Decompositions of $I-\pi g$-continuity

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Abstract: In this paper, we introduce the notions of $I-\pi$-open sets, $I-\pi g$-open sets, $I-\pi g\alpha$-open sets, $I-\pi gp$-open sets, $I-E_r$-sets and $I-E_r^*$-sets in ideal topological spaces and investigate some of their properties and using these notions we obtain three decompositions of $I-\pi g$-continuity.

Keywords: $I-\pi g\alpha$-continuity, $I-\pi gp$-continuity, $I-E_r$-continuity, $I-E_r^*$-continuity and $I-\pi g$-continuity.

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1 Introduction and Preliminaries

In 1968, Zaitsev [12] introduced the concept of $\pi$-closed sets and in 1970, Levine [5] initiated the study of so called $g$-closed sets in topological spaces. The concept of $g$-continuity was introduced and studied by Balachandran et. al. in 1991 [1]. Dontchev and Noiri [2] defined the notions of $\pi g$-closed sets and $\pi g$-continuity in topological spaces. Quite Recently Ravi et. al. [9] obtained three different decompositions of $\pi g$-continuity in topological spaces by providing two types of weaker forms of continuity, namely $E_r$-continuity and $E_r^*$-continuity and in [10], they also obtained three different decompositions of $\pi g$-continuity via idealization.

Recently, Rajamani et. al. [7] introduced $I-g$-open sets, $I-gp$-open sets, $I-gs$-open sets and obtained three different decompositions of $I-g$-continuity and in [8], they also introduced $I-rg$-open sets, $I-ga^{**}$-open sets, $I-gpr$-open sets and obtained three different decompositions of $I-rg$-continuity. In this paper, we introduce the notions of $I-\pi$-open sets, $I-\pi g$-open sets, $I-\pi g\alpha$-open sets, $I-\pi gp$-open sets, $I-E_r$-sets and $I-E_r^*$-sets to obtain three decompositions of $I-\pi g$-continuity.

Let $(X, \tau)$ be a topological space. An ideal is defined as a nonempty collection $I$ of subsets of $X$ satisfying the following two conditions:

(i) If $A \in I$ and $B \subseteq A$, then $B \in I$

(ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

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For a subset $A \subseteq X$, $A^*(I) = \{ x \in X/U \cap A \notin I \text{ for each neighborhood } U \text{ of } x \}$ is called the local function of $A$ with respect to $I$ and $\tau$ \[4\]. We simply write $A^*$ instead of $A^*(I)$ in case there is no chance for confusion. $X^*$ is often a proper subset of $X$.

For every ideal topological space $(X, \tau, I)$ there exists a topology $\tau^*(I)$, finer than $\tau$, generated by $\beta(I, \tau) = \{ U \setminus I : U \in \tau \text{ and } I \in I \}$, but in general $\beta(I, \tau)$ is not always a topology \[11\]. Also, $\text{cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$ \[11\].

Additionally, $\text{cl}^*(A) \subseteq \text{cl}(A)$ for any subset $A$ of $X$ \[3\]. Throughout this paper, $X$ denotes the ideal topological space $(X, \tau, I)$ and also $\text{cl}(A)$ and $\text{int}(A)$ denote the closure of $A$ and the interior of $A$ in $(X, \tau)$, respectively. $\text{int}^*(A)$ will denote the interior of $A$ in $(X, \tau^*, I)$.

**Definition 1.1.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be

(i) $I$-pre-open \[6\] if $A \subseteq \text{int}^*(\text{cl}^*(A))$,

(ii) $I$-\(\alpha\)-open \[6\] if $A \subseteq \text{int}^*(\text{cl}^*(\text{int}^*(A)))$,

(iii) a $I$-t-set \[7\] if $\text{int}^*(\text{cl}^*(A)) = \text{int}^*(A)$,

(iv) an $I$-\(\alpha\)-set \[7\] if $\text{int}^*(\text{cl}^*(\text{int}^*(A))) = \text{int}^*(A)$,

(v) $I$-regular closed \[8\] if $A = \text{cl}^*(\text{int}^*(A))$.

The complement of $I$-regular closed set is $I$-regular open \[8\].

Also, we have $I$-\(\alpha\)-int$(A) = A \cap \text{int}^*(\text{cl}^*(\text{int}^*(A)))$ \[6\] and $I$-\(\alpha\)-int$(A) = A \cap \text{int}^*(\text{cl}^*(A))$ \[6\], where $I$-\(\alpha\)-int$(A)$ denotes the $I$-\(\alpha\)-interior of $A$ in $(X, \tau, I)$ which is the union of all $I$-\(\alpha\)-open sets of $(X, \tau, I)$ contained in $A$. $I$-\(\alpha\)-int$(A)$ has similar meaning.

## 2 $I$-$\pi g$-open sets, $I$-$\pi g\alpha$-open sets and $I$-$\pi gp$-open sets

**Definition 2.1.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is called

(i) $I$-$\pi$-open if the finite union of $I$-regular open sets,

The complement of $I$-$\pi$-open set is $I$-$\pi$-closed,

(ii) $I$-$\pi g$-open if $F \subseteq \text{int}^*(A)$ whenever $F \subseteq A$ and $F$ is $I$-$\pi$-closed in $X$,

(iii) $I$-$\pi g\alpha$-open if $F \subseteq I$-$\alpha$-int$(A)$ whenever $F \subseteq A$ and $F$ is $I$-$\pi$-closed in $X$,

(iv) $I$-$\pi gp$-open if $F \subseteq I$-pint$(A)$ whenever $F \subseteq A$ and $F$ is $I$-$\pi$-closed in $X$.

**Proposition 2.2.** For a subset of an ideal topological space, the following hold:

(i) Every $I$-$\pi g$-open set is $I$-$\pi g\alpha$-open.

(ii) Every $I$-$\pi g\alpha$-open set is $I$-$\pi gp$-open.

(iii) Every $I$-$\pi g$-open set is $I$-$\pi gp$-open.

Proof.
(i) Let \( A \) be an \( \mathcal{I} \)-\( \pi g \)-open. Then, for any \( \mathcal{I} \)-\( \pi \)-closed set \( F \) with \( F \subseteq A \), we have \( F \subseteq \text{int}^*(A) \subseteq \text{int}^*((\text{int}^*(A))^*) \cup \text{int}^*(A) = \text{int}^*((\text{int}^*(A))^*) \cup \text{int}^*(\text{int}^*(A)) \subseteq \text{int}^*((\text{int}^*(A))^* \cup \text{int}^*(A)) = \text{int}^*(\text{cl}^*(\text{int}^*(A))). \) That is, \( F \subseteq A \cap \text{int}^*(\text{cl}^*(\text{int}^*(A))) = \text{I-aint}(A) \) which implies that \( A \) is \( \mathcal{I} \)-\( \pi g \alpha \)-open.

(ii) Let \( A \) be \( \mathcal{I} \)-\( \pi g \alpha \)-open. Then, for any \( \mathcal{I} \)-\( \pi \)-closed set \( F \) with \( F \subseteq A \), we have \( F \subseteq \text{I-alpha}(A) = A \cap \text{int}^*(\text{cl}^*(\text{int}^*(A))) \subseteq \text{int}^*(A) \cap \text{I-pint}(A) \) which implies that \( A \) is \( \mathcal{I} \)-\( \pi g p \)-open.

(iii) It is an immediate consequence of (i) and (ii).

\[ \boxed{\mathcal{I} \text{-}\pi g \alpha \text{-open}} \]

Remark 2.3. The converses of Proposition 2.2 are not true, in general.

Example 2.4. Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \) and \( \mathcal{I} = \{\emptyset, \{d\}\} \). Then \( \{a, b, d\} \) is \( \mathcal{I} \)-\( \pi g p \)-open set but not \( \mathcal{I} \)-\( \pi g \alpha \)-open.

Example 2.5. In Example 2.4, \( \{a, b, d\} \) is \( \mathcal{I} \)-\( \pi g p \)-open set but not \( \mathcal{I} \)-\( \pi g \)-open.

Example 2.6. Let \( X = \{a, b, c, d, e\} \), \( \tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\} \) and \( \mathcal{I} = \{\emptyset\} \). Clearly \( \{a, c, d, e\} \) is \( \mathcal{I} \)-\( \pi g \alpha \)-open set but not \( \mathcal{I} \)-\( \pi g \)-open.

Remark 2.7. By Proposition 2.2, we have the following diagram. In this diagram, there is no implication which is reversible as shown by examples above.

\[ \boxed{\mathcal{I} \text{-}\pi g \alpha \text{-open} \uparrow \mathcal{I} \text{-}\pi g \text{-open} \downarrow \mathcal{I} \text{-}\pi g p \text{-open}} \]

3 \( \mathcal{I} \)-\( E_r \)-sets and \( \mathcal{I} \)-\( E_r^* \)-sets

Definition 3.1. A subset \( A \) of an ideal topological space \((X, \tau, \mathcal{I})\) is called

(i) a \( \mathcal{I} \)-\( E_r \)-set if \( A = U \cap V \), where \( U \) is \( \mathcal{I} \)-\( \pi g \)-open and \( V \) is a \( \mathcal{I} \)-\( t \)-set,

(ii) a \( \mathcal{I} \)-\( E_r^* \)-set if \( A = U \cap V \), where \( U \) is \( \mathcal{I} \)-\( \pi g \)-open and \( V \) is an \( \mathcal{I} \)-\( \alpha^* \)-set.

We have the following proposition:

Proposition 3.2. For a subset of an ideal topological space, the following hold:

(i) Every \( \mathcal{I} \)-\( t \)-set is an \( \mathcal{I} \)-\( \alpha^* \)-set [?] and a \( \mathcal{I} \)-\( E_r \)-set.

(ii) Every \( \mathcal{I} \)-\( \alpha^* \)-set is a \( \mathcal{I} \)-\( E_r^* \)-set.

(iii) Every \( \mathcal{I} \)-\( E_r \)-set is a \( \mathcal{I} \)-\( E_r^* \)-set.

(iv) Every \( \mathcal{I} \)-\( \pi g \)-open set is both \( \mathcal{I} \)-\( E_r \)-set and \( \mathcal{I} \)-\( E_r^* \)-set.
From Proposition 3.2, we have the following diagram.

\[
\begin{array}{ccc}
\mathcal{I}-\pi g\text{-open set} & \longrightarrow & \mathcal{I}-E_\tau\text{-set} \\
\downarrow & & \downarrow \\
\mathcal{I}-E_\tau^*\text{-set} & \longleftarrow & \mathcal{I}-\alpha^*\text{-set}
\end{array}
\]

**Remark 3.3.** The converses of implications in Diagram II need not be true as the following examples show.

**Example 3.4.** Let \(X = \{a, b, c, d\}\), \(\tau = \emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\) and \(\mathcal{I} = \emptyset, \{c\}\). Then \(\{a, b\}\) is \(\mathcal{I}-E_\tau\text{-set but not a }\mathcal{I}-\pi g\text{-open set.}\)

**Example 3.5.** In Example 3.4, \(\{b, c, d\}\) is \(\mathcal{I}-E_\tau\text{-set but not an }\mathcal{I}-\pi g\text{-open set.}\)

**Example 3.6.** In Example 2.4, \(\{a, b, d\}\) is \(\mathcal{I}-E_\tau^*\text{-set but not a }\mathcal{I}-E_\tau\text{-set.}\)

**Example 3.7.** In Example 3.4, \(\{a, b\}\) is \(\mathcal{I}-E_\tau^*\text{-set but not an }\mathcal{I}-\alpha^*\text{-set.}\)

**Example 3.8.** In Example 2.4, \(\{b\}\) is \(\mathcal{I}-\alpha^*\text{-set but not a }\mathcal{I}-\tau\text{-set.}\)

**Proposition 3.9.** A subset \(A\) of \(X\) is \(\mathcal{I}-\pi g\text{-open}\) if and only if it is both \(\mathcal{I}-\pi gp\text{-open}\) and a \(\mathcal{I}-E_\tau\text{-set in }\)

\(X\).

**Proof.** Necessity is trivial. We prove the sufficiency. Assume that \(A\) is \(\mathcal{I}-\pi gp\text{-open}\) and a \(\mathcal{I}-E_\tau\text{-set in }\)

\(X\). Let \(F \subseteq A\) and \(F\) is \(\mathcal{I}-\pi\text{-closed in }\)

\(X\). Since \(A\) is a \(\mathcal{I}-E_\tau\text{-set in }\)

\(X\), \(A = U \cap V\), where \(U\) is \(\mathcal{I}-\pi g\text{-open}\)

and \(V\) is a \(\mathcal{I}-\tau\text{-set.}\) Since \(A\) is \(\mathcal{I}-\pi g\text{-open}\), \(F \subseteq \mathcal{I}\text{-pint}(A) = A \cap \text{int}*(\text{cl}^*(A)) = (U \cap V) \ni \text{int}*(\text{cl}^*(U \cap V)) \subseteq (U \cap V) \cap \text{int}*(\text{cl}^*(U) \cap \text{cl}^*(V)) = (U \cap V) \cap \text{int}*(\text{cl}^*(U)) \cap \text{int}*(\text{cl}^*(V)).\) This implies \(F \subseteq \text{int}*(\text{cl}^*(V)) = \text{int}*(V)\) since \(V\) is a \(\mathcal{I}-\tau\text{-set.}\) Since \(F\) is \(\mathcal{I}-\pi\text{-closed, }U\) is \(\mathcal{I}-\pi g\text{-open and }F \subseteq U\), we have \(F \subseteq \text{int}(U).\) Therefore, \(F \subseteq \text{int}(U) \ni \text{int}*(V) = \text{int}*(U \cap V) = \text{int}*(A).\) Hence \(A\) is \(\mathcal{I}-\pi g\text{-open in }\)

\(X.\)

**Corollary 3.10.** A subset \(A\) of \(X\) is \(\mathcal{I}-\pi g\text{-open}\) if and only if it is both \(\mathcal{I}-\pi gp\text{-open}\) and a \(\mathcal{I}-E_\tau\text{-set in }\)

\(X.\)

**Proof.** This is an immediate consequence of Proposition 3.9.

**Proposition 3.11.** A subset \(A\) of \(X\) is \(\mathcal{I}-\pi g\text{-open}\) if and only if it is both \(\mathcal{I}-\pi gp\text{-open}\) and a \(\mathcal{I}-E_\tau^*\text{-set in }\)

\(X.\)

**Proof.** Necessity is trivial. We prove the sufficiency. Assume that \(A\) is \(\mathcal{I}-\pi gp\text{-open}\) and a \(\mathcal{I}-E_\tau^*\text{-set in }\)

\(X.\) Let \(F \subseteq A\) and \(F\) is \(\mathcal{I}-\pi\text{-closed in }\)

\(X.\) Since \(A\) is a \(\mathcal{I}-E_\tau^*\text{-set in }\)

\(X, A = U \cap V\), where \(U\) is \(\mathcal{I}-\pi g\text{-open and }V\) is a \(\mathcal{I}-\alpha^*\text{-set.}\) Now since \(F\) is \(\mathcal{I}-\pi\text{-closed, }F \subseteq U\) and \(U\) is \(\mathcal{I}-\pi g\text{-open, }F \subseteq \text{int}*(U).\) Since \(A\) is \(\mathcal{I}-\pi gp\text{-open, }F \subseteq \mathcal{I}\text{-oint}(A) = A \ni \text{int}*(\text{cl}^*(\text{int}*(A))) = (U \cap V) \ni \text{int}*(\text{cl}^*(\text{int}*(U \cap V))) = (U \cap V) \ni \text{int}*(\text{cl}^*(U) \cap \text{int}*(V)) \subseteq (U \cap V) \ni \text{int}*(\text{cl}^*(U)) \cap \text{int}*(\text{cl}^*(V)) = (U \cap V) \ni \text{int}*(\text{cl}^*(U)) \cap \text{int}*(\text{cl}^*(V)) \ni \text{int}*(\text{cl}^*(V'))) = (U \cap V) \ni \text{int}*(\text{cl}^*(U)) \cap \text{int}*(\text{cl}^*(V)),\) since \(V\) is an \(\mathcal{I}-\alpha^*\text{-set.}\) This implies \(F \subseteq \text{int}*(V).\) Therefore, \(F \subseteq \text{int}*(U) \ni \text{int}*(V) = \text{int}*(U \cap V) = \text{int}*(A).\) Hence \(A\) is \(\mathcal{I}-\pi g\text{-open in }\)

\(X.\)

**Remark 3.12.**

(i) The concepts of \(\mathcal{I}-\pi gp\text{-open sets and }\mathcal{I}-E_\tau\text{-sets are independent of each other.}\)
(ii) The concepts of $I$-$\pi g\alpha$-open sets and $I$-$E_r$-sets are independent of each other.

(iii) The concepts of $I$-$\pi g\alpha$-open sets and $I$-$E^*_r$-sets are independent of each other.

**Example 3.13.** Consider the Example 2.4. Then

(i) $\{d\}$ is $I$-$E_r$-set but not a $I$-$\pi g\alpha$-open.

(ii) $\{a, b, d\}$ is $I$-$\pi g\alpha$-open but not a $I$-$E_r$-set.

**Example 3.14.**

(i) In Example 2.4, $\{b, c, d\}$ is $I$-$E_r$-set but not an $I$-$\pi g\alpha$-open set.

(ii) In Example 2.6, $\{a, c, d, e\}$ is $I$-$\pi g\alpha$-open set but not a $I$-$E_r$-set.

**Example 3.15.**

(i) In Example 2.5, $\{b, c, d\}$ is $I$-$E^*_r$-set but not an $I$-$\pi g\alpha$-open set.

(ii) In Example 2.6, $\{a, c, d, e\}$ is $I$-$\pi g\alpha$-open set but not a $I$-$E^*_r$-set.

4 Decompositions of $I$-$\pi g$-continuity

**Definition 4.1.** A mapping $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $I$-$\pi g\alpha$-continuous (resp. $I$-$\pi g$-continuous, $I$-$\pi g\pi$-continuous, $I$-$E_r$-continuous and $I$-$E^*_r$-continuous) if for every $V \in \sigma$, $f^{-1}(V)$ is $I$-$\pi g\alpha$-open (resp. $I$-$\pi g$-open, $I$-$\pi g\pi$-open, a $I$-$E_r$-set and a $I$-$E^*_r$-set) in $(X, \tau, I)$. From Propositions 3.9 and 3.11 and Corollary 3.10 we have the following decompositions of $I$-$\pi g$-continuity.

**Theorem 4.2.** Let $(X, \tau, I)$ be an ideal topological space. For a mapping $f : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:

(i) $f$ is $I$-$\pi g$-continuous;

(ii) $f$ is $I$-$\pi g\pi$-continuous and $I$-$E_r$-continuous;

(iii) $f$ is $I$-$\pi g\alpha$-continuous and $I$-$E_r$-continuous;

(iv) $f$ is $I$-$\pi g\alpha$-continuous and $I$-$E^*_r$-continuous.

**Remark 4.3.**

(i) The concepts of $I$-$\pi g\pi$-continuity and $I$-$E_r$-continuity are independent of each other.

(ii) The concepts of $I$-$\pi g\alpha$-continuity and $I$-$E_r$-continuity are independent of each other.

(iii) The concepts of $I$-$\pi g\alpha$-continuity and $I$-$E^*_r$-continuity are independent of each other.

**Example 4.4.**

(i) Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$, $I = \{\emptyset, \{d\}\}$ and $\sigma = \{\emptyset, Y, \{b, c, d\}\}$. Let $f : (X, \tau, I) \to (Y, \sigma)$ be the identity function. Then $f$ is $I$-$E_r$-continuous but not $I$-$\pi g\pi$-continuous.
(ii) In Example 4.4(1), if \( \sigma \) is replaced by \( \sigma = \{\emptyset, Y, \{a, b, d\}\} \), then \( f \) is \( \mathcal{I}-\pi gp \)-continuous but not \( \mathcal{I}-E_r \)-continuous.

Example 4.5.

(i) In Example 4.4(1), \( f \) is \( \mathcal{I}-E_r \)-continuous but not \( \mathcal{I}-\pi g\alpha \)-continuous.

(ii) Let \( X = Y = \{a, b, c, d, e\} \), \( \tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\} \), \( \mathcal{I} = \{\emptyset\} \) and \( \sigma = \{\emptyset, Y, \{a, c, d, e\}\} \). Let \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) be the identity function. Then \( f \) is \( \mathcal{I}-\pi g\alpha \)-continuous but not \( \mathcal{I}-E_r \)-continuous.

Example 4.6.

(i) In Example 4.4(1), \( f \) is \( \mathcal{I}-E_r^* \)-continuous but not \( \mathcal{I}-\pi g\alpha \)-continuous.

(ii) In Example 4.5(2), \( f \) is \( \mathcal{I}-\pi g\alpha \)-continuous but not \( \mathcal{I}-E_r^* \)-continuous.

5 Conclusion

Recently, Rajamani et. al. [7] introduced \( \mathcal{I} \)-g-open sets, \( \mathcal{I} \)-gp-open sets, \( \mathcal{I} \)-gs-open sets and obtained three different decompositions of \( \mathcal{I} \)-g-continuity and in [8], they also introduced \( \mathcal{I} \)-rg-open sets, \( \mathcal{I} \)-ga**-open sets, \( \mathcal{I} \)-gpr-open sets and obtained three different decompositions of \( \mathcal{I} \)-rg-continuity. In this paper, we introduced the notions of \( \mathcal{I} \)-\( \pi \)-open sets, \( \mathcal{I} \)-\( \pi gp \)-open sets, \( \mathcal{I} \)-\( \pi g\alpha \)-open sets, \( \mathcal{I} \)-\( \pi gp \)-open sets, \( \mathcal{I} \)-\( E_r \)-sets and \( \mathcal{I} \)-\( E_r^* \)-sets to obtain three decompositions of \( \mathcal{I} \)-\( \pi g \)-continuity.

References


