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b-chromatic Number for the Graphs Obtained by Duplicating Edges

Research Article

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Abstract: A b-colouring of a graph G is a proper vertex colouring of G such that each colour class contains a vertex that has atleast one neighbour in every other colour class and b-chromatic number of a graph G is the largest integer $\phi(G)$ for which G has a b-colouring with $\phi(G)$ colours. In this paper, we have obtained the b-chromatic number of the graphs E_n, F_n and the graphs obtained by duplicating all the edges of path, cycle, complete graph, wheel graph, Ladder graph L_n by vertices.

MSC: 05C15, 05C38.

Keywords: b-colouring, b-chromatic number.

1. Introduction

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A proper k-colouring of graph G is a function C defined from $V(G)$ onto a set of colours $\{1, 2, \ldots, k\}$ such that any two adjacent vertices have different colours. For every i, $1 \leq i \leq k$, the set C_i is an independent set of vertices which is called as the colour class of the colour i. Let P_n be a path with n vertices and $n-1$ edges. Let C_n be a cycle with n vertices and n edges.

For $n \geq 2$, E_n denotes a graph consisting of $(n-1)$ 3-sided faces, $(n-1)$ 5-sided faces and one external infinite face, embedded in the plane and labeled as in Figure 1 [3]. For $n \geq 2$, F_n denotes a graph consisting of $(n-1)$ 3-sided faces,

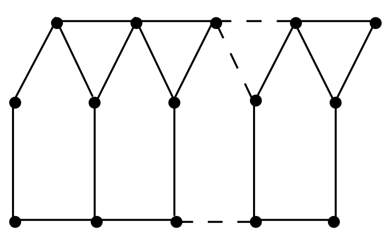

Figure 1.

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$(n-1)$ 5-sided faces, $(n-1)$ 6-sided faces and one external infinite face, embedded in the plane and labeled as in Figure 2 [3].

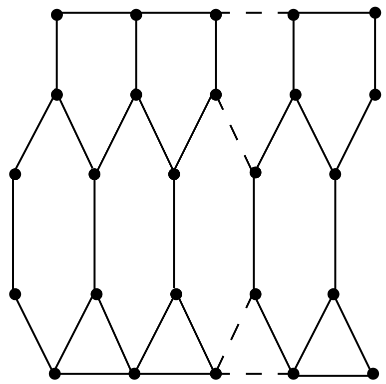

Figure 2.

The Ladder graph L_n is $P_2 \times P_n$. Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$ [8]. We denote the graph obtained by duplicating all the edges of G by vertices as $EV(G)$.

The b-chromatic number of a graph was introduced by R. W. Irving and D. F. Manlove when considering minimal proper colouring with respect to a partial order defined on the set of all partition of vertices of graph. The b-chromatic number of a graph G, denoted by $\phi(G)$ is the largest positive integer t such that there exists a proper colouring for G with t colours in which every colour class contains at least one vertex adjacent to some vertex in all the other colour classes. Such a colouring is called a b-colouring.

In [7], K. Thilagavathi et. al. obtained the b-colouring of the central graphs of path, cycle and complete bipartite graph. Motivated by these works, we have obtained the b-chromatic number of the graphs E_n, F_n and the graphs obtained by duplicating all the edges of path, cycle, complete graph, wheel graph, Ladder graph L_n by vertices.

2. Main Results

Proposition 2.1. $\phi(E_n) = \begin{cases} 3, & n = 1 \\ 4, & 2 \leq n \leq 5 \\ 5, & n \geq 6 \end{cases}$

Proof. In E_n, $deg(v) \leq 3$ when $n = 1$ and $deg(v) \leq 4$ otherwise. So $\phi(E_n) \leq 4$ when $n = 1$ and $\phi(E_n) \leq 5$ otherwise. When $n \geq 6$, at least 5 vertices are of degree 4. So $\phi(E_n) \leq 5$.

Let $u_1, u_2, \ldots, u_{n+1}$ and $v_1, v_2, \ldots, v_{n+1}$ be the vertices on the path of length n and $x_1, x_2, \ldots, x_{n+1}$ be the remaining vertices of E_n so that $x_iu_i, x_iv_i \in E(E_n), i = 1, 2, \ldots, n+1$ and $u_ix_{i+1} \in E(E_n), i = 1, 2, \ldots, n$. Assign the colours for the vertices of E_n as follows:

$$C(u_i) = i(mod\ 5), \quad 1 \leq i \leq n+1$$

$$C(x_i) = (i+2)(mod\ 5), \quad 1 \leq i \leq n+1 \text{ and}$$

$$C(v_i) = (i+3)(mod\ 5), \quad 1 \leq i \leq n+1.$$

Then u_2, u_3, u_4, u_5 and u_6 are the members of the colour classes $1, 2, 3, 4$ and 0 respectively in which they are adjacent to at least one member of all the remaining colour classes. Thus $\phi(E_n) = 5$ for $n \geq 6$.

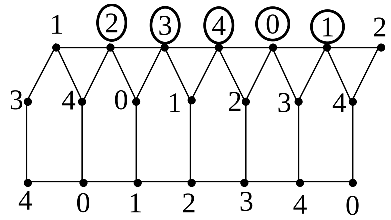

Figure 3. **A b-colouring of E_6 with $\phi = 5$.**

When $2 \leq n \leq 5$, only $n - 1$ vertices are of degree 4. So $\phi(E_n) < 5$ when $2 \leq n \leq 5$. In this case, assign the colours for the vertices of E_n as follows:

For $1 \leq i \leq n+1$,

$$C(u_i) = \begin{cases} 0, & i \text{ is odd} \\ 1, & i \text{ is even} \end{cases}$$

$$C(x_i) = \begin{cases} 3, & i \text{ is odd} \\ 2, & i \text{ is even} \end{cases}$$

$$\text{and } C(v_i) = \begin{cases} 2, & i \text{ is odd} \\ 3, & i \text{ is even.} \end{cases}$$

Then u_1, u_2, x_2 and x_3 are the members of colour classes $0, 1, 2$ and 3 respectively with the required property. Thus $\phi(E_n) = 4$ for $2 \leq n \leq 5$. □

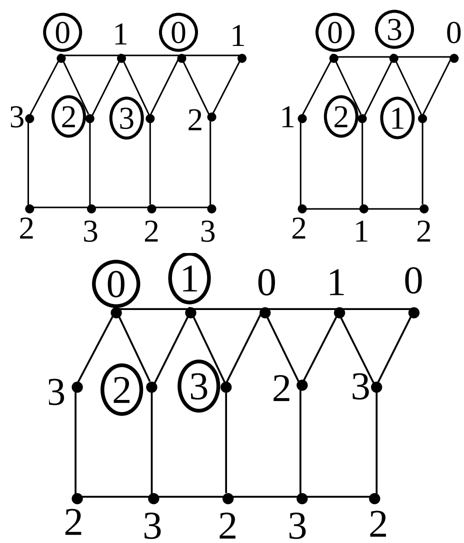

Figure 4.

Proposition 2.2. $\phi(F_n) = \begin{cases} 4, & 1 \leq n \leq 5 \\ 5, & n \geq 6 \end{cases}$

Proof. In F_n, $deg(v) \leq 3$ when $n = 1$ and $deg(v) \leq 4$ otherwise. So $\phi(F_n) \leq 4$ when $1 \leq n \leq 5$ and $\phi(F_n) \leq 5$ otherwise. When $n \geq 6$, at least 5 vertices are of degree 4. So $\phi(F_n) \leq 5$.

Let $u_1, u_2, \ldots, u_{n+1}$ and $v_1, v_2, \ldots, v_{n+1}$ be the vertices on the paths of length n. Let $x_i, y_i, z_i, 1 \leq i \leq n+1$ be the vertices so that $u_i x_i, x_i y_i, y_i z_i, z_i v_i \in E(F_n), 1 \leq i \leq n+1$ and $x_i y_{i+1}, v_i z_{i+1} \in E(F_n), 1 \leq i \leq n$. When $n \geq 6$, assign the colours to the vertices of F_n as follows:

For $1 \leq i \leq n+1$,

$$C(u_i) = (i-1)(mod\ 5),$$

$$C(x_i) = (i+1)(mod\ 5),$$

$$C(y_i) = (i+2)(mod\ 5),$$

$$C(z_i) = (i+1)(mod\ 5) \text{ and}$$

$$C(v_i) = (i-1)(mod\ 5).$$

Then v_2, v_3, v_4, v_5 and v_6 are the members of the colour classes $1, 2, 3, 4$ and 0 respectively in which they are having all the remaining colours as neighbouring colours. Thus $\phi(F_n) = 5$ for $n \geq 6$.

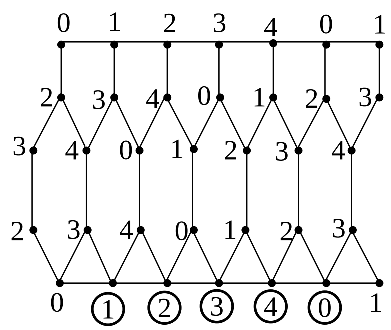

Figure 5. **A b-colouring of F_6 with $\phi = 5$.**

When $1 \leq n \leq 5$, only $n-1$ vertices are of degree 4. So $\phi(F_n) < 5$. By assigning the colours as

$$C(u_i) = \begin{cases} 0, & i \text{ is odd} \\ 1, & i \text{ is even} \end{cases}$$

$$C(x_i) = \begin{cases} 3, & i \text{ is odd} \\ 0, & i \text{ is even} \end{cases}$$

$$C(y_i) = \begin{cases} 2, & i \text{ is odd} \\ 1, & i \text{ is even} \end{cases}$$

$$C(z_i) = \begin{cases} 1, & i \text{ is odd} \\ 2, & i \text{ is even} \end{cases}$$

$$\text{and } C(v_i) = \begin{cases} 0, & i \text{ is odd} \\ 3, & i \text{ is even,} \end{cases}$$

for each $i, 1 \leq i \leq n$, the vertices x_1, y_2, z_2 and v_1 are the members of the colour classes $3, 1, 2$ and 0 respectively with the required property. This implies that $\phi(F_n) = 4$ for $1 \leq n \leq 5$.

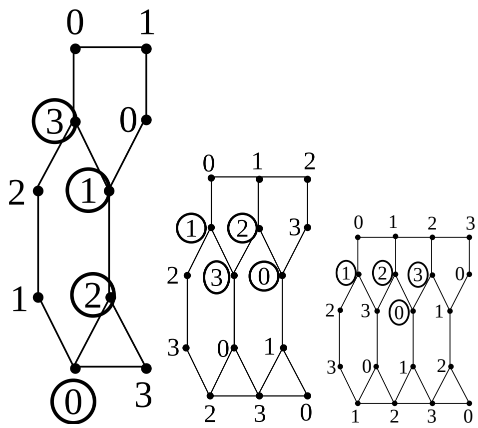

Figure 6. **A b-colouring of F_n, $1 \leq n \leq 5$ with $\phi = 4$.**

□

Proposition 2.3. $\phi(EV(P_n)) = \begin{cases} 3, & 2 \leq n \leq 5 \\ 4, & n = 6 \\ 5, & n \geq 7 \end{cases}$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on the path and $x_1, x_2, \ldots, x_{n-1}$ be the vertices corresponding to the edges of P_n so that $x_iv_i, x_iv_{i+1} \in E(EV(P_n)), 1 \leq i \leq n-1$. When $n \geq 7$, at least five vertices are of degree $\Delta = 4$ and hence $\phi(EV(P_n)) \leq 5$. Colour the vertices as follows:

$$C(v_i) = (i+3)(mod\ 5), \quad 1 \leq i \leq n \text{ and}$$
$$C(x_i) = (i+1)(mod\ 5), \quad 1 \leq i \leq n-1.$$

Then v_2, v_3, v_4, v_5 and v_6 are the members of the colour classes $0, 1, 2, 3$ and 4 respectively so that each one having all the remaining colours in its neighbours. Thus $\phi(EV(P_n)) = 5$ for $n \geq 7$.

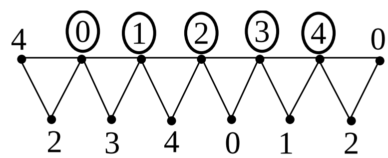

Figure 7. **A b-colouring of $EV(P_7)$ with $\phi = 5$.**

When $n = 6$, only 4 vertices are of degree $\Delta = 4$ and hence $\phi(EV(P_n)) < 5$. Colour the vertices as follows:

$$C(v_i) = (i + 2)(mod\ 4), \quad 1 \leq i \leq n \text{ and}$$
$$C(x_i) = (i + 1)(mod\ 4), \quad 1 \leq i \leq n - 1.$$

Then v_2, v_3, v_4 and v_5 are the members of the colour classes $0, 1, 2$ and 3 respectively with the required property. Hence $\phi(EV(P_n)) = 4$.

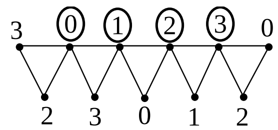

Figure 8. **A b-colouring of $EV(P_6)$ with $\phi = 4$.**

When $3 \leq n \leq 5$, only $n-2$ vertices are of degree $\Delta = 4$ and no vertex is of degree 3. So $\phi(EV(P_n)) < 4$. Colour the vertices as follows:

$$C(v_i) = (i - 1)(mod\ 3), \quad 1 \leq i \leq n \text{ and}$$
$$C(x_i) = (i + 1)(mod\ 3), \quad 1 \leq i \leq n - 1.$$

Then v_1, v_2 and v_3 are the members of the colour classes $0, 1$ and 2 respectively with the required property.

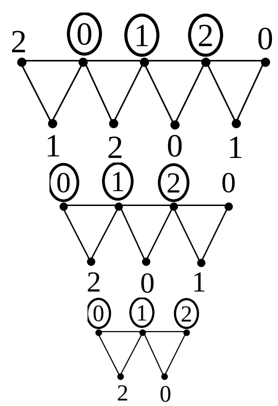

Figure 9. **A b-colouring of $EV(P_n), 3 \leq n \leq 5$ with $\phi = 3$.**

While $n = 2$, $EV(P_n) = K_3$ and $\phi(K_3) = 3$. Hence $\phi(EV(P_n)) = 3$ for $2 \leq n \leq 5$.

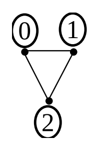

Figure 10. **A b-colouring of $EV(P_2)$ with $\phi = 3$.**

□

Proposition 2.4. *For any $n \geq 3$,*

$$\phi(EV(C_n)) = \begin{cases} 3, & n = 3 \\ 4, & n = 4 \\ 5, & n \geq 5 \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on the cycle and $x_1, x_2, \ldots, x_n$ be the vertices corresponding to the cycles of C_n so that $v_i x_i, x_i v_{i+1} \in E(EV(C_n)), 1 \leq i \leq n$ where $v_{n+1} = v_1$. When $n \geq 5$, at least five vertices are of degree $\Delta = 4$. So $\phi(EV(C_n)) \leq 5$. For $n \geq 7$, assign the colours $4, 0, 1, 2, 3, 4, 0$ to the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ respectively and $2, 3, 4, 0, 1, 2$ to the vertices $x_1, x_2, x_3, x_4, x_5, x_6$ respectively and colour the remaining vertices with a proper colouring using the colours $0, 1, 2, 3$ and 4, the vertices v_2, v_3, v_4, v_5 and v_6 are the members of the colour classes $0, 1, 2, 3, 4$ respectively in which they are adjacent to at least one number of the remaining colour classes. Hence $\phi(EV(C_n)) = 5$ for $n \geq 7$. The b-colouring of $EV(C_5)$ and $EV(C_6)$ are given in Figure 11. So that $\phi(EV(C_5)) = \phi(EV(C_6)) = 5$. When $n = 3, 4$ the

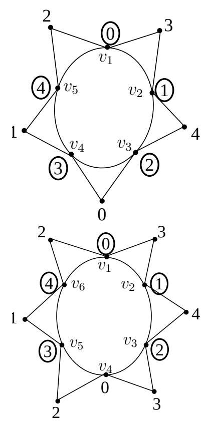

Figure 11. **A b-colouring of $EV(C_n), n = 5, 6$ with $\phi = 5$.**

number of vertices of degree $\Delta = 4$ is n and hence $\phi(EV(C_n)) < 5$. Also the number of vertices of degree $\Delta = 4$ in $EV(C_3)$ is 3. So $\phi(EV(C_3)) < 4$ and $\phi(EV(C_4)) < 5$. The b-colouring of $EV(C_3)$ and $EV(C_4)$ are given in Figure 12 so that $\phi(EV(C_3)) = 3$ and $\phi(EV(C_4)) = 4$.

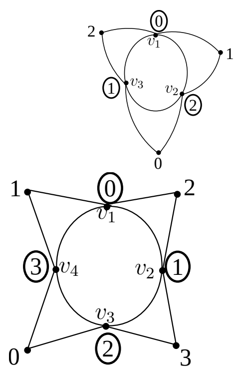

Figure 12. **A b-colouring of $EV(C_n), n = 3, 4$.**

□

Proposition 2.5. *For any $n \geq 1$, $\phi(EV(K_n)) = n$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n in $EV(K_n)$ and $x_1, x_2, \ldots, x_m$, $m = \binom{n}{2}$, be the vertices corresponding to the edges of K_n. Since $EV(K_n)$ has n vertices of degree $\Delta = 2n-2$, $\phi(EV(K_n)) \leq n$.

By assigning the colours $0, 1, 2, \ldots, (n-1)$ to the vertices $v_1, v_2, \ldots, v_n$ respectively and giving the proper colouring to the remaining vertices, it follows that $\phi(EV(K_n)) = n$.

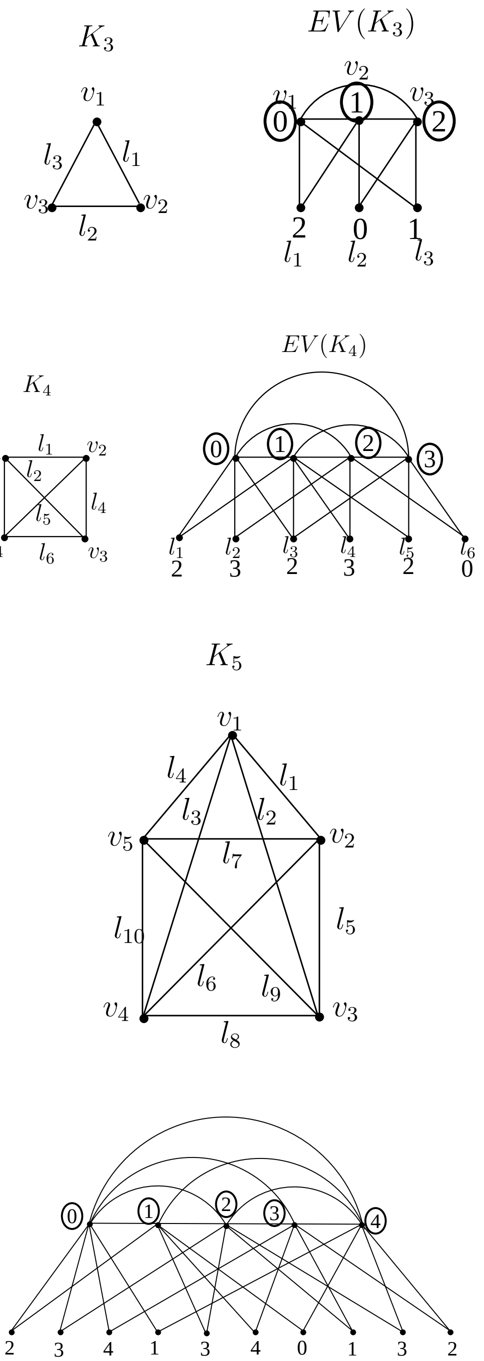

Figure 13. **A b-colouring of $EV(K_5)$ with $\phi = 5$.**

□

Proposition 2.6. *For any $n \geq 2$, $\phi(EV(K_{1,n})) = 3$.*

Proof. Let v_0 be the central vertex and $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. Let x_i be the vertex corresponding to the edge $v_0v_i, 1 \leq i \leq n$ in $EV(K_{1,n})$. In $EV(K_{1,n}), 2n$ vertices are of degree 2 and v_0 is the only vertex with degree $2n$. Hence $\phi(EV(K_{1,n})) \leq 3$. By assigning the colour 0 to v_0, 1 to v_i's and 2 for all x_i's, the result follows:

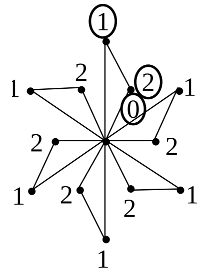

Figure 14. **A b-colouring of $EV(K_{1,6})$ with $\phi = 3$.**

□

Proposition 2.7. $\phi(EV(W_n)) = \begin{cases} 7, & \textit{for } n \geq 6 \\ n+1, & \textit{for } 3 \leq n \leq 5 \end{cases}$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on the cycle and v_0 be the central vertex of W_n. Let x_i be the vertex corresponding to the edge $v_iv_{i+1}, 1 \leq i \leq n-1$, x_n be the vertex corresponding to the edge v_nv_1 and $x_{n+i}, 1 \leq i \leq n$ be the vertex corresponding to the edge v_0v_i in $EV(W_n)$. Assume that $n \geq 6$. In $EV(W_n)$, one vertex namely v_0 is of degree $2n$, n vertices are of degree 6 and the remaining $2n$ vertices are of degree 2. So $\phi(EV(W_n)) \leq 7$. Colour the vertex v_0 by $0, v_i, 1 \leq i \leq 6$ by i and the remaining v_i's by the sequence of colours $3, 4, 5, 6, 3, 4, 5, 6, \ldots,$ colour the vertices $x_n, x_1, x_2, \ldots, x_{n-1}$ by $5, 6, 5, 6, 1, 2, 1, 2, \ldots, 1, 2$, while n is even and $3, 4, 5, 6, 1, 2, 1, 2, \ldots, 2, 1$ while n is odd. Colour the vertices $x_{n+1}, x_{n+2}, \ldots, x_{n+8}$ by $5, 6, 1, 2, 3, 4, 5, 6$ while n is even and $4, 4, 1, 2, 3, 4, 5, 6$ while n is odd and the remaining x_i's, $n+9 \leq i \leq 2n$ are assigned by a proper colouring. Then $v_0, v_1, \ldots, v_6$ be the members of the respective colour classes of the colours $0, 1, 2, 3, 4, 5, 6$ so that it has exactly one neighbour in the remaining colour classes. Therefore $\phi(EV(W_n)) = 7$ for $n \geq 6$.

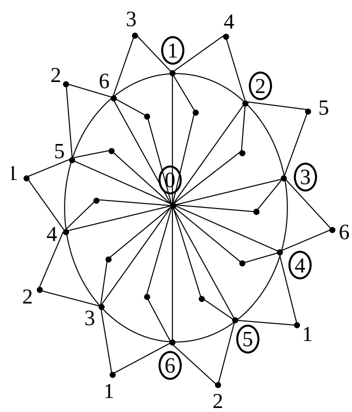

Figure 15. **A b-colouring of $EV(W_{10})$ with $\phi = 6$.**

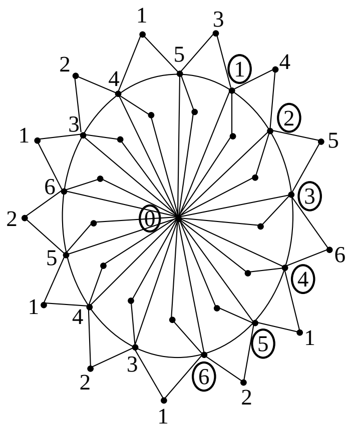

Figure 16. **A b-colouring of $EV(W_{13})$ with $\phi = 6$.**

When $3 \leq n \leq 5$ the number of vertices with degree 6 is n and one vertex is of degree $2n$. So $\phi(EV(W_n)) \leq n + 1$. A b-colouring for $EV(W_n), 3 \leq n \leq 5$ is shown in Figure 17. Hence $\phi(EV(W_n)) = n + 1$, for $3 \leq n \leq 5$.

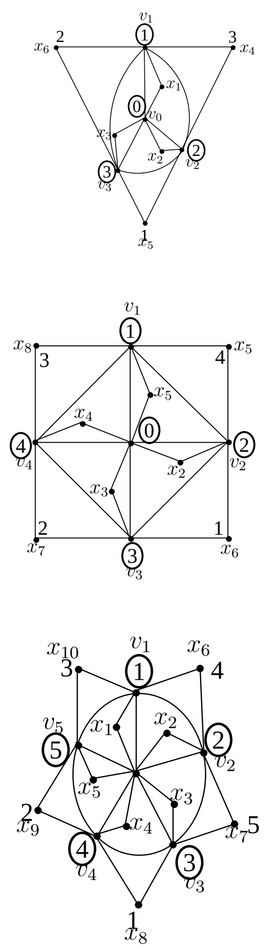

Figure 17. **A b-colouring of $EV(W_n), 3 \leq n \leq 5$ with $\phi = n + 1$.**

□

Proposition 2.8. $\phi(EV(L_n)) = \begin{cases} 7, & for\ n \geq 6 \\ 6, & for\ n = 5 \\ 5, & for\ n = 3, 4 \\ 4, & for\ n = 2. \end{cases}$

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the path of length $n-1$. Let x_i and y_i be the duplicating vertices of the edges u_iu_{i+1} and v_iv_{i+1} respectively, $1 \leq i \leq n-1$ and z_i be the duplicating vertex of the edge $u_iv_i, 1 \leq i \leq n$. When $n \geq 3$, the maximum degree of $EV(L_n)$ is 6 and the number of vertices having the degree 6 is $2(n-1)$. Hence $\phi(EV(L_n)) \leq 7$. Assume that $n \geq 9$. Assign the colours to the vertices as follows:

$$C(u_i) = (i+5)(mod\ 7), \quad 1 \leq i \leq n$$

$$C(v_i) = (i+2)(mod\ 7), \quad 1 \leq i \leq n$$

$$C(x_i) = (i+1)(mod\ 7), \quad 1 \leq i \leq n-1$$

$$C(y_i) = C(x_i), \quad 1 \leq i \leq n-1 \text{ and}$$

$$C(z_i) = (i+3)(mod\ 7), \quad 1 \leq i \leq n.$$

By assigning these colours, the vertices $v_2, v_3, \ldots, v_8$ are the members of the respective colour classes $0, 1, 2, \ldots, 7$ in which they are adjacent to at least one member of all the remaining colour classes. Hence $\phi(EV(L_n)) = 7$ for all $n \geq 9$.

When $6 \leq n \leq 8$, at least 7 vertices are of degree 6 and the b-colouring for these values of n are given in Figure 18. Hence $\phi(EV(L_n)) = 6, 6 \leq n \leq 8$.

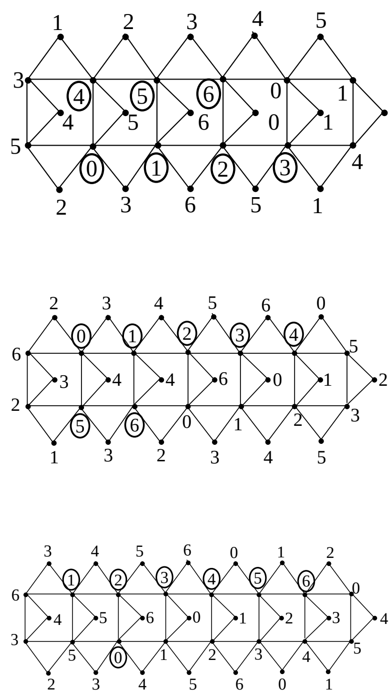

Figure 18. **A b-colouring of $EV(L_n), 6 \leq n \leq 8$.**

When $n = 5$, since there are only 6 vertices are of degree 6, $\phi(EV(L_n)) < 7$. A b-colouring with 6 colours for $EV(L_5)$ is given in Figure 19. Hence $\phi(EV(L_5)) = 6$.

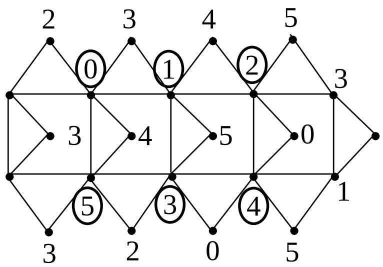

Figure 19. **A b-colouring of $EV(L_5)$ with $\phi = 6$.**

When $n = 4(or\ 3), 4(or\ 2)$ vertices are having degree 6 and 4 vertices are of degree 4. So b-colouring with 6 colours is not possible. A b-colouring with 5 colours is given in Figure 20. Hence $\phi(EV(L_n)) = 5, n = 3, 4$.

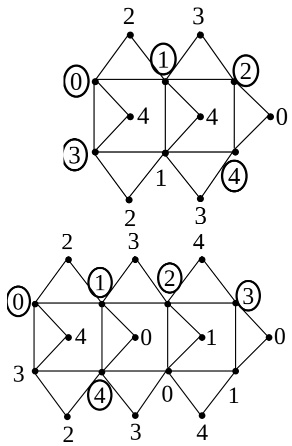

Figure 20. **A b-colouring of $EV(L_n), n = 3, 4$ with $\phi = 5$.**

When $n = 2$, 4 vertices are having the maximum degree 4. Hence $\phi(EV(L_2)) \leq 4$. A b-colouring with 4 colours is given in Figure 21.

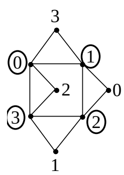

Figure 21. **A b-colouring of $EV(L_2)$ with $\phi = 4$.**

□

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