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Associated Properties of γ-$\pi g\gamma$-closed Functions

Research Article

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Abstract: The concept of γ-open sets was introduced in [3, 7, 11]. The primary purpose of this paper is to introduce and study pre-$\pi g\gamma$-closed functions by using γ-open sets.

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1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of γ-open sets introduced in [3, 7, 11].

In 1970, Levine [12] defined and studied generalized closed sets in topological spaces. In 1982, Malghan [15] defined generalized closed functions and obtained some preservation theorems of normality and regularity. In 1990, Arya and Nour [5] defined generalized semi-open sets and used them to obtain characterizations of s-normal spaces due to Maheshwari and Prasad [13]. In 1993, Devi et.al. [6] defined and studied generalized semi-closed functions and showed that the continuous generalized semi-closed surjective image of a normal space is s-normal. In 1998, Noiri et.al. [16] defined generalized pre-closed sets and introduced generalized pre-closed functions and showed that the continuous generalized pre-closed surjective image of normal space is prenormal [17](or p-normal [19]). Recently, Tahiliani [22] has defined generalized β-closed functions and has shown that the continuous generalized β-closed surjective images of normal (resp. regular) spaces are β-normal [14] (resp. β-regular [2]). Further, it has shown that β-regularity is preserved under continuous pre-β-open [14] β-$g\beta$-closed

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[22] surjections. Recently, Sreeja and Janaki [20] has defined $\pi g\gamma$-closed sets and studied properties and characterizations of them.

Throughout this paper, X and Y refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, cl(A) and int(A) denote the closure of A and the interior of A in X, respectively. A subset A of X is said to be regular open [21] (resp. regular closed [21]) if A = int(cl(A)) (resp. A = cl(int(A))). The finite union of regular open sets is said to be π-open [23]. The complement of a π-open set is said to be π-closed [23]. A subset A of X is said to be β-open [1] (= semi pre-open [4]) if A $\subseteq$ cl(int(cl(A))). A subset A of X is said to be γ-open [11] or sp-open [7] or b-open [3] if A $\subseteq$ cl(int(A)) $\cup$ int(cl(A)). The complement of γ-open (resp. regular open) set is called γ-closed (= b-closed) (resp. regular closed). The intersection of all γ-closed sets of X containing A is called the γ-closure [11] (= b-closure) of A and is denoted by γcl(A) (= bcl(A)). It is evident that a set A is γ-closed if and only if γcl(A) = A [18]. The γ-interior [11] of A, γint(A) or bint(A), is the union of all γ-open sets contained in A. A subset A of X is said to be γ-clopen [9] or b-regular [18] if it is γ-open and γ-closed.

The family of all γ-open (resp. γ-closed, γ-clopen, β-open, regular open, regular closed) sets of X is denoted by BO(X) or γO(X) (resp. BC(X) or γC(X), BR(X) or γCO(X), βO(X), RO(X), RC(X)). A subset A of a topological space (X, τ) is called a generalized γ-closed [8] (briefly gγ-closed) set of X if γcl(A) $\subseteq$ U holds whenever A $\subseteq$ U and U is open in X. A will be called gγ-open if X\A is gγ-closed. A subset A of a topological space (X, τ) is called $\pi g\gamma$-closed [20] set of X if γcl(A) $\subseteq$ U holds whenever A $\subseteq$ U and U is π-open in X. A will be called $\pi g\gamma$-open if X\A is $\pi g\gamma$-closed.

Lemma 1.1 ([20]). *A subset A of a space X is $\pi g\gamma$-open in X if and only if F $\subseteq$ γint(A) whenever F $\subseteq$ A and F is π-closed in X.*

Remark 1.2 ([11]). *Every open set is γ-open but not conversely.*

Remark 1.3 ([8]). *Every γ-open set is gγ-open but not conversely.*

Remark 1.4 ([20]). *Every gγ-open set is $\pi g\gamma$-open but not conversely.*

Remark 1.5 ([11]). *Every γ-open set is β-open but not conversely.*

Theorem 1.6 ([4]). *For any subset A of a topological space X, the following conditions are equivalent:*

(1) A $\in$ βO(X);

(2) A $\subseteq$ cl(int(cl(A)));

(3) cl(A) $\in$ RC(X).

2. Pre-$\pi g\gamma$-closed Functions

Definition 2.1. *A function $f : X \to Y$ is said to be pre-$\pi g\gamma$-closed (= γ-$\pi g\gamma$-closed) (resp. regular $\pi g\gamma$-closed, almost $\pi g\gamma$-closed) if for each F $\in$ γC(X) (resp. F $\in$ BR(X), F $\in$ RC(X)), f(F) is $\pi g\gamma$-closed in Y.*

Definition 2.2. *A function $f : X \to Y$ is said to be $\pi g\gamma$-closed if for each closed set F of X, f(F) is $\pi g\gamma$-closed in Y.*

From the above definitions, we obtain the following diagram:

$$\begin{array}{ccc} \text{pre-}\pi g\gamma\text{-closed} & \longrightarrow & \text{regular } \pi g\gamma\text{-closed} \\ \downarrow & & \downarrow \\ \pi g\gamma\text{-closed} & \longrightarrow & \text{almost } \pi g\gamma\text{-closed} \end{array}$$

Remark 2.3. *None of all implications in the above diagram is reversible as the following examples show.*

Example 2.4. *Let X = Y = $\{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{b, d\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, Y, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is both regular $\pi g\gamma$-closed and $\pi g\gamma$-closed but it is not pre-$\pi g\gamma$-closed.*

Example 2.5. *Let X = Y = $\{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, Y, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is almost $\pi g\gamma$-closed but not $\pi g\gamma$-closed.*

Example 2.6. *Let X = Y = $\{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$ and $\sigma = \{\emptyset, Y, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is almost $\pi g\gamma$-closed but not regular $\pi g\gamma$-closed.*

Lemma 2.7. *A surjective function $f : X \to Y$ is pre-$\pi g\gamma$-closed (resp. regular $\pi g\gamma$-closed) if and only if for each subset B of Y and each $U \in \gamma O(X)$ (resp. $U \in BR(X)$) containing $f^{-1}(B)$, there exists a $\pi g\gamma$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Corollary 2.8. *If a surjective function $f : X \to Y$ is pre-$\pi g\gamma$-closed (resp. regular $\pi g\gamma$-closed), then for each π-closed set K of Y and each $U \in \gamma O(X)$ (resp. $U \in BR(X)$) containing $f^{-1}(K)$, there exists $V \in \gamma O(Y)$ containing K such that $f^{-1}(V) \subseteq U$.*

Proof. Suppose that $f : X \to Y$ is pre-$\pi g\gamma$-closed (resp. regular $\pi g\gamma$-closed). Let K be any π-closed set of Y and U $\in$ γO(X) (resp. U $\in$ BR(X)) containing f^{-1}(K). By Lemma 2.7, there exists a $\pi g\gamma$-open set G of Y such that K $\subseteq$ G and f^{-1}(G) $\subseteq$ U. Since K is π-closed, by Lemma 1.1, K $\subseteq$ γint(G). Put V = γint(G). Then, K $\subseteq$ V $\in$ γO(Y) and f^{-1}(V) $\subseteq$ U. □

Definition 2.9 ([10]). *A function $f : X \to Y$ is said to be*

(1) π-irresolute if $f^{-1}(F)$ is π-closed in X for every π-closed set F of Y.

(2) m-π-closed if f(F) is π-closed in Y for every π-closed set F of X.

Lemma 2.10. *A function $f : X \to Y$ is π-irresolute if and only if $f^{-1}(F)$ is π-open in X for every π-open set F of Y.*

Theorem 2.11. *If $f : X \to Y$ is π-irresolute pre-$\pi g\gamma$-closed bijection, then f(H) is $\pi g\gamma$-closed in Y for each $\pi g\gamma$-closed set H of X.*

Proof. Let H be any $\pi g\gamma$-closed set of X and V an π-open set of Y containing f(H). Since f^{-1}(V) is an π-open set of X containing H, γcl(H) $\subseteq$ f^{-1}(V) and hence f(γcl(H)) $\subseteq$ V. Since f is pre-$\pi g\gamma$-closed and γcl(H) $\in$ γC(X), f(γcl(H)) is $\pi g\gamma$-closed in Y. We have γcl(f(H)) $\subseteq$ γcl(f(γcl(H))) $\subseteq$ V. Therefore, f(H) is $\pi g\gamma$-closed in Y. □

Definition 2.12. *A function $f : X \to Y$ is said to be pre-$\pi g\gamma$-continuous or γ-$\pi g\gamma$-continuous if $f^{-1}(K)$ is $\pi g\gamma$-closed in X for every $K \in \gamma C(Y)$.*

It is obvious that a function $f : X \to Y$ is pre-$\pi g\gamma$-continuous if and only if $f^{-1}(V)$ is $\pi g\gamma$-open in X for every $V \in \gamma O(Y)$.

Theorem 2.13. *If $f : X \to Y$ is m-π-closed pre-$\pi g\gamma$-continuous bijection, then $f^{-1}(K)$ is $\pi g\gamma$-closed in X for each $\pi g\gamma$-closed set K of Y.*

Proof. Let K be $\pi g\gamma$-closed set of Y and U an π-open set of X containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then V is an π-open in Y, $K \subseteq V$ and $f^{-1}(V) \subseteq U$. Therefore, we have $\gamma cl(K) \subseteq V$ and hence $f^{-1}(K) \subseteq f^{-1}(\gamma cl(K)) \subseteq f^{-1}(V) \subseteq U$. Since f is pre-$\pi g\gamma$-continuous and $\gamma cl(K)$ is γ-closed in Y, $f^{-1}(\gamma cl(K))$ is $\pi g\gamma$-closed in X and hence $\gamma cl(f^{-1}(K)) \subseteq \gamma cl(f^{-1}(\gamma cl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $\pi g\gamma$-closed in X. □

Recall that a function $f : X \to Y$ is said to be γ-irresolute [9] if $f^{-1}(V) \in \gamma O(X)$ for every $V \in \gamma O(Y)$.

Remark 2.14. *Every γ-irresolute function is pre-$\pi g\gamma$-continuous but not conversely.*

Proof. Let $A \in \gamma O(Y)$. Since f is γ-irresolute, $f^{-1}(A) \in \gamma O(X)$ and so, by Remarks 1.3 and 1.4, $f^{-1}(A)$ is $\pi g\gamma$-open in X. Hence f is pre-$\pi g\gamma$-continuous. □

Example 2.15. *Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b, d\}, \{a, b, d\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is pre-$\pi g\gamma$-continuous but not γ-irresolute.*

Corollary 2.16. *If $f : X \to Y$ is m-π-closed γ-irresolute bijection, then $f^{-1}(K)$ is $\pi g\gamma$-closed in X for each $\pi g\gamma$-closed set K of Y.*

Proof. It is obtained from Theorem 2.13 and Remark 2.14. □

For the composition of pre-$\pi g\gamma$-closed functions, we have the following Theorems.

Theorem 2.17. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $gof : X \to Z$ is pre-$\pi g\gamma$-closed if f is pre-$\pi g\gamma$-closed and g is π-irresolute pre-$\pi g\gamma$-closed bijection.*

Proof. The proof follows immediately from Theorem 2.11. □

Theorem 2.18. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $gof : X \to Z$ be pre-$\pi g\gamma$-closed. Then the following hold:*

(1) If f is a γ-irresolute surjection, then g is pre-$\pi g\gamma$-closed;

(2) If g is a m-π-closed pre-$\pi g\gamma$-continuous injection, then f is pre-$\pi g\gamma$-closed.

Proof.

(1) Let $K \in \gamma C(Y)$. Since f is γ-irresoulte and surjective, $f^{-1}(K) \in \gamma C(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Therefore, g(K) is $\pi g\gamma$-closed in Z and hence g is pre-$\pi g\gamma$-closed.

(2) Let $H \in \gamma C(X)$. Then (gof)(H) is $\pi g\gamma$-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 2.13, f(H) is $\pi g\gamma$-closed in Y and hence f is pre-$\pi g\gamma$-closed.

□

Lemma 2.19. *A surjective function $f : X \to Y$ is almost $\pi g\gamma$-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a $\pi g\gamma$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Corollary 2.20. *If a surjective function $f : X \to Y$ is almost $\pi g\gamma$-closed, then for each π-closed set K of Y and each $U \in RO(X)$ containing $f^{-1}(K)$, there exists $V \in \gamma O(Y)$ such that $K \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. The proof is similar to that of Corollary 2.8. □

Recall that a topological space (X, τ) is said to be quasi-normal [23] if for every disjoint π-closed sets A and B of X, there exist disjoint sets U, V $\in \tau$ such that A $\subseteq$ U and B $\subseteq$ V.

Definition 2.21. *A topological space (X, τ) is said to be quasi-γ-normal if for every disjoint π-closed sets A and B of X, there exist disjoint sets U, $V \in \gamma O(X)$ such that $A \subseteq U$ and $B \subseteq V$.*

Theorem 2.22. *Let $f : X \to Y$ be a π-irresolute almost $\pi g\gamma$-closed surjection. If X is quasi-normal, then Y is quasi-γ-normal.*

Proof. Let K_1 and K_2 be any disjoint π-closed sets of Y. Since f is π-irresolute, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint π-closed sets of X. By the quasi-normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(K_i) \subseteq U_i$, where i=1,2. Now, put $G_i = \mathrm{int}(\mathrm{cl}(U_i))$ for i=1, 2, then $G_i \in RO(X)$, $f^{-1}(K_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \emptyset$. By Corollary 2.20, there exists $V_i \in \gamma O(Y)$ such that $K_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, i= 1, 2. Since $G_1 \cap G_2 = \emptyset$, f is surjective we have $V_1 \cap V_2 = \emptyset$. This shows that Y is quasi-γ-normal. □

Definition 2.23 ([8]). *A function $f : X \to Y$ is said to be γ-open (resp. γ-closed), if $f(U) \in \gamma O(Y)$ (resp. $f(U) \in \gamma C(Y)$) for every open (resp. closed) set U of X.*

Definition 2.24 ([8]). *A function $f : X \to Y$ is said to be $g\gamma$-closed if $f(U)$ is $g\gamma$-closed in Y for every closed set U of X.*

The following four Corollaries are immediate consequences of Theorem 2.22.

Corollary 2.25. *If $f : X \to Y$ is a π-irresolute $\pi g\gamma$-closed surjection and X is quasi-normal, then Y is quasi-γ-normal.*

Corollary 2.26. *If $f : X \to Y$ is a π-irresolute $g\gamma$-closed surjection and X is quasi-normal, then Y is quasi-γ-normal.*

Corollary 2.27. *If $f : X \to Y$ is a π-irresolute γ-closed surjection and X is quasi-normal, then Y is quasi-γ-normal.*

Corollary 2.28. *If $f : X \to Y$ is a π-irresolute closed surjection and X is quasi-normal, then Y is quasi-γ-normal.*

Definition 2.29 ([8]). *A function $f : X \to Y$ is said to be strongly γ-closed (resp. strongly γ-open) if for each $F \in \gamma C(X)$ (resp. $F \in \gamma O(X)$), $f(F) \in \gamma C(Y)$ (resp. $f(F) \in \gamma O(Y)$).*

Remark 2.30. *Every strongly γ-closed function is γ-closed but not conversely.*

Proof. Let A be a closed set of X. Then A is γ-closed set of X. Since f is strongly γ-closed, f(A) $\in \gamma$C(Y). Hence f is γ-closed. □

Example 2.31. *Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is γ-closed but not strongly γ-closed.*

Theorem 2.32 ([18]). *Let A be a subset of a topological space X. Then*

(1) $A \in BO(X)$ if and only if $bcl(A) \in BR(X)$.

(2) $A \in BC(X)$ if and only if $bint(A) \in BR(X)$.

Theorem 2.33. *Let $f : X \to Y$ be a π-irresolute regular $\pi g\gamma$-closed surjection. If X is quasi-γ-normal, then Y is quasi-γ-normal.*

Proof. Although the proof is similar to that of Theorem 2.22, we will state it for the convenience of the reader. Let K_1 and K_2 be any disjoint π-closed sets of Y. Since f is π-irresolute, $f^{-1}(\mathrm{K}_1)$ and $f^{-1}(\mathrm{K}_2)$ are disjoint π-closed sets of X. By the quasi-γ-normality of X, there exist disjoint sets U_1, $\mathrm{U}_2 \in \gamma\mathrm{O}(\mathrm{X})$ such that $f^{-1}(\mathrm{K}_i) \subseteq \mathrm{U}_i$, for i=1,2. Now, put $\mathrm{G}_i = \gamma\mathrm{cl}(\mathrm{U}_i)$ for i=1, 2, then by Theorem 2.32, $\mathrm{G}_i \in \mathrm{BR}(\mathrm{X})$, $f^{-1}(\mathrm{K}_i) \subseteq \mathrm{U}_i \subseteq \mathrm{G}_i$ and $\mathrm{G}_1 \cap \mathrm{G}_2 = \emptyset$. By Corollary 2.8, there exists $\mathrm{V}_i \in \gamma\mathrm{O}(\mathrm{Y})$ such that $\mathrm{K}_i \subseteq \mathrm{V}_i$ and $f^{-1}(\mathrm{V}_i) \subseteq \mathrm{G}_i$, where i= 1, 2. Since f is surjective and $\mathrm{G}_1 \cap \mathrm{G}_2 = \emptyset$, we obtain $\mathrm{V}_1 \cap \mathrm{V}_2 = \emptyset$. This shows that Y is quasi-γ-normal. □

Corollary 2.34. *Let $f : X \to Y$ be a π-irresolute pre-$\pi g\gamma$-closed surjection. If X is quasi-γ-normal, then Y is quasi-γ-normal.*

Remark 2.35. *Every strongly γ-closed function is pre-$\pi g\gamma$-closed but not conversely.*

Proof. Let $\mathrm{F} \in \gamma\mathrm{C}(\mathrm{X})$. Since f is strongly γ-closed, $\mathrm{f(F)} \in \gamma\mathrm{C}(\mathrm{Y})$ and so, by Remarks 1.3 and 1.4, f(F) is $\pi g\gamma$-closed in Y. Hence f is pre-$\pi g\gamma$-closed. □

Example 2.36. *Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, Y, \{b, d\}, \{a, b, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is pre-$\pi g\gamma$-closed but not strongly γ-closed.*

Corollary 2.37. *If $f : X \to Y$ is a π-irresolute strongly γ-closed surjection and X is quasi-γ-normal, then Y is quasi-γ-normal.*

Theorem 2.38. *Let $f : X \to Y$ be a m-π-closed pre-$\pi g\gamma$-continuous injection. If Y is quasi-γ-normal, then X is quasi-γ-normal.*

Proof. Let H_1 and H_2 be any disjoint π-closed sets of X. Since f is a m-π-closed injection, $f(\mathrm{H}_1)$ and $f(\mathrm{H}_2)$ are disjoint π-closed sets of Y. By the quasi-γ-normality of Y, there exist disjoint sets V_1, $\mathrm{V}_2 \in \gamma\mathrm{O}(\mathrm{Y})$ such that $f(\mathrm{H}_i) \subseteq \mathrm{V}_i$, for i=1,2. Since f is pre-$\pi g\gamma$-continuous $f^{-1}(\mathrm{V}_i)$ and $f^{-1}(\mathrm{V}_i)$ are disjoint $\pi g\gamma$-open sets of X and $\mathrm{H}_i \subseteq f^{-1}(\mathrm{V}_i)$ for i=1,2. Now, put $\mathrm{U}_i = \gamma\mathrm{int}(f^{-1}(\mathrm{V}_i))$ for i=1,2. Then $\mathrm{U}_i \in \gamma\mathrm{O}(\mathrm{X})$, $\mathrm{H}_i \subseteq \mathrm{U}_i$ and $\mathrm{U}_1 \cap \mathrm{U}_2 = \emptyset$. This shows that X is quasi-γ-normal. □

Corollary 2.39. *If $f : X \to Y$ is a m-π-closed γ-irresolute injection and Y is quasi-γ-normal, then X is quasi-γ-normal.*

Proof. This is an immediate consequence of Theorem 2.38, since every γ-irresoulte function is pre-$\pi g\gamma$-continuous. □

Definition 2.40. *A topological space X is said to be quasi-regular if for each π-closed set F and each point $x \in X - F$, there exist disjoint U, $V \in \tau$ such that $x \in U$ and $F \subseteq V$.*

Theorem 2.41. *For a topological space X, the following properties are equivalent:*

(1) X is quasi-regular;

(2) For each π-open set U in X and each $x \in U$, there exists $V \in \tau$ such that $x \in V \subseteq cl(V) \subseteq U$;

(3) For each π-open set U in X and each $x \in U$, there exists a clopen set V such that $x \in V \subseteq U$.

Proof. (1) $\Rightarrow$ (2): Let U be an π-open set of X containing x. Then X\U is a π-closed set not containing x. By (1), there exist disjoint X\cl(V), $V \in \tau$ such that $x \in V$ and $X\backslash U \subseteq X\backslash cl(V)$. Then we have $V \in \tau$ such that $x \in V \subseteq cl(V) \subseteq U$.
(2) $\Rightarrow$ (3): Let U be an π-open set of X containing x. By (2), there exists $V \in \tau$ such that $x \in V \subseteq cl(V) \subseteq U$. Take V = cl(V). Thus V is closed and so V is clopen. Hence we have V is clopen set such that $x \in V \subseteq U$.
(3) $\Rightarrow$ (1): Let $F = X\backslash U$ be a π-closed set not containing x. Then U is an π-open set of X containing x. By (3), there exists a clopen set V such that $x \in V \subseteq U$. Then there exist disjoint $G = X\backslash V$, $V \in \tau$ such that $x \in V$ and $F = X\backslash U \subseteq G = X\backslash V$. Hence X is quasi-regular. □

Definition 2.42. *A topological space X is said to be quasi-γ-regular if for each π-closed set F and each point $x \in X-F$, there exist disjoint U, $V \in \gamma O(X)$ such that $x \in U$ and $F \subseteq V$.*

Theorem 2.43. *For a topological space X, the following properties are equivalent:*

(1) X is quasi-γ-regular;

(2) For each π-open set U in X and each $x \in U$, there exists $V \in \gamma O(X)$ such that $x \in V \subseteq \gamma cl(V) \subseteq U$;

(3) For each π-open set U in X and each $x \in U$, there exists $V \in BR(X)$ such that $x \in V \subseteq U$.

Theorem 2.44. *Let $f : X \rightarrow Y$ be a π-irresolute γ-open almost $\pi g\gamma$-closed surjection. If X is quasi-regular, then Y is quasi-γ-regular.*

Proof. Let $y \in Y$ and V be an π-open neighbourhood of y. Take a point $x \in f^{-1}(y)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is π-open in X. By the quasi-regularity of X, there exists an π-open set U of X such that $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(cl(U)) \subseteq V$. Also, since U is open set of X and f is γ-open, $f(U) \in \gamma O(Y)$. Moreover, since U is β-open, by Theorem 1.6, cl(U) is regular closed set of X. Since f is almost $\pi g\gamma$-closed, f(cl(U)) is $\pi g\gamma$-closed in Y. Therefore, we obtain $y \in f(U) \subseteq \gamma cl(f(U)) \subseteq \gamma cl(f(cl(U))) \subseteq V$. It follows from Theorem 2.43 that Y is quasi-γ-regular. □

Corollary 2.45. *If $f : X \rightarrow Y$ is a π-irresolute γ-open $\pi g\gamma$-closed surjection and X is quasi-regular, then Y is quasi-γ-regular.*

Corollary 2.46. *If $f : X \rightarrow Y$ is a π-irresolute γ-open γ-closed surjection and X is quasi-regular, then Y is quasi-γ-regular.*

Theorem 2.47. *Let $f : X \rightarrow Y$ be a π-irresolute strongly γ-open regular $\pi g\gamma$-closed surjection. If X is quasi-γ-regular, then Y is quasi-γ-regular.*

Proof. Let F be any π-closed set of Y and $y \in Y - F$. Then $f^{-1}(F)$ is π-closed in X and $f^{-1}(F) \cap f^{-1}(y) = \emptyset$. Take a point $x \in f^{-1}(y)$. Since X is quasi-γ-regular, there exist disjoint sets U_1, $U_2 \in \gamma O(X)$ such that $x \in U_1$ and $f^{-1}(F) \subseteq U_2$. Therefore, we have $f^{-1}(F) \subseteq U_2 \subseteq \gamma cl(U_2)$, by Theorem 2.32, $\gamma cl(U_2) \in BR(X)$ and $U_1 \cap \gamma cl(U_2) = \emptyset$. Since f is regular $\pi g\gamma$-closed, by Corollary 2.8, there exists $V \in \gamma O(Y)$ such that $F \subseteq V$ and $f^{-1}(V) \subseteq \gamma cl(U_2)$. Since f is strongly γ-open, we have $f(U_1) \in \gamma O(Y)$. Moreover, $U_1 \cap f^{-1}(V) = \emptyset$ and hence $f(U_1) \cap V = \emptyset$. Consequently, we obtain $y \in f(U_1) \in \gamma O(Y)$, $F \subseteq V \in \gamma O(Y)$ and $f(U_1) \cap V = \emptyset$. This shows that Y is quasi-γ-regular. □

Corollary 2.48. *If $f : X \to Y$ is a π-irresolute strongly γ-open pre-$\pi g\gamma$-closed surjection and X is quasi-γ-regular, then Y is quasi-γ-regular.*

References

[1] M.E.Abd El-Monsef, S.N.El-Deeb and R.A.Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ., 12(1983), 77-90.

[2] M.E.Abd El-Monsef, A.N.Geaisa and R.A.Mahmoud, *β-regular spaces*, Proc. Math. Phys. Soc. Egypt., 60(1985), 47-52.

[3] D.Andrijevic, *On b-open sets*, Mat. Vesnik, 48(1-2)(1996), 59-64.

[4] D.Andrijevic, *Semi-preopen sets*, Mat. Vesnik, 38(1986), 24-32.

[5] S.P.Arya and T.M.Nour, *Characterizations of s-normal spaces*, Indian J. Pure Appl. Math., 21(8)(1990), 717-719.

[6] R.Devi, K.Balachandran and H.Maki, *Semi-generalized closed maps and generalized semi-closed maps*, Mem. Fac. Kochi Univ. Ser. A. Math., 14(1993), 41-54.

[7] J.Dontchev and M.Przemski, *On the various decompositions of continuous and some weakly continuous functions*, Acta Math. Hungar., 71(1-2)(1996), 109-120.

[8] E.Ekici, *On γ-normal spaces*, Bull. Math. Soc. Sci. Math. Roumanie (N. S), 50(98)(2007), 259-272.

[9] E.Ekici and M.Caldas, *Slightly γ-continuous functions,* Bol. Soc. Paran. Mat., (3s.)22(2)(2004), 63-74.

[10] E.Ekici and C.W.Baker, *On πg-closed sets and continuity*, Kochi J. Math., 2(2007), 35-42.

[11] A.A.El-Atik, *A study on some types of mappings on topological spaces*, Master's Thesis, Tanta University, Egypt, (1997).

[12] N.Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, 19(2)(1970), 89-96.

[13] S.N.Maheshwari and R.Prasad, *On s-normal spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, 22(70)(1978), 27-29.

[14] R.A.Mahmoud and M.E.Abd El-Monsef, *β-irresolute and β-topological invariant*, Proc. Math. Pakistan. Acad. Sci., 27(1990), 285-296.

[15] S.R.Malghan, *Generalized closed maps*, J. Karnataka Univ. Sci., 27(1982), 82-88.

[16] T.Noiri, H.Maki and J.Umehara, *Generalized preclosed functions*, Mem. Fac. Sci. Kochi Univ. Ser. A Math., 19(1998), 13-20.

[17] T.M.J.Nour, *Contributions to the theory of bitopological spaces*, Ph. D Thesis, Delhi University, India, (1989).

[18] J.H.Park, *Strongly θ-b-continuous functions,* Acta Math. Hungar., 110(4)(2006), 347-359.

[19] Paul and Bhattacharyya, *On p-normal spaces*, Soochow J. Math., 21(3)(1995), 273-289.

[20] D.Sreeja and C.Janaki, *On πgb-closed sets in topological spaces*, International Journal of Mathematical Archive, 2(8)(2011), 1314-1320.

[21] M.H.Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., 41(1937), 375-481.

[22] S.Tahiliani, *Generalized β-closed functions*, Bull. Cal. Math. Soc., 98(4)(2006), 367-376.

[23] V.Zaitsev, *On certain classes of topological spaces and their bicompactifications*, Dokl. Akad. Nauk. SSSR, 178(1968), 778-779.