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Graceful Labeling for One Point Union for Path of Graphs

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Abstract : This paper contains some results on graceful labeling. We have proved that P_n^t $(tn \cdot P_r \times P_s)$, P_n^t $(tn \cdot K_{1,m})$, P_n^t $(tn \cdot C_m)$ and $S(t \cdot C_n)$ are graceful graphs.

Keywords : Graceful labeling, grid graph, star graph, cycle, one point union for path of graphs and open star of graphs.

AMS Subject Classification: 05C78.

1 Introduction

The graceful labeling was introduced by A. Rosa [6] during 1967. He proved that the cycle C_n is graceful, when $n \equiv 0, 3 \pmod 4$. Acharya and Gill [1] have investigated graceful labeling for the grid graph $(P_n \times P_m)$. Kaneria and Makadia [4] discussed gracefulness of $(P_n \times P_m) \cup (P_r \times P_s)$, $(P_n \times P_m) \cup (P_r \times P_s) \cup C_{2f+3}$, tensor product $P_2(T_p)P_n$ and star of cycle $C_n^\star$ $(n \equiv 0 \pmod 4)$. Kaneria et al. [5] proved that join sum of $K_{m,n}$, path union of $K_{m,n}$ and star of $K_{m,n}$ are graceful graphs. Gallian [2] provide a dynamic survey of graph labeling. It contains bibliographic references on the concept of graph labeling. Graph labeling has various applications within mathematics, computer science and communication network, such applications have been studied by Yegnanarayanan and Vaidhyanathan [8]. We begin with a simple, undirected, finite graph $G = (V, E)$ on p vertices and q edges. For all terminology and notations we follows Harary [3]. We give a brief summary of definitions which are useful in this paper.

Definition 1.1. *A function f is called* graceful labeling *of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, \dots, q\}$ is injective and the induce function $f^\star : E \longrightarrow \{1, 2, \dots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called* graceful graph *if it admits a graceful labeling.*

Definition 1.2. *Let G be a graph and $G_1, G_2, \dots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} $(i = 1, 2, \dots n-1)$ is called* path union of graph G.

Definition 1.3. *A graph G is obtained by replacing each edge of $K_{1,t}$ by a path P_n of length n on $n+1$ vertices is called* one point union for t copies of path P_n. *We shall denote such graph G by P_n^t.*

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Definition 1.4. *A graph G is obtained by replacing each vertices of P_n^t except the central vertex by the graphs G_1, G_2, $\dots$, G_{tn} is known as* one point union for path of graphs. *We shall denote such graph G by $P_n^t(G_1, G_2, \dots, G_{tn})$, where P_n^t is the one point union of t copies of path P_n. If we replace each vertices of P_n^t except the central vertex by a graph H, i.e. $G_1 = G_2 = \cdots = G_{tn} = H$, such one point union of path graph, we shall denote it by $P_n^t\ (tn \cdot H)$.*

Definition 1.5. *A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the graphs $G_1, G_2, \dots, G_n$ is known as open star of graphs. We shall denote such graph by $S(G_1, G_2, \dots, G_n)$. If we replace each vertices of $K_{1,n}$ except the apex vertex by a graph G, i.e. $G_1 = G_2 = \cdots G_n = G$, such open star of a graph, we shall denote by $S(n \cdot G)$.*

2 Main Results

Theorem 2.1. *$P_n^t\ (tn \cdot P_r \times P_s)$ is graceful.*

Proof. Let G be a graph obtained by replacing each vertices of P_n^t except the central vertex by the grid graph $P_r \times P_s$. i.e. G is the graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex by the path union of n copies of the grid graph $P_r \times P_s$. Let u_0 be the central vertex of the graph G. Let $u_{i,j,k}$ $(1 \leq k \leq rs)$ be the vertices of $P_r \times P_s$, which is j^{th} copy of the path union of n copies of $P_r \times P_s$ lies in i^{th} branch of the graph G, $\forall\ j = 1, 2, \dots, n$ and $\forall\ i = 1, 2, \dots, t$.

First we shall join $u_{i,j,rs}$ with the vertex $u_{i,j,1}$ by an edge to form the path union of n copies of $P_r \times P_s$ for i^{th} branch of G, $\forall\ j = 1, 2, \dots, n$ and $\forall\ i = 1, 2, \dots, t$. Now we shall join $u_{i,1,1}$ with the vertex u_0 by an edge to form the one point union of path graphs G, $\forall\ i = 1, 2, \dots, t$.

We know that the grid graph $P_r \times P_s$ is a graceful graph on rs vertices and $2rs - (r+s)$ edges. Let v_{ab} $(1 \leq a \leq r, 1 \leq b \leq s)$ be vertices of the grid graph $P_r \times P_s$.

Let $f : V(P_r \times P_s) \longrightarrow \{0, 1, \dots, 2rs - (r+s)\}$ be the graceful labeling with two sequences of labels, among one is increasing another one is decreasing, which start by $f(v_{1,1}) = 2rs - (r+s)$ and end with $f(v_{r,s}) = \lfloor \frac{2rs-(r+s)}{2} \rfloor$. If we take $V(P_r \times P_s) = \{u_1 = v_{1,1}, u_2 = v_{1,2}, \dots, u_r = v_{1,r}, u_{r+1} = v_{2,1}, \dots, u_{2r} = v_{2,r}, \dots, u_{rs} = v_{r,s}\}$, then we have $f(u_1) = 2rs - (r+s)$ and $f(u_{rs}) = \lfloor \frac{2rs-(r+s)}{2} \rfloor$.

We shall define labeling function g for the first copy(branch) of the path union of n copies of $P_r \times P_s$ as follows. Let $z = 2rs - (r+s)$. To define labeling function $g : V(\textit{path union of } n \textit{ copies of } P_r \times P_s) \longrightarrow \{0, 1, \dots, q\}$, where $q = n(z+1) - 1$. We shall consider following two cases.

Case−I : $z = 2rs - (r+s)$ is odd.

$$
\begin{aligned}
g(u_{1,1,k}) &= f(u_k), && \text{when } f(u_k) < \tfrac{z}{2}, \forall\ k = 1, 2, \dots, rs, \\
&= f(u_k) + (q - z), && \text{when } f(u_k) > \tfrac{z}{2}, \forall\ k = 1, 2, \dots, rs; \\
g(u_{1,j,k}) &= g(u_{1,j-1,k}) - (\tfrac{z+1}{2}), && \text{when } g(u_{1,j-1,k}) > \tfrac{q}{2},\ \forall\ k = 1, 2, \dots, rs, \forall\ j = 2, 3, \dots, n \\
&= g(u_{1,j-1,k}) + (\tfrac{z+1}{2}), && \text{when } g(u_{1,j-1,k}) < \tfrac{q}{2},\ \forall\ k = 1, 2, \dots, rs, \forall\ j = 2, 3, \dots, n;
\end{aligned}
$$

Case−II : $z = 2rs - (r+s)$ is even.

$$
\begin{aligned}
g(u_{1,1,k}) &= f(u_k), && \text{when } f(u_k) < \tfrac{z}{2}, \forall\ k = 1, 2, \dots, rs, \\
&= f(u_k) + (q - z), && \text{when } f(u_k) \geq \tfrac{z}{2}, \forall\ k = 1, 2, \dots, rs; \\
g(u_{1,2,k}) &= g(u_{1,1,rs+1-k}) + (q - z), && \text{when } g(u_{1,1,rs+1-k}) < \tfrac{q}{2}, \forall\ k = 1, 2, \dots, rs, \\
&= g(u_{1,1,rs+1-k}) - (q - z), && \text{when } g(u_{1,1,rs+1-k}) > \tfrac{q}{2}, \forall\ k = 1, 2, \dots, rs; \\
g(u_{1,j,k}) &= g(u_{1,j-2,k}) + (z+1), && \text{when } g(u_{1,j-2,k}) < \tfrac{q}{2},\ \forall\ k = 1, 2, \dots, rs, \forall\ j = 3, 4, \dots, n \\
&= g(u_{1,j-2,k}) - (z+1), && \text{when } g(u_{1,j-2,k}) > \tfrac{q}{2},\ \forall\ k = 1, 2, \dots, rs, \forall\ j = 3, 4, \dots, n;
\end{aligned}
$$

Above labeling pattern give rise graceful labeling to the path union of n copies of $P_r \times P_s$, which lies in first branch of the graph G. Now we shall define labeling function $h : V(G) \longrightarrow \{0, 1, \ldots, Q\}$, where $Q = t(q+1) = tn(z+1) = tn(2rs - (r+s) + 1)$ as follows:

$h(u_0) = 0;$

$h(u_{1,j,k}) = g(u_{1,j,k}) + 1,$ when $g(u_{1,j,k}) < \frac{q}{2}$, $\forall\, k = 1, 2, \ldots, rs, \forall\, j = 1, 2, \ldots, n$

$\quad = g(u_{1,j,k}) + (Q - q),$ when $g(u_{1,j,k}) \geq \frac{q}{2}$, $\forall\, k = 1, 2, \ldots, rs, \forall\, j = 1, 2, \ldots, n;$

$h(u_{2,j,k}) = h(u_{1,j,k}) + (Q - (q+1)),$ when $h(u_{1,j,k}) < \frac{Q}{2}$, $\forall\, k = 1, 2, \ldots, rs, \forall\, j = 1, 2, \ldots, n$

$\quad = h(u_{1,j,k}) - (Q - (q-1)),$ when $h(u_{1,j,k}) > \frac{Q}{2}$, $\forall\, k = 1, 2, \ldots, rs, \forall\, j = 1, 2, \ldots, n;$

$h(u_{i,j,k}) = h(u_{i-2,j,k}) + (q+1),$ when $h(u_{i-2,j,k}) < \frac{Q}{2}$,

$\quad = h(u_{i-2,j,k}) - (q+1),$ when $h(u_{i-2,j,k}) > \frac{Q}{2}$,

$\forall\, k = 1, 2, \ldots, rs, \forall\, j = 1, 2, \ldots, n; \forall\, i = 3, 4, \ldots, t;$

Above labeling pattern give rise graceful labeling to the graph G and so it is a graceful graph. □

Illustration 2.2. *$P_3^3(9 \cdot P_3 \times P_3)$ and its graceful labeling shown in figure–1.*

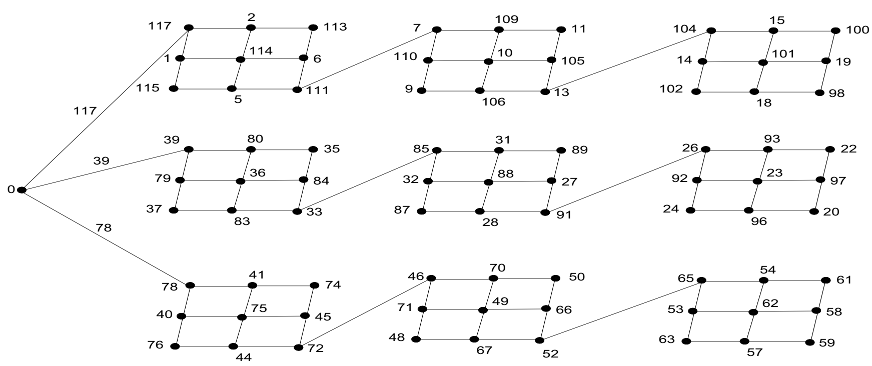

Figure 1: A graph obtained by one point union for path of grid graph $P_3 \times P_3$ and its graceful labeling

Theorem 2.3. *$P_n^t(tn \cdot K_{1,m})$ is graceful tree.*

Proof. Let G be a graph obtained by replacing each vertices of P_n^t except the central vertex by the graph $K_{1,m}$. i.e. G is the graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex by the path union of n copies of the graph $K_{1,m}$. Let u_0 be the central vertex for the graph G. Let $u_{s,l,i}$ $(0 \leq i \leq m)$ be the vertices of $K_{1,t}$ which is l^{th} copy of the path union of n copies of $K_{1,m}$ lies in s^{th} branch of the graph G, $\forall\, l = 1, 2, \ldots \ldots, n$ and $\forall\, s = 1, 2, \ldots, t$.

First we shall join $u_{s,l,0}$ with the vertex $u_{s,l+1,0}$ by an edge to form the path union of n copies of $K_{1,m}$ for s^{th} branch of G, $\forall\, l = 1, 2, \ldots, n-1$ and $\forall\, s = 1, 2, \ldots, t$. Now we shall join $u_{s,1,0}$ with the vertex u_0 by an edge to form the one point union of path graphs G, $\forall\, s = 1, 2, \ldots, t$. We shall define labeling function f for the first copy (branch) of the path union of n copies of $K_{1,m}$ as follows: $f : V(\textit{path union of } n \textit{ copies of } K_{1,m}) \longrightarrow \{0, 1, \ldots, q\}$, where $q = n(m+1) - 1$, defined by,

$f(u_{1,1,0}) = q,$

$f(u_{1,1,i}) = i - 1,$ $\forall\, i = 1, 2, \ldots, m;$

$f(u_{1,2,0}) = f(u_{1,1,m}) + 1,$

$f(u_{1,2,i}) = q - i,$ $\forall\, i = 1, 2, \ldots, m;$

$f(u_{1,l,0}) = f(u_{1,l-2,0}) + (-1)^l[m+1],$ $\forall\, l = 3, 4, \ldots, n,$

$f(u_{1,l,i}) = f(u_{1,l-2,i}) + (-1)^{l-1}[m+1], \qquad \forall\ l = 3,4,\ldots,n,\ \forall\ i = 1,2,\ldots,m.$

Above labeling pattern give rise graceful labeling to the path union of n copies of $K_{1,m}$,which lies in first branch of G. Now we shall define labeling function $g : V(G) \longrightarrow \{0,1,\ldots,Q\}$, where $Q = t(q+1) = t(n(m+1))$ as follows:

$g(u_0) = 0;$

$$
\begin{aligned}
g(u_{1,l,i}) &= f(u_{1,l,i}) + 1, && \text{when } f(u_{1,l,i}) \le m\lceil \tfrac{n}{2}\rceil + \tfrac{n-2}{2},\\
&= f(u_{1,l,i}) + (Q-q), && \text{when } f(u_{1,l,i}) > m\lceil \tfrac{n}{2}\rceil + \tfrac{n-2}{2},
\end{aligned}
$$

$\forall\ i = 0,1,\ldots,m,\ \forall\ l = 1,2,\ldots,n;$

$$
\begin{aligned}
g(u_{2,l,i}) &= g(u_{1,l,i}) + (Q-(q+1)), && \text{when } g(u_{1,l,i}) < \tfrac{Q}{2}, \forall\ i = 0,1,\ldots,m,\ \forall\ l = 1,2,\ldots,n\\
&= g(u_{1,l,i}) - (Q-(q+1)), && \text{when } g(u_{1,l,i}) > \tfrac{Q}{2}, \forall\ i = 0,1,\ldots,m,\ \forall\ l = 1,2,\ldots,n;\\
g(u_{s,l,i}) &= g(u_{s-2,l,i}) - (-1)^s(q+1), && \text{when } g(u_{s-2,l,i}) < \tfrac{Q}{2},\\
&= g(u_{s-2,l,i}) + (-1)^s(q+1), && \text{when } g(u_{s-2,l,i}) > \tfrac{Q}{2},
\end{aligned}
$$

$\forall\ s = 3,4,\ldots,t,\ \forall\ i = 0,1,\ldots,m,\ \forall\ l = 1,2,\ldots,n;$

Above labeling pattern give rise graceful labeling to the graph G and so it is a graceful graph. □

Illustration 2.4. *$P_3^3(9 \cdot K_{1,8})$ and its graceful labeling shown in figure–2.*

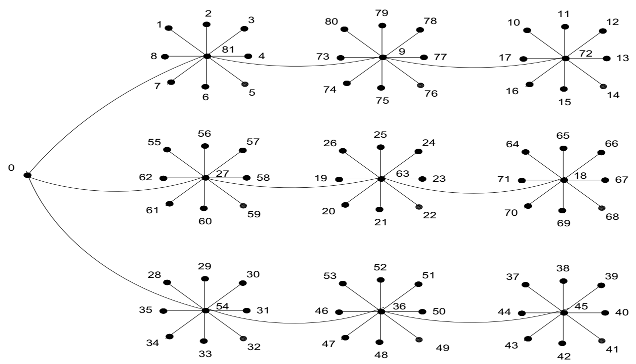

Figure 2: A graph obtained by one point union for path of star graphs and its graceful labeling

Theorem 2.5. *$S(t \cdot C_n)$ is graceful graph, where $n \equiv 0$ (mod 4).*

Proof. Let G be a graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex by the cycle $C_n(n \equiv 0(mod4))$. Let u_0 be the central vertex for the graph G. Let $u_{s,i}$ $(1 \le i \le n)$ be the vertices of s^{th} copy of C_n in G, $\forall\ s = 1,2,\ldots t$.

We shall join $u_{s,1}$ with the vertex u_0 by an edge to form the open star of cycles $G = S(t \cdot C_n)$, $\forall\ s = 1,2,\ldots,t$. The open star of cycles $G = S(t \cdot C_n)$ has $P = tn+1$ vertices and $Q = t(n+1)$ edges. We define labeling function $f : V(G) \longrightarrow \{0,1,\ldots,Q\}$ as follows:

$f(u_0) = 0;$

$$
\begin{aligned}
f(u_{1,i}) &= Q - (\tfrac{i-1}{2}), && \forall\ i = 1,3,\ldots,n-1,\\
&= \tfrac{i}{2}, && \forall\ i = 2,4,\ldots,\tfrac{n}{2},\\
&= \tfrac{i}{2} + 1, && \forall\ i = \tfrac{n}{2}+2, \tfrac{n}{2}+4,\ldots,n;\\
f(u_{2,i}) &= f(u_{1,i}) - (n+1)(t-1), && \text{when } f(u_{1,i}) > \tfrac{Q}{2}, \forall\ i = 1,2,\ldots,n\\
&= f(u_{1,i}) + (n+1)(t-1), && \text{when } f(u_{1,i}) < \tfrac{Q}{2}, \forall\ i = 1,2,\ldots,n;\\
f(u_{s,i}) &= f(u_{s-2,i}) - (n+1), && \text{when } f(u_{s-2,i}) > \tfrac{Q}{2}, \forall\ i = 1,2,\ldots,n,\ \forall\ s = 3,4,\ldots,t\\
&= f(u_{s-2,i}) + (n+1), && \text{when } f(u_{s-2,i}) < \tfrac{Q}{2}, \forall\ i = 1,2,\ldots,n,\ \forall\ s = 3,4,\ldots,t.
\end{aligned}
$$

Above labeling pattern give rise graceful labeling to the graph G and so it is a graceful graph. □

Illustration 2.6. *Open star of* 4 *copies of* C_8 *and its graceful labeling shown in figure–3.*

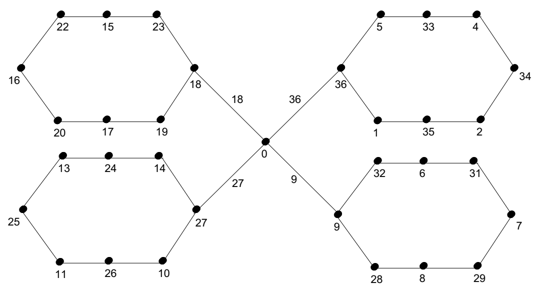

Figure 3: A graph obtained by open star of C_8 and its graceful labeling

Theorem 2.7. $P_n^t(tn \cdot C_m)$ *is graceful, where* $m \equiv 0$ *(mod* 4*).*

Proof. Let G be a graph obtained by replacing each vertices of P_n^t except the central vertex by the graph C_m, where $m \equiv 0 \pmod 4$. i.e. G is the graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex by the path union of n copies of the graph C_m ($m \equiv 0 \pmod 4$). Let u_0 be the central vertex for the graph G. Let $u_{s,l,i}$ $(1 \leq i \leq m)$ be the vertices of C_m^l, which is l^{th} copy of the path union of n copies of C_m lies in s^{th} branch of the graph G, $\forall\ l = 1, 2, \ldots \ldots, n$ and $\forall\ s = 1, 2, \ldots, t$.

First we shall join $u_{s,l,m}$ with the vertex $u_{s,l+1,1}$ by an edge to form the path union of n copies of C_m, where $m \equiv 0 \pmod 4$ for s^{th} branch of G, $\forall\ l = 1, 2, \ldots, n-1$ and $\forall\ s = 1, 2, \ldots, t$. Now we shall join $u_{s,1,1}$ with the vertex u_0 by an edge to form the one point union of path graphs G, $\forall\ s = 1, 2, \ldots, t$. We shall define labeling function f for the first copy (branch) of the path union of n copies of C_m ($m \equiv 0 \pmod 4$) as follows: $f : V(path\ union\ of\ n\ copies\ of\ C_m) \longrightarrow \{0, 1, \ldots, q\}$, where $q = nm + n - 1$, defined by,

$$
\begin{aligned}
f(u_{1,1,i}) &= q - (\tfrac{i-1}{2}), && \forall\ i = 1, 3, \ldots, m-1,\\
&= (\tfrac{i-2}{2}), && \forall\ i = 2, 4, \ldots, \tfrac{m}{2},\\
&= \tfrac{i}{2}, && \forall\ i = \tfrac{m}{2} + 2, \tfrac{m}{2} + 4, \ldots, m;\\
f(u_{1,2,i}) &= f(u_{1,1,i}) + (q - m), && \text{when } f(u_{1,1,i}) < \tfrac{q}{2}, \forall\ i = 1, 2, \ldots, m\\
&= f(u_{1,1,i}) - (q - m), && \text{when } f(u_{1,1,i}) > \tfrac{q}{2}, \forall\ i = 1, 2, \ldots, m;\\
f(u_{1,l,i}) &= f(u_{1,l-2,i}) + (m + 1), && \text{when } f(u_{1,l-2,i}) < \tfrac{q}{2}, \forall\ i = 1, 2, \ldots, m, \forall\ l = 3, 4, \ldots, n\\
&= f(u_{1,l-2,i}) - (m + 1), && \text{when } f(u_{1,l-2,i}) > \tfrac{q}{2}, \forall\ i = 1, 2, \ldots, m, \forall\ l = 3, 4, \ldots, n.
\end{aligned}
$$

Above labeling pattern give rise graceful labeling to the path union of n copies of C_m ($m \equiv 0(mod4)$), which lies in first branch of G. Now we shall define labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, Q\}$, where $Q = t(q + 1) = t(n(m + 1))$ as follows:

$$
\begin{aligned}
g(u_0) &= 0;\\
g(u_{1,l,i}) &= f(u_{1,l,i}) + 1, && \forall\ i = 2, 4, \ldots, m,\ \imath\forall\ l = 1, 2, \ldots, n,\\
&= f(u_{1,l,i}) + (Q - q), && \forall\ i = 1, 3, \ldots, m-1,\ \forall\ l = 1, 2, \ldots, n;\\
g(u_{2,l,i}) &= g(u_{1,l,i}) + (Q - (q + 1)), && \forall\ i = 1, 3, \ldots, m-1,\ \forall\ l = 1, 2, \ldots, n,\\
&= g(u_{1,l,i}) - (Q - (q + 1)), && \forall\ i = 2, 4, \ldots, m,\ \imath\forall\ l = 1, 2, \ldots, n;\\
g(u_{s,l,i}) &= g(u_{s-2,l,i}) - (-1)^i (q + 1), && \forall\ i = 1, 2, \ldots, m,\ \forall\ l = 1, 2, \ldots, n,\ \forall\ s = 3, 4, \ldots, t.
\end{aligned}
$$

Above labeling pattern give rise graceful labeling to the graph G and so it is a graceful graph. $\square$

Illustration 2.8. *$P_4^3(12 \cdot C_8)$ and its graceful labeling shown in figure–4.*

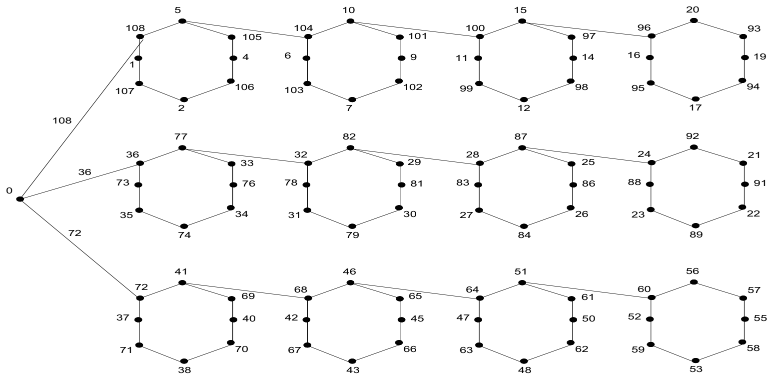

Figure 4: A graph obtained by one point union for path of cycles and its graceful labeling

3 Concluding Remarks

Labeled graph is the topics of current interest. We discussed here graceful labeling to $P_n^t(tn \cdot; P_r \times P_s)$, $P_n^t(tn \cdot K_{1,n})$, P_n^t $(tn \cdot C_m)$ and $S(t \cdot C_n)$. The present work contribute four new results. The labeling pattern is illustrated for derived results. This work throws some light on the gracefulness of one point union for path of graphs.

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