International Journal of Mathematics And its Applications
Volume 3, Issue 1 (2015), 33–38.
ISSN: 2347-1557

Graceful Labeling for Double Step Grid Graph

V.J.Kaneria[†] **and H.M.Makadia**[‡,1]

[†] *Department of Mathematics, Saurashtra University, Rajkot- 360005, India.*

[‡] *Department of Mathematics, Government Engineering College, Rajkot- 360005, India.*

Abstract : We investigate a new graph which is called double step grid graph. We proved that the double step grid graph is graceful. We have investigated some double step grid graph related families of connected graceful graphs. We proved that path union of double step grid graph, cycle of double step grid graph and star of double step grid graph are graceful.

Keywords : Graceful labeling, double step grid graph, path union of graphs, cycle of graphs, star of a graph.

AMS Subject Classification : 05C78.

---

1 Introduction

The graceful labeling was introduced by A. Rosa [1] during 1967. Golomb [2] named such labeling as graceful labeling, which was called earlier as $\beta-$valuation. In this work we introduce a new graph which is called double step grid graph and it is denoted by DSt_n. We begin with a simple, undirected finite graph $G = (V, E)$ with $|V| = p$ vertices and $|E| = q$ edges. For all terminology and notations we follows Harary [3]. Here are some of the definitions which are useful in this paper.

Definition 1.1. *A function f is called* graceful labeling *of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, \dots, q\}$ is injective and the induced function $f^\star : E \longrightarrow \{1, 2, \dots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called* graceful graph *if it admits a graceful labeling.*

Definition 1.2. *Let G be a graph and $G_1, G_2, \dots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} $(1 \leq i \leq n-1)$ is called* path union *of G.*

Definition 1.3 ([4])**.** *For a cycle C_n, each vertex of C_n is replaced by connected graphs $G_1, G_2, \dots, G_n$ and is known as* cycle of graphs. *We shall denote it by $C(G_1, G_2, \dots, G_n)$. If we replace each vertex by a graph G, i.e. $G_1 = G$, $G_2 = G$, $\dots$, $G_n = G$, such cycle of a graph G is denoted by $C(n \cdot G)$.*

Definition 1.4 ([5])**.** *Let G be a graph on n vertices. The graph obtained by replacing each vertex of the star $K_{1,n}$ by a copy of G is called a star of G and is denoted by $G^\star$.*

[1]`Corresponding author E-Mail: makadia.hardik@yahoo.com (H.M.Makadia)`

Definition 1.5 ([6]). *Take $P_n, P_n, P_{n-1}, \dots, P_2$ paths on $n, n, n-1, n-2, \dots, 3, 2$ vertices and arrange them vertically. A graph obtained by joining horizontal vertices of given successive paths is known as a* step grid graph *of size n, where $n \geq 3$. It is denoted by St_n. Obviously $|V(St_n)| = \frac{1}{2}(n^2+3n-2)$ and $|E(St_n)| = n^2+n-2$.*

Definition 1.6. *Take $P_n, P_n, P_{n-2}, P_{n-4}, \dots, P_4, P_2$ paths on $n, n, n-2, n-4, \dots, 4, 2$ vertices and arrange them centrally horizontal. where $n \equiv 0$ (mod 2), $n \neq 2$. A graph obtained by joining vertical vertices of given successive paths is known as a* double step grid graph *of size n. It is denoted by DSt_n. Obviously $|V(DSt_n)| = \frac{n}{4}(n+6)$ and $|E(DSt_n)| = \frac{n^2+3n-2}{2}$.*

In this paper we introduced gracefulness of double step grid graph, path union of double step grid graph, cycle of double step grid graph and star of double step grid graph. For detail survey of graph labeling we refer Gallian [7].

2 Main Results

Theorem 2.1. *A double step grid graph DSt_n is a graceful graph, where $n \equiv 0$ (mod 2), $n \neq 2$.*

Proof. Let $G = DSt_n$ be any double step grid graph of size n, where $n \equiv 0$ (mod 2), $n \neq 2$. We mention each vertices of first row like $u_{1,j}$ $(1 \leq j \leq n)$ and 2^{nd} row like $u_{2,j}$ $(1 \leq j \leq n)$ and 3^{rd} row like $u_{3,j}$ $(1 \leq j \leq n-2)$ and 4^{th} row like $u_{4,j}$ $(1 \leq j \leq n-4)$ similarly the last row like $u_{\frac{n}{2}+1,j}$ $(1 \leq j \leq 2)$. We see that number of vertices in G is $|V(G)| = p = \frac{n}{4}(n+6)$ and the number of edges in G is $|E(G)| = q = \frac{n^2+3n-2}{2}$. We define labeling function $f : V(G) \longrightarrow \{0, 1, \dots, q\}$ as follows

$$
\begin{aligned}
&f(u_{1,1}) = q; && \\
&f(u_{i,1}) = i^2 - 2i, && \forall\, i = 2, 3, \dots, \frac{n}{2}+1 \\
&f(u_{i,2}) = q - i(i-1), && \forall\, i = 2, 3, \dots, \frac{n}{2}+1 \\
&f(u_{i,j}) = f(u_{i+1,j-1}) - (-1)^{i+j}, && \forall\, i = 1, 2, \qquad \forall\, j = i+1, i+2, \dots, n \\
&f(u_{i,j}) = f(u_{i-1,j+2}) + (-1)^{j}, && \forall\, i = 3, 4, \dots, \frac{n}{2}, \qquad \forall\, j = 3, 4, \dots, n-2(i-2).
\end{aligned}
$$

Above labeling patten give rise a graceful labeling to the graph G. So G is a graceful graph. ☐

Illustration 2.2. *DSt_8 and its graceful labeling shown in figure 2.2.*

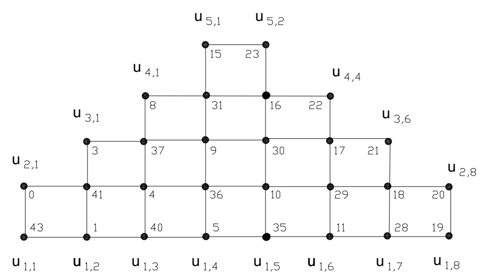

Figure 2.2: DSt_8, double step grid graph with $n = 8$ and its graceful labeling.

Theorem 2.3. *Path union of finite copies of the double step grid graph DSt_n is a graceful graph, where $n \equiv 0$ (mod 2), $n \neq 2$.*

Proof. Let G be a path union of r copies for the double step grid graph DSt_n, where $n \equiv 0$ (mod 2), $n \neq 2$. Let f be the graceful labeling of DSt_n as we mentioned in Theorem 2.1. In graph G, we see that the vertices $|V(G)| = P = \frac{rn}{4}(n+6)$ and the edges $|E(G)| = Q = \frac{rn(n+3)}{2} - 1$. Let $u_{k,i,j}$ $(i = 1, 2, \ldots, \frac{n}{2}+1$, $j = 1, 2, \ldots, \min\{n, n+4-2i\})$ be vertices of k^{th} copy of DSt_n, $\forall\ k = 1, 2, \ldots, r$. Where the vertices of k^{th} copy of DSt_n is $p = \frac{n}{4}(n+6)$ and edges of k^{th} copy of DSt_n is $q = \frac{n^2+3n-2}{2}$. Join the vertices $u_{k,1,n}$ to $u_{k+1,1,1}$ for $k = 1, 2, \ldots, r-1$ by an edge to from the path union of r copies of double step grid graph. We define labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, Q\}$ as follows

$$
\begin{aligned}
g(u_{1,i,j}) &= f(u_{i,j}) && \text{if } f(u_{i,j}) < \frac{q}{2};\\
&= f(u_{i,j}) + (Q-q) && \text{if } f(u_{i,j}) > \frac{q}{2},\\
\forall\ i &= 1, 2, \ldots, \frac{n}{2}+1,\ j = 1, 2, \ldots, min\{n, n+4-2i\};\\
g(u_{k,i,j}) &= g(u_{k-1,i,j}) + \lceil \frac{q}{2} - 1 \rceil && \text{if } g(u_{k-1,i,j}) < \frac{Q}{2},\\
&= g(u_{k-1,i,j}) - \lfloor \frac{q}{2} + 2 \rfloor && \text{if } g(u_{k-1,i,j}) > \frac{Q}{2},\\
\forall\ i &= 1, 2, \ldots, \frac{n}{2}+1,\ j = 1, 2, \ldots, min\{n, n+4-2i\},\ k = 2, 3, \ldots, r
\end{aligned}
$$

Above labeling patten give rise a graceful labeling to given graph G. So path union of finite copies of the double step grid graph is graceful graph. □

Illustration 2.4. *Path union of 3 copies of DSt_4 and its graceful labeling shown in figure 2.4*

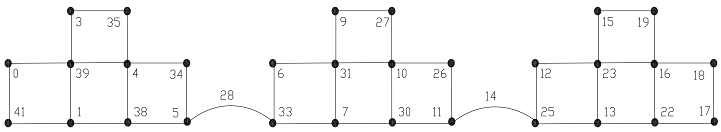

Figure-2.4: A Path union of 3 copies of DSt_4 and its graceful labeling.

Theorem 2.5. *Cycle of r copies of double step grid graph $C(r \cdot DSt_n)$ is a graceful graph, where $n \equiv 0$ (mod 2), $n \neq 2$ and $r \equiv 0, 3$ (mod 4) is graceful.*

Proof. Let $G = C(r \cdot DSt_n)$ be a cycle of double step grid graph DSt_n. Let f be the graceful labeling for DSt_n as we mentioned in Theorem 2.1. In graph G, we see that the vertices $|V(G)| = P = \frac{rn}{4}(n+6)$ and the edges $|E(G)| = Q = \frac{rn(n+3)}{2}$. Let $u_{k,i,j}$ $(i = 1, 2, \ldots, \frac{n}{2}+1$, $j = 1, 2, \ldots, \min\{n, n+4-2i\})$ be the vertices of k^{th} copy of DSt_n, $\forall\ k = 1, 2, \ldots, r$. Where the vertices of k^{th} copy of DSt_n is $p = \frac{n}{4}(n+6)$ and edges of k^{th} copy of DSt_n is $q = \frac{n^2+3n-2}{2}$. Join the vertices $u_{k,1,n}$ with $u_{k+1,1,n}$ for $k = 1, 2, \ldots, r-1$ and $u_{r,1,n}$ with $u_{1,1,n}$ by an edge to from $C(r \cdot DSt_n)$. We define labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, Q\}$ as follows

$$
\begin{aligned}
g(u_{1,i,j}) &= f(u_{i,j}) && \text{if } f(u_{i,j}) < \frac{q}{2},\\
&= f(u_{i,j}) + (Q-q) && \text{if } f(u_{i,j}) > \frac{q}{2},\\
\forall\ i &= 1, 2, \ldots, \frac{n}{2}+1, \forall\ j = 1, 2, \ldots, \min\{n, n+4-2i\};
\end{aligned}
$$

$$
\begin{aligned}
g(u_{2,i,j}) &= g(u_{1,i,j}) + (Q-q) && \text{if } g(u_{1,i,j}) < \frac{Q}{2},\\
&= g(u_{1,i,j}) - (Q-q) && \text{if } g(u_{1,i,j}) > \frac{Q}{2},\\
&\forall\, i = 1,2,\dots,\frac{n}{2}+1, \forall\, j = 1,2,\dots,\min\{n, n+4-2i\};\\
g(u_{k,i,j}) &= g(u_{k-2,i,j}) - (q+1) && \text{if } g(u_{k-2,i,j}) > \frac{Q}{2},\\
&= g(u_{k-2,i,j}) + (q+1) && \text{if } g(u_{k-2,i,j}) < \frac{Q}{2},\\
&\forall\, i = 1,2,\dots,\frac{n}{2}+1,\ \forall\, j = 1,2,\dots,\min\{n, n+4-2i)\}, \quad \forall\, k = 3,4,\dots,\lceil\frac{r}{2}\rceil;\\
g(u_{\lceil\frac{r}{2}\rceil+1,i,j}) &= g(u_{\lceil\frac{r}{2}\rceil-1,i,j}) + (q+2) && \text{if } g(u_{\lceil\frac{r}{2}\rceil-1,i,j}) < \frac{Q}{2},\\
&= g(u_{\lceil\frac{r}{2}\rceil-1,i,j}) - (q+1) && \text{if } g(u_{\lceil\frac{r}{2}\rceil-1,i,j}) > \frac{Q}{2},\\
&\forall\, i = 1,2,\dots,\frac{n}{2}+1, \forall\, j = 1,2,\dots,\min\{n, n+4-2i\};\\
g(u_{\lceil\frac{r}{2}\rceil+2,i,j}) &= g(u_{\lceil\frac{r}{2}\rceil,i,j}) + (q+2) && \text{if } g(u_{\lceil\frac{r}{2}\rceil,i,j}) < \frac{Q}{2},\\
&= g(u_{\lceil\frac{r}{2}\rceil,i,j}) - (q+1) && \text{if } g(u_{\lceil\frac{r}{2}\rceil,i,j}) > \frac{Q}{2},\\
&\forall\, i = 1,2,\dots,\frac{n}{2}+1, \forall\, j = 1,2,\dots,\min\{n, n+4-2i\};\\
g(u_{k,i,j}) &= g(u_{k-2,i,j}) - (q+1) && \text{if } g(u_{k-2,i,j}) > \frac{Q}{2},\\
&= g(u_{k-2,i,j}) + (q+1) && \text{if } g(u_{k-2,i,j}) < \frac{Q}{2},\\
&\forall\, i = 1,2,\dots,\frac{n}{2}+1, \forall\, j = 1,2,\dots,\min\{n, n+4-2i\}, \quad \forall\, k = \lceil\frac{r}{2}\rceil+3, \lceil\frac{r}{2}\rceil+4,\dots,r.
\end{aligned}
$$

Above labeling patten give rise a graceful labeling to cycle of r copies for double step grid graph. □

Illustration 2.6. *$C(4 \cdot DSt_6)$ and its graceful labeling shown in figure 2.6.*

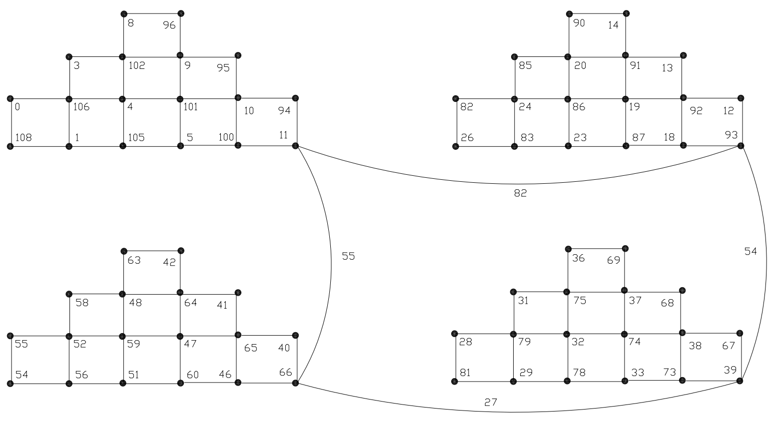

Figure 2.6: A cycle of four copies for DSt_6 and its graceful labeling.

Theorem 2.7. *Star of double step grid graph $(DSt_n)^\star$ is graceful,where $n \equiv 0$ (mod 2), $n \neq 2$.*

Proof. Let $G = (DSt_n)^\star$ be a star of double step grid graph DSt_n, $n \equiv 0$ (mod 2), $n \neq 2$. let f be the graceful labeling for DSt_n as we mention in Theorem 2.1. In graph G, we see that the vertices $|V(G)| = P = p(p+1)$ and the edges $|E(G)| = Q = (p+1)q + p$, where $p = \frac{n}{4}(n+6)$ and $q = \frac{n^2+3n-2}{2}$. Let $u_{k,i,j}$ ($i = 1,2,\dots,\frac{n}{2}+1$, $j = 1,2,\dots$,min$\{n, n+4-2i\}$) be the vertices of k^{th} copy of DSt_n, $\forall$ $k = 1,2,\dots,p$. Where the vertices of k^{th} copy of DSt_n is $p = \frac{n}{4}(n+6)$ and edges of k^{th} copy of DSt_n is $q = \frac{n^2+3n-2}{2}$. We mention that central copy of $(DSt_n)^\star$ is $(DSt_n)^{(0)}$ and other copies of $(DSt_n)^\star$ is

$(DSt_n)^{(k)}$, $\forall\ k = 1, 2, \ldots, p$. We define labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, Q\}$ as follows

$$
\begin{aligned}
g(u_{0,i,j}) &= f(u_{i,j}) && \text{if } f(u_{i,j}) < \frac{q}{2},\\
&= f(u_{i,j}) + (Q - q) && \text{if } f(u_{i,j}) > \frac{q}{2},\\
&\forall\ i = 1, 2, \ldots, \frac{n}{2} + 1, \forall\ j = 1, 2, \ldots, \min\{n, n + 4 - 2i\};\\
g(u_{1,i,j}) &= g(u_{0,i,j}) + p(q + 1) && \text{if } g(u_{0,i,j}) < \frac{Q}{2},\\
&= g(u_{0,i,j}) - p(q + 1) && \text{if } g(u_{0,i,j}) > \frac{Q}{2},\\
&\forall\ i = 1, 2, \ldots, \frac{n}{2} + 1,\ \forall\ j = 1, 2, \ldots, \min\{n, n + 4 - 2i\};\\
g(u_{k,i,j}) &= g(u_{k-2,i,j}) + (q + 1) && \text{if } g(u_{k-2,i,j}) < \frac{Q}{2},\\
&= g(u_{k-2,i,j}) - (q + 1) && \text{if } g(u_{k-2,i,j}) > \frac{Q}{2},\\
&\forall\ i = 1, 2, \ldots, \frac{n}{2} + 1,\ \forall\ j = 1, 2, \ldots, \min\{n, n + 4 - 2i\},\quad \forall\ k = 2, 3, \ldots, p.
\end{aligned}
$$

We see that difference of vertices for the central copy $(DSt_n)^{(0)}$ of G and its other copies $(DSt_n)^{(k)}$ $(1 \le k \le p)$ is precisely following sequence

$$
\begin{gathered}
p(q + 1)\\
(q + 1)\\
(p - 1)(q + 1)\\
\vdots\\
\lfloor \tfrac{p}{2} \rfloor (q + 1).
\end{gathered}
$$

Using this sequence we can produce required edge label by joining corresponding vertices of $(DSt_n)^{(0)}$ with its other copy $(DSt_n)^{(k)}$ $(1 \le k \le p)$ in G. Thus G admits graceful labeling. □

Illustration 2.8. *Star graph of DSt_4 and its graceful labeling shown in figure 2.8.*

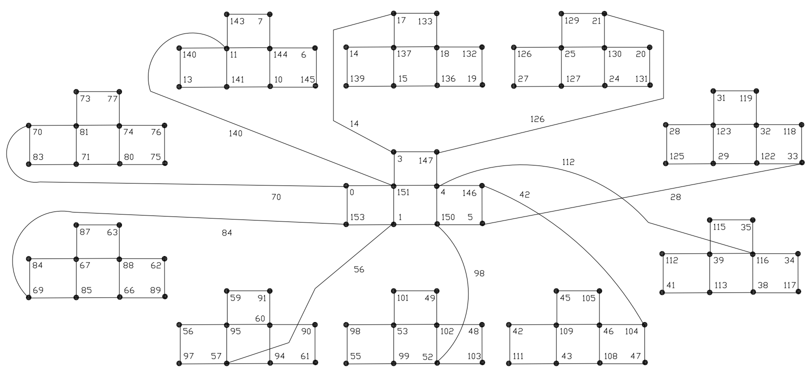

Figure 2.8: A star graph of DSt_4 and its graceful labeling.

3 Concluding Remarks

Here we introduced a new graph is called double step grid graph. Present work contributes some new results. We discussed gracefulness of double step grid graphs, path union of double step grid graph, cycle of Double step graph and star of double step grid graph. The labeling patten is demonstrated by means of illustrations which provide better understanding to derived results.

References

[1] A. Rosa, On certain valuation of graph, Theory of Graphs (Rome, July 1966), Goden and Breach, N. Y. and Paris, (1967), 349–355.

[2] S. W. Golomb, How to number a graph. In: Graph Theory and Computing (R. C. Read. Ed.) Academic Press. New York, (1972), 23–37.

[3] F. Harary, *Graph theory*, Addition Wesley, Massachusetts, (1972).

[4] V. J. Kaneria, H. M. Makadia and M. M. Jariya, *Graceful labeling for cycle of graphs*, Int. J. of Math. Res., 6(2)(2014), 135–139.

[5] S. K. Vaidya, S. Srivastav, V. J. Kaneria and G. V. Ghodasara, *Cordial and* $3-$*equitable labeling of star of a cycle*, Mathematics Today 24(2008), 54–64.

[6] V. J. Kaneria and H. M. Makadia, *Graceful labeling for Step Grid Graphs*, J. of Adv. in Math., 9(5)(2014), 2647–2654.

[7] J.A.Gallian, *A Dynamic Survey of Graph Labeling* ,The Electronics Journal of Combinatorics, 17(2014), #DS6.