5	0.1	0.5
0.5	0.5	0.5	0.5	0.1

$\mu(xy)$

$\mu(x):\mu(y)$	0.5	0.1	0.5	0.1
0.5	0.5	0.5	0.5	0.5
0.1	0.5	0.1	0.1	0.5
0.5	0.5	0.1	0.5	0.1
0.1	0.5	0.5	0.1	0.1

Clearly $\mu(xy) \nleq \mu(x) \vee \mu(y)$.
Therefore μ *is not an anti-fuzzy ideal.*

$$\theta^{-}(\mu)(x) = \begin{cases} 0.5 & \textit{if } x{=}\ 0,\ 1,\ 2,\ 3 \end{cases}$$

Obviously, $\theta^{-}(\mu)(x-y) \leq \theta^{-}(\mu)(x) \vee \theta^{-}\mu)(y)$
Also, $\theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \vee \theta^{-}(\mu)(y)$ *and* $\theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \wedge \theta^{-}(\mu)(y)$.
Therefore $\theta^{-}(\mu)$ *is an anti-fuzzy ideal.*

Theorem 3.4. *Let* θ *be a complete congruence relation on* R. *If* μ *is an anti-fuzzy left (right) ideal of* R, *then* $\theta_{-}(\mu)$ *is an anti-fuzzy left (right) ideal of* R.

Proof. For $x,y \in R$,

$$\begin{aligned} \theta_{-}(\mu)(x-y) &= \bigwedge_{z\in[x-y]_\theta} \mu(z) \\ &= \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a-b) \\ &\leq \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a) \vee \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy left ideal }) \\ &= \bigwedge_{a\in[x]_\theta} \mu(a) \vee \bigwedge_{b\in[y]_\theta} \mu(b) \\ &= \theta_{-}(\mu)(x) \vee \theta_{-}(\mu)(y) \end{aligned}$$

Hence $\theta_{-}(\mu)(x-y) \leq \theta_{-}(\mu)(x) \vee \theta_{-}(\mu)(y)$.
Also we have,

$$\begin{aligned} \theta_{-}(\mu)(xy) &= \bigwedge_{z\in[xy]_\theta} \mu(z) \\ &= \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(ab) \\ &\leq \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a) \vee \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy left ideal }) \\ &= \bigwedge_{a\in[x]_\theta} \mu(a) \vee \bigwedge_{b\in[y]_\theta} \mu(b) \\ &= \theta_{-}(\mu)(x) \vee \theta_{-}(\mu)(y) \end{aligned}$$

Hence $\theta_-(\mu)(xy) \leq \theta_-(\mu)(x) \vee \theta_-(\mu)(y)$.

Again we have,

$$\begin{aligned}\theta_-(\mu)(xy) &= \bigwedge_{z\in[xy]_\theta} \mu(z) \\ &= \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(ab) \\ &\leq \bigwedge_{b\in[y]_\theta} (\mu(b)) \quad (\because \mu \text{ is an anti-fuzzy left ideal }) \\ &= \theta_-(\mu)(y)\end{aligned}$$

Hence $\theta_-(\mu)(xy) \leq \theta_-(\mu)(y)$. Therefore, $\theta_-(\mu)$ is an anti-fuzzy left ideal of R. Similarly we can prove the other case also, completing the proof. □

Corollary 3.5. *Let θ be a complete congruence relation on R. If μ is an anti-fuzzy left (right) ideal of R, then μ is a rough anti-fuzzy left (right) ideal of R.*

Proof. This follows from Theorems (3.3) and (3.4). □

Remark. *If θ is a complete congruence relation on R and μ is an anti-fuzzy ideal of R, then μ is a rough anti-fuzzy ideal of R.*

4 Homomorphism and Anti-homomorphism on a Rough Anti-Fuzzy Ideal

In this section we study about the properties of homomorphic/anti-homomorphic image/pre-image of an upper/lower rough anti-fuzzy left/right ideal in a ring.

Definition 4.1. *Let R and R' be any two rings. Then the function $f : R \to R'$ is said to be a homomorphism (anti-homomorphism) if for all $x, y \in R$*

$$f(x+y) = f(x) + f(y) \text{ and } f(xy) = f(x)f(y) \quad (f(xy) = f(y)f(x))$$

Theorem 4.2 ([6]). *Let f be a homomorphism (anti-homomorphism) from ring R_1 onto a ring R_2 and let μ be a fuzzy subset of R_1 . Then*

1. $f(\theta_1^-(\mu)) = \theta_2^-(f(\mu))$
2. $f(\theta_{1-}(\mu)) \subseteq \theta_{2-}(f(\mu))$. *If f is one to one, then* $f(\theta_{1-}(\mu)) = \theta_{2-}(f(\mu))$

Remark 4.3 ([10]). *Let f be a homomorphism (anti-homomorphism) from ring R_1 onto a ring R_2 and let σ be a fuzzy subset of R_2. Then $f^{-1}(\sigma)$ is a fuzzy subset of R_1. Hence by theorem 4.2, we get $f(\theta_1^-(f^{-1}(\sigma)) = \theta_2^-(f(f^{-1}(\sigma)))$. If f is one to one and onto, $\theta_1^-(f^{-1}(\sigma)) = f^{-1}(\theta_2^-(\sigma))$.*

Theorem 4.4. *Let f be a homomorphism (anti-homomorphism) from a ring R_1 onto a ring R_2 and let μ be a fuzzy subset of R_1 . Then*

1. $f_-(\theta_1^-(\mu)) \supseteq \theta_2^-(f_-(\mu))$
2. $f_-(\theta_{1-}(\mu)) = \theta_{2-}(f_-(\mu))$. *If f is one to one* $f_-(\theta_1^-(\mu)) = \theta_2^-(f_-(\mu))$

Proof. For $x \in R_2$

$$\begin{aligned} f_-(\theta_1^-(\mu))(x) &= \bigwedge_{f(a)=x} \theta_1^-(\mu)(a) = \bigwedge_{f(a)=x} \bigvee_{z\in[a]_{\theta_1}} \mu(z) \\ &= \bigwedge_{f(a)=x} \bigvee_{a\in[z]_{\theta_1}} \mu(a) \geq \bigvee_{a\in[z]_{\theta_1}} \bigwedge_{f(a)=x} \mu(a) \\ &= \bigvee_{a\in[z]_{\theta_1}} f_-(\mu)(x) = \bigvee_{f(a)\in[f(z)]_{\theta_2}} f_-(\mu)(x) \\ &= \bigvee_{x\in[f(z)]_{\theta_2}} f_-(\mu)(x) = \bigvee_{f(z)\in[x]_{\theta_2}} f_-(\mu)(f(z)) \\ &= \theta_2^- f_-(\mu)(x) \end{aligned}$$

Therefore, $f_-(\theta_1^-(\mu)) \supseteq \theta_2^-(f_-(\mu))$.

If f is one to one, $f_-(\theta_1^-(\mu)) = \theta_2^-(f_-(\mu))$ is clear.

$$\begin{aligned} f_-(\theta_{1-}(\mu))(x) &= \bigwedge_{f(a)=x} \theta_{1-}(\mu)(a) = \bigwedge_{f(a)=x} \bigwedge_{z\in[a]_{\theta_1}} \mu(z) \\ &= \bigwedge_{f(a)=x} \bigwedge_{a\in[z]_{\theta_1}} \mu(a) \\ &= \bigwedge_{a\in[z]_{\theta_1}} \bigwedge_{f(a)=x} \mu(a) = \bigwedge_{a\in[z]_{\theta_1}} f_-(\mu)(x) \\ &= \bigwedge_{f(a)\in[f(z)]_{\theta_2}} f_-(\mu)(x) = \bigwedge_{x\in[f(z)]_{\theta_2}} f_-(\mu)(x) \\ &= \bigwedge_{f(z)\in[x]_{\theta_2}} f_-(\mu)(f(z)) = \theta_{2-} f_-(\mu)(x) \end{aligned}$$

Therefore, $f_-(\theta_{1-}(\mu)) = \theta_{2-}(f_-(\mu))$. □

Example. *Consider the onto ring homomorphism $f : Z_2 \to \{0\}$. Clearly f is not one-one. Define a fuzzy set $\mu : Z_2 \to [0,1]$ such that $\mu(0) = 0$ and $\mu(1) = 0.1$. Define an equivalence relation θ_1 on Z_2 and θ_2 on $\{0\}$ as $\theta_1 = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\theta_2 = \{(0,0)\}$ respectively*

For $x \in \{0\}$

$$\begin{aligned} f_-({\theta_1}^-(\mu))(x) =& f_-({\theta_1}^-(\mu))(0) = \bigwedge_{f(a)=0} \bigvee_{(a,z)\in\theta_1} \mu(z) = \bigwedge_{a=0,1} \bigvee_{(a,z)\in\theta_1} \mu(z) \\ &= \bigwedge\{ \bigvee_{(0,z)\in\theta_1} \mu(z), \bigvee_{(1,z)\in\theta_1} \mu(z)\} = \bigwedge\{\mu(1),\mu(1)\} = \mu(1) = 0.1 \\ {\theta_2}^- f_-(\mu)(x) =& {\theta_2}^- f_-(\mu)(0) = \bigvee_{(z,0)\in\theta_2} \bigwedge_{f(a)=z} \mu(a) = \bigvee_{(0,0)\in\theta_2} \bigwedge_{f(a)=0} \mu(a) \\ &= \bigwedge_{a=0,1} \mu(a) = \mu(0) = 0 \end{aligned}$$

This shows that f is not one-one and $f_-({\theta_1}^-(\mu))(x) \neq {\theta_2}^-(f_-(\mu))(x)$.

Theorem 4.5 ([7])**.** *Let f be an isomorphism from a ring R_1 onto a ring R_2 and let μ be a rough f-invariant anti-fuzzy subring of R_1. Then $f(\mu)$ is a rough anti-fuzzy subring of R_2.*

Theorem 4.6. *Isomorphic pre-image of a rough anti-fuzzy left (right) ideal is a rough anti-fuzzy left (right) ideal. Moreover isomorphic pre-image of a rough anti-fuzzy ideal is a rough anti-fuzzy ideal.*

Proof. Let f be an isomorphism from a ring R_1 onto a ring R_2 and let σ be a rough anti-fuzzy left ideal of R_2. Then $\theta_2^-(\sigma)$ and $\theta_{2-}(\sigma)$ are anti-fuzzy left ideals of R_2. For $x, y \in R_1$,

$$\begin{aligned} f^{-1}(\theta_2^-(\sigma))(x-y) &= \theta_2^-(\sigma)f(x-y) \\ &= \theta_2^-(\sigma)(f(x)-f(y)) \quad (\because\ f \text{ is a homomorphism}) \\ &\leq \theta_2^-(\sigma)f(x) \vee \theta_2^-(\sigma)f(y) \quad (\because \theta_2^-(\sigma) \text{ is an anti-fuzzy left ideal }) \\ &= f^{-1}(\theta_2^-(\sigma))(x) \vee f^{-1}(\theta_2^-(\sigma))(y) \end{aligned}$$

Therefore, $f^{-1}(\theta_2^-(\sigma))(x-y) \leq f^{-1}(\theta_2^-(\sigma))(x) \vee f^{-1}(\theta_2^-(\sigma))(y)$.
Also

$$\begin{aligned} f^{-1}(\theta_2^-(\sigma))(xy) &= \theta_2^-(\sigma)f(xy) \\ &= \theta_2^-(\sigma)(f(x)f(y)) \quad (\because\ f \text{ is a homomorphism}) \\ &\leq \theta_2^-(\sigma)f(x) \vee \theta_2^-(\sigma)f(y) \quad (\because \theta_2^-(\sigma) \text{ is an anti-fuzzy left ideal }) \\ &= f^{-1}(\theta_2^-(\sigma))(x) \vee f^{-1}(\theta_2^-(\sigma))(y) \end{aligned}$$

Therefore $f^{-1}(\theta_2^-(\sigma))(xy) \leq f^{-1}(\theta_2^-(\sigma))(x) \vee f^{-1}(\theta_2^-(\sigma))(y)$.
Again

$$\begin{aligned} f^{-1}(\theta_2^-(\sigma))(xy) &= \theta_2^-(\sigma)f(xy) \\ &= \theta_2^-(\sigma)(f(x)f(y)) \quad (\because\ f \text{ is a homomorphism}) \\ &\leq \theta_2^-(\sigma)f(y) \quad (\because \theta_2^-(\sigma) \text{ is an anti-fuzzy left ideal }) \\ &= f^{-1}(\theta_2^-(\sigma))(y) \end{aligned}$$

Therefore $f^{-1}(\theta_2^-(\sigma))(xy) \leq f^{-1}(\theta_2^-(\sigma))(y)$.
Thus $f^{-1}(\theta_2^-(\sigma))$ is an anti-fuzzy left ideal of R_1. Similarly we can prove that $f^{-1}(\theta_{2-}(\sigma))$ is an anti-fuzzy left ideal of R_1.By remark (4.3), $\theta_1^-(f^{-1}(\sigma))$ and $\theta_{1-}(f^{-1}(\sigma))$ are anti-fuzzy left ideals of R_1. Therefore, $f^{-1}(\sigma)$ is a rough anti-fuzzy left ideal of R_1. Similarly we can prove the other case also. Hence the theorem is proved. □

Theorem 4.7. *Let f be a homomorphism from a ring R_1 onto a ring R_2 and let μ be an upper rough f-invariant anti-fuzzy left (right) ideal of R_1. Then $f(\mu)$ is an upper rough anti-fuzzy left (right) ideal of R_2. Moreover homomorphic image of an upper rough f-invariant anti-fuzzy ideal is an upper rough anti-fuzzy ideal.*

Proof. Let μ be an upper rough anti-fuzzy left ideal of R_1. Then $\theta_1^-(\mu)$ is an anti-fuzzy left ideal of R_1. For $y_1, y_2 \in R_2$, $\exists\ x_1, x_2 \in R_1$ such that $f(x_1) = y_1$ and $f(x_2)= y_2$. By theorem (4.5), $f(\theta_1^-(\mu))$ is an anti-fuzzy subring of R_2. Now

$$\begin{aligned} f(\theta_1^-(\mu))[y_1y_2] &= \sup_{t\in f^{-1}(y_1y_2)} \theta_1^-(\mu)(t) \\ &= \sup_{x_1\in f^{-1}(y_1),\ x_2\in f^{-1}(y_2)} \theta_1^-(\mu)(x_1x_2) \quad (\because \theta_1^-(\mu) \text{ is } f\text{-invariant}) \\ &\leq \sup_{x_2\in f^{-1}(y_2)} (\theta_1^-(\mu)(x_2)) \quad (\because \theta_1^-(\mu) \text{ is an anti-fuzzy left ideal}) \\ &= f(\theta_1^-(\mu))(y_2) \end{aligned}$$

Therefore, $f(\theta_1^-(\mu))(y_1y_2) \leq f(\theta_1^-(\mu))y_2$. Therefore, $f(\theta_1^-(\mu))$ is an anti-fuzzy left ideal of R_2. By theorem (4.2), $f(\theta_1^-(\mu))= \theta_2^-(f(\mu))$ is an anti-fuzzy left ideal of R_2. Hence $f(\mu)$ is an upper rough anti-fuzzy left ideal of R_2. Similarly we can establish the other case also. This proves the theorem. □

Theorem 4.8. *Let f be an isomorphism from a ring R_1 onto a ring R_2 and let μ be a lower rough f-invariant anti-fuzzy left (right) ideal of R_1. Then $f(\mu)$ is a lower rough anti-fuzzy left (right) ideal of R_2. Moreover isomorphic image of a lower rough f-invariant anti-fuzzy ideal is a lower rough anti-fuzzy ideal.*

Proof. The proof is similar to that of theorem (4.7). □

Corollary 4.9. *Let f be an isomorphism from a ring R_1 onto a ring R_2 and let μ be a rough f-invariant anti-fuzzy left (right) ideal of R_1. Then $f(\mu)$ is a rough anti-fuzzy left (right) ideal of R_2. Moreover isomorphic image of a rough f-invariant anti-fuzzy ideal is a rough anti-fuzzy ideal.*

Proof. This follows from theorems (4.7) and (4.8). □

Theorem 4.10. *Let f be an isomorphism from a ring R_1 onto a ring R_2 and let μ be an upper rough f-invariant anti-fuzzy left (right) ideal of R_1. Then $f_-(\mu)$ is an upper rough anti-fuzzy left (right) ideal of R_2. Moreover isomorphic anti-image of an upper rough f-invariant anti-fuzzy ideal is an upper rough anti-fuzzy ideal.*

Proof. Let μ be an upper rough anti-fuzzy left ideal of R_1. Then $\theta_1^-(\mu)$ is an anti-fuzzy left ideal of R_1. For $y_1, y_2 \in R_2$, $\exists\ x_1, x_2 \in R_1$ such that $f(x_1) = y_1$ and $f(x_2)= y_2$.

$$\begin{aligned}
f_-(\theta_1^-(\mu))(y_1 - y_2) &= \inf_{t\in f^{-1}(y_1-y_2)} \theta_1^-(\mu)(t) \\
&= \inf_{x_1\in f^{-1}(y_1),\ x_2\in f^{-1}(y_2)} \theta_1^-(\mu)(x_1 - x_2) \quad (\because \theta_1^-(\mu) \text{ is } f\text{-invariant}) \\
&\leq \inf_{x_1\in f^{-1}(y_1),\ x_2\in f^{-1}(y_2)} (\theta_1^-(\mu)(x_1) \vee \theta_1^-(\mu)(x_2)) \quad (\because \theta_1^-(\mu) \text{ is an anti-fuzzy ideal}) \\
&= \inf_{x_1\in f^{-1}(y_1)} \theta_1^-(\mu)(x_1) \vee \inf_{x_2\in f^{-1}(y_2)} \theta_1^-(\mu)(x_2) \\
&= f_-(\theta_1^-(\mu))(y_1) \vee f_-(\theta_1^-(\mu))(y_2)
\end{aligned}$$

Therefore, $f_-(\theta_1^-(\mu))(y_1 - y_2) \leq f_-(\theta_1^-(\mu))(y_1)\ \vee\ f_-(\theta_1^-(\mu))(y_2)$.
Similarly, $f_-(\theta_1^-(\mu))(y_1y_2) \leq f_-(\theta_1^-(\mu))(y_1)\ \vee\ f_-(\theta_1^-(\mu))(y_2)$

$$\begin{aligned}
\text{Also} f_-(\theta_1^-(\mu))(y_1y_2) &= \inf_{t\in f^{-1}(y_1y_2)} \theta_1^-(\mu)(t) \\
&= \inf_{x_1\in f^{-1}(y_1),\ x_2\in f^{-1}(y_2)} \theta_1^-(\mu)(x_1x_2) \quad (\because \theta_1^-(\mu) \text{ is } f\text{-invariant}) \\
&\leq \inf_{x_2\in f^{-1}(y_2)} (\theta_1^-(\mu)(x_2)) \quad (\because \theta_1^-(\mu) \text{ is an anti-fuzzy left ideal}) \\
&= f_-(\theta_1^-(\mu))(y_2)
\end{aligned}$$

Hence, $f_-(\theta_1^-(\mu))(y_1y_2) \leq f_-(\theta_1^-(\mu))(y_2)$.
Therefore, $f_-(\theta_1^-(\mu))$ is an anti-fuzzy left ideal of R_2. By theorem (4.4), $f_-(\theta_1^-(\mu))= \theta_2^-(f_-(\mu))$ is an anti-fuzzy left ideal of R_2. Hence $f_-(\mu)$ is an upper rough anti-fuzzy left ideal of R_2. Similarly we can establish the other part also. This completes the proof. □

Theorem 4.11. *Let f be a homomorphism from a ring R_1 onto a ring R_2 and let μ be a lower rough f-invariant anti-fuzzy left (right) ideal of R_1. Then $f_-(\mu)$ is a lower rough anti-fuzzy left (right) ideal* *of R_2. Moreover homomorphic anti-image of a lower rough f-invariant anti-fuzzy ideal is a lower rough anti-fuzzy ideal.*

Theorem 4.12. *Let f be an isomorphism from a ring R_1 onto a ring R_2 and let μ be a rough f-invariant anti-fuzzy left (right) ideal of R_1. Then $f_{-}(\mu)$ is a rough anti-fuzzy left (right) ideal of R_2. Moreover isomorphic anti-image of a rough f-invariant anti-fuzzy ideal is a rough anti-fuzzy ideal.*

The following theorems in anti-homomorphisms can be proved in similar way as the corresponding theorems with homomorphism.

Theorem 4.13. *Anti-homomorphic image of an upper rough f-invariant anti-fuzzy left (right) ideal is an upper rough anti-fuzzy right (left) ideal. Moreover anti-homomorphic image of an upper rough f-invariant anti-fuzzy ideal is an upper rough anti-fuzzy ideal.*

Theorem 4.14. *Anti-isomorphic image of a lower rough f-invariant anti-fuzzy left (right) ideal is a lower rough anti-fuzzy right (left) ideal. Moreover anti-isomorphic image of a lower rough f-invariant anti-fuzzy ideal is a lower rough anti-fuzzy ideal.*

Corollary 4.15. *Anti-isomorphic image of a rough f-invariant anti-fuzzy left (right) ideal is a rough anti-fuzzy right (left) ideal. Moreover anti-isomorphic image of a rough f-invariant anti-fuzzy ideal is a rough anti-fuzzy ideal.*

Theorem 4.16. *Anti-isomorphic pre-image of a rough anti-fuzzy left (right) ideal is a rough anti-fuzzy right (left) ideal. Moreover anti-isomorphic pre-image of a rough anti-fuzzy ideal is a rough anti-fuzzy ideal.*

Theorem 4.17. *Anti-isomorphic anti-image of an upper rough f-invariant anti-fuzzy left (right) ideal is an upper rough anti-fuzzy right (left) ideal. Moreover anti-isomorphic anti-image of an upper rough f-invariant anti-fuzzy ideal is an upper rough anti-fuzzy ideal.*

Theorem 4.18. *Anti-homomorphic anti-image of a lower rough f-invariant anti-fuzzy left (right) ideal is a lower rough anti-fuzzy right (left) ideal. Moreover anti-homomorphic anti-image of a lower rough f-invariant anti-fuzzy ideal is a lower rough anti-fuzzy ideal.*

Corollary 4.19. *Anti-isomorphic anti-image of a rough f-invariant anti-fuzzy left (right) ideal is a rough anti-fuzzy right (left) ideal. Moreover anti-isomorphic anti-image of a rough f-invariant anti-fuzzy ideal is a rough anti-fuzzy ideal.*

5 Conclusion

In this paper, we have shown that the theory of rough sets can be extended to ideals in rings. We discussed the concept of rough anti-fuzzy ideal. Also, we discussed homomorphic and anti-homomorphic properties of rough anti-fuzzy ideals. In a similar fashion the theory of rough sets can be extended to other topics in ring theory.

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