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Perov Type Results in Gauge Spaces and its Applications to Systems of Integral Equations

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Abstract: In this paper we present Perov type fixed point theorems for contractive mappings in Gheorghiu's sense on spaces endowed with a family of vector valued pseudo-metrics. Applications to systems of integral equations are given to illustrate the theory. The examples also prove the advantage of using vector valued pseudo-metrics and matrices that are convergent to zero, for the study of systems of equations.

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1. Introduction

In this paper we are concerning with the solvability of the semilinear operator system

$$\left.\begin{aligned} A_1(x,y,z,w) &= x \\ A_2(x,y,z,w) &= y \\ A_3(x,y,z,w) &= z \\ A_4(x,y,z,w) &= w \end{aligned}\right\} \tag{1}$$

in a complete gauge space X (space endowed with a family of pseudo-metrics). Here $A_1, A_2, A_3, A_4 : X^4 \to X$ are given non-linear operators. Systems of this type arise from mathematical modelling of many interaction, competitive and cooperative processes from a variety of disciplines, including physics, biology, chemistry, engineering and other sciences. For instance, the system

$$\left.\begin{aligned} x(t) &= \int_{t-\tau_1}^{t} f_1(s, x(\sigma_1(s)), y(\sigma_2(s)), z(\sigma_3(s)), w(\sigma_1(s)))ds \\ y(t) &= \int_{t-\tau_2}^{t} f_2(s, x(\sigma_1(s)), y(\sigma_2(s)), z(\sigma_3(s)), w(\sigma_1(s)))ds \\ z(t) &= \int_{t-\tau_3}^{t} f_3(s, x(\sigma_1(s)), y(\sigma_2(s)), z(\sigma_3(s)), w(\sigma_1(s)))ds \\ w(t) &= \int_{t-\tau_4}^{t} f_1(s, x(\sigma_1(s)), y(\sigma_2(s)), z(\sigma_3(s)), w(\sigma_1(s)))ds \end{aligned}\right\} \tag{2}$$

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is a mathematical model for the spread of two interacted infectious diseases with contact rates that vary seasonally. In these equations $x(t), y(t), z(t), w(t)$ represent the proportion of infectives in a population at time t, for each of the four epidemics; $\tau_1, \tau_2, \tau_3, \tau_4$ stand for the length of time an individual remains infectious of each one of the diseases; and f_1, f_2, f_3, f_4 are the proportion of new infectives per unit time for the four epidemics. The modified arguments $\sigma_1(t), \sigma_2(t), \sigma_3(t), \sigma_4(t)$ can be of retarded type, when $\sigma_1(t), \sigma_2(t), \sigma_3(t), \sigma_4(t) \leq t$, or of advanced type, if $\sigma_1(t), \sigma_2(t), \sigma_3(t), \sigma_4(t) \geq t$. For only one disease, and without argument deviations, such a model was introduced by Cooke and Kaplan [2] (see also Precup [16]).

It is obvious that system 1 can be viewed as a fixed point problem,

$$T(u) = u \tag{3}$$

in the space X^4, where $u = (x, y, z, w)$ and $T = (A_1, A_2, A_3, A_4)$. Therefore, we may think to apply to 3, in X^4 endowed with the gauge structure induced by that of X, different abstract existences results from the theory of nonlinear operators on gauge spaces. Such a result is the contraction principle extended to gauge spaces by Colojoara (1961) [1], Gheorghiu (1967) [5] and Tarafdar (1974) [23]. However, as pointed out by Perov and Kibenko [15] in connection with Banach's contraction principle and Precup [17] Mishra et al., [9] and cited their in, for other abstract principles (Schauder's, Leray-Schauder's and Krasnoselskii's cone theorems), we may expect that better results can be obtained for system 1 if X^4 is endowed with a family of vector-valued pseudo-metrics. Of course, in this situation the contraction condition has to be expressed in terms of a matrix instead of scalar Lipschitz constants allowing the two mappings A_1, A_2, A_3 and A_4 to satisfy more relaxed Lipschitz conditions.

Our first goal in this paper is to present Perov type fixed point theorems for contractive mappings in Gheorghiu's sense on spaces endowed with a family of vector-valued pseudo-metrics. Then we present applications to system 2 with parameter standardization $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1$ in two cases:

(a). for advanced arguments $\tau_1 = \tau_2 = \tau_3 = \tau_4 = t + 4$,

(b). for unmodified arguments $\tau_1 = \tau_2 = \tau_3 = \tau_4 = t$.

The use of a gauge structure is motivated by our interest in discussing long term behaviour of the system, i.e., $t \in [0, \infty)$, while the advanced arguments in the first example lead to Gheorghiu's contraction notion. Our abstract results are new and complement the existing literature in fixed point theory in gauge/uniform spaces. In addition, compared to previous applications in Precup [18] and Precup-Viorel [19], our new applications give to the vector approach a new asset for its use in the treatment of systems.

In order to make clear the connection of our results to the existing literature, we conclude this introductory section recalling some definitions and results (details can be found in Precup [16]). By a vector-valued metric on a set X one means a map $d : X \times X \to \mathbb{R}^n$ with the following properties: $d(x, y) \geq 0$ for all $x, y \in X$ and if $d(x, y) = 0$ then $x = y$; $d(x, y) = d(y, x)$ for all $x, y \in X$; $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Here, if $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, then by $a \leq b$ we mean that $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. A set X endowed with a vector-valued metric d is said to be a generalized metric space. For the generalized metric spaces, the notions of a convergent sequence, Cauchy sequence and completeness are similar to those for usual metric spaces.

Let (X, d) be a generalized metric space. A map $T : X \to X$ is said to be a generalized contraction if there exists a matrix $M \in \mathbb{M}_{n\times n}(\mathbb{R}_+)$ such that

$$M^k \to 0 \ \ as \ \ k \to \infty \tag{4}$$

and

$$d(T(x),T(y)) \leq Md(x,y) \text{ for all } x,y \in X.$$

The Lipschitz matrix M satisfying 4 is said to be convergent to zero. The extension of Banach's fixed point theorem to generalized contractions on spaces with a vector-valued metric is due to Perov and Kibenko (see [15] and [16]).

Theorem 1.1. *(Perov) Let (X,d) be a complete generalized metric space and $T: X \to X$ be a generalized contraction with Lipschitz matrix M. Then T has a unique fixed point x^* and for each $x_0 \in X$, one has*

$$d(T^k(x_0),x^*)) \leq M^k(I-M)^{-1}d(x_0,T(x_0))$$

for all $k \in \mathbb{N}$.

As concerns matrices which are convergent to zero, we mention the following equivalent characterizations (see Precup [17]): If M be a square matrix of nonnegative numbers, then the following statements are equivalent:

(i). the matrix M is convergent to zero;

(ii). $I-M$ is non-singular and $(I-M)^{-1} = I + M + M^2 + \dots$;

(iii). $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M-\lambda I) = 0$;

(iv). $I-M$ is non-singular and $(I-M)^{-1}$ has nonnegative elements.

Finally we recall basic definitions and results of the theory of gauge spaces. A map $d: X \times X \to \mathbb{R}^+$ is said to be a pseudo-metric, or a gauge on the set X, if it has the following properties: $d(x,x) = 0$, $d(x,y) = d(y,x)$ and $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$. A family $\mathcal{P} = \{d_\alpha\}_{\alpha \in \Lambda}$ of pseudometrics on X (or a gauge structure on X) is said to be separating if for each pair of points $x,y \in X$ with $x = y$, there is a $d_\alpha \in \mathcal{P}$ such that $d_\alpha(x,y) = 0$. A pair $(X,\mathcal{P})$ of a nonempty set X and a separating gauge structure $\mathcal{P}$ on X is called a gauge space.

It is well-known (see Dugundji [5: pp. 198–204]) that any family $\mathcal{P}$ of pseudometrics on a set X induces on X a structure U of uniform space and conversely, any uniform structure on X is induced by a family of pseudo-metrics on X. In addition, U is separating (or Hausdorff) if and only if $\mathcal{P}$ is separating. Hence we may identify gauge spaces to Hausdorff uniform spaces. We now recall the notion of contraction on a gauge space, introduced by Gheorghiu [7] (see also Chi¸s-Precup [2] and Angelov [1]). Let $(X,\mathcal{P})$ be a gauge space with $\mathcal{P} = \{d_\alpha\}_{\alpha \in \Lambda}$. A map $T: D(T) \subset X \to X$ is a contraction if there exists a function $\varphi: \Lambda \to \Lambda$ and $a \in \mathbb{R}_+^\Lambda$, $a = \{a_\alpha\}_{\alpha \in \Lambda}$ such that

$$d_\alpha(T(x),T(y)) \leq a_\alpha d_{\varphi(\alpha)}(x,y)$$

for all $\alpha \in \Lambda$ and $x,y \in D(T)$ and

$$\sum_{i=1}^{\infty} a_\alpha a_{\varphi(\alpha)} a_{\varphi^2(\alpha)} \dots a_{\varphi^{i-1}(\alpha)} d_{\varphi^i(\alpha)}(x,y) < \infty$$

for every $\alpha \in \Lambda$ and $x,y \in D(T)$. Here φ^i is the $i-th$ iterate of φ.

We note that this notion was first introduced by Marinescu [8] in locally convex spaces assuming that $\varphi^2 = \varphi$ and then by Colojoară [1] in uniform spaces, under the same condition. The case $\varphi = 1_\Lambda$ (identity) was considered by Tarafdar [23] (see also Frigon [4]). Also note that a somewhat different notion of contraction in a uniform space was defined by Knill [7] in terms of entourages.

Theorem 1.2 ([5])**.** *Let $(X,\mathcal{P})$ be a complete gauge space and let $T: X \to X$ be a contraction. Then T has a unique fixed point which can be obtained by successive approximations starting from any element of X.*

2. Main Abstract Results

In this section we introduce the notions of a vector-valued pseudo-metric, generalized gauge space and generalized contraction. Then Gheorghiu's theorem is extended for generalized contractions on complete generalized gauge spaces. A second result is concerning with mappings which are contractive in Gheorghiu's sense only on one of its orbits. The results are Perov-Gheorghiu mixtures and have the advantages of both approaches.

Definition 2.1. *Let Z be a set. A vector-valued map $D : Z\times Z \to \mathbb{R}^n_+$ is said to be a vector-valued pseudo-metric, or a vector-valued gauge on Z, if it has the following properties: $D(u,u) = 0$; $D(u,v) = D(v,u)$; and $D(u,v) \leq D(u,w)+D(w,v)$ for all $u,v,w \in Z$. Here again, if $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, then by $a \leq b$ we mean $a_i \leq b_i$ for $i = 1, 2, \ldots, n$.*

A family $\mathcal{G} = \{D_\alpha\}_{\alpha\in\Lambda}$ of vector-valued pseudo-metrics on Z (or a generalized gauge structure on Z) is said to be separating if for each pair of points $u, v \in Z$ with $u = v$, there is a $D_\alpha \in \mathcal{G}$ such that $D_\alpha(u,v) = 0$. A pair $(Z,\mathcal{G})$ of a nonempty set Z and a separating generalized gauge structure $\mathcal{G}$ on Z is called a generalized gauge space. For the generalized gauge spaces, the notions of a convergent sequence, Cauchy sequence and completeness are similar to those for usual gauge spaces. By analogy, we can introduce the vector version of Gheorghiu's notion of contraction.

Definition 2.2. *Let $(Z,\mathcal{G})$ be a generalized gauge space with $\mathcal{G} = \{D\alpha\}\alpha \in \Lambda$. A map $T : D(T) \subset Z \to Z$ is a generalized contraction if there exists a function $\varphi : \Lambda \to \Lambda$ and $M \in M_{n\times n}(\mathbb{R}_+)^\Lambda$, $M = \{M_\alpha\}_{\alpha\in\Lambda}$ such that*

$$D\alpha(T(u),T(v)) \leq M_\alpha D_{\varphi(\alpha)}(u,v) \quad \text{for all} \quad \alpha \in \Lambda \quad \text{and} \quad u,v \in D(T) \tag{5}$$

and

$$\sum_{i=1}^{\infty} M_\alpha M_{\varphi(\alpha)} M_{\varphi^2(\alpha)} \ldots M_{\varphi^{i-1}(\alpha)} D_{\varphi^i(\alpha)}(u,v) < \infty \tag{6}$$

for every $\alpha \in \Lambda$ and $u, v \in D(T)$.

Now the Perov type analog for generalized contractions of Gheorghiu's fixed point theorem reads as follows:

Theorem 2.3. *Let $(Z,\mathcal{G})$ be a complete generalized gauge space and let $T : Z \to Z$ be a generalized contraction. Then T has a unique fixed point which can be obtained by successive approximations starting from any element of Z.*

Proof. Let u_0 be an arbitrary element of Z. Define a sequence (u_k) by

$$u_{k+1} = T(u_k), \quad k \in \mathbb{N}. \tag{7}$$

Then using 5 we have

$$\begin{aligned}
D_\alpha(u_k,u_{k+1}) &= D\alpha(T(u_{k-1}),T(u_k)) \\
&\leq M_\alpha D_{\varphi(\alpha)}(u_{k-1},u_k) \\
&= M_\alpha D_{\varphi(\alpha)}(T(u_{k-2}),T(u_{k-1})) \\
&\leq M_\alpha M_{\varphi(\alpha)} D_{\varphi^2(\alpha)}(u_{k-2},u_{k-1}) \\
&\vdots \\
&\leq M_\alpha M_{\varphi(\alpha)} \ldots M_{\varphi^{k-1}(\alpha)} D_{\varphi^k(\alpha)}(u_0,u_1)
\end{aligned}$$

for every $\alpha \in \Lambda$ and $k = 1, 2, \ldots$. As a consequence we have

$$\begin{aligned} D_\alpha(u_k, u_{k+m}) &= D\alpha(u_k, u_{k+1}) + \cdots + D_\alpha(u_{k+m-1}, u_{k+m}) \\ &\leq \sum_{n=0}^{m-1} M_\alpha M_{\varphi(\alpha)} \ldots M_{\varphi^{k+n-1}(\alpha)} D_{\varphi^{k+n}(\alpha)}(u_0, u_1) \\ &= \sum_{i=k}^{k+m-1} M_\alpha M_{\varphi(\alpha)} \ldots M_{\varphi^{i-1}(\alpha)} D_{\varphi^i(\alpha)}(u_0, u_1). \end{aligned}$$

Hence, according to 6, (u_k) is a Cauchy sequence. Let u^* be its limit. Then, letting $k \to \infty$ in 7 gives $u^* = T(u^*)$. For uniqueness, assume that u_1, u_2 are two fixed points of T. Then

$$\begin{aligned} D_\alpha(u_1, u_2) &= D_\alpha(T(u_1), T(u_2)) \\ &\leq M_\alpha D_{\varphi(\alpha)}(u_1, u_2) \\ &\leq M_\alpha M_{\varphi(\alpha)} D_{\varphi^2(\alpha)}(u_1, u_2) \\ &\vdots \\ &\leq M_\alpha M_{\varphi(\alpha)} \ldots M_{\varphi^{k-1}(\alpha)} D_{\varphi^k(\alpha)}(u_1, u_2) \end{aligned}$$

and using 6 we obtain that $D_\alpha(u_1, u_2) = 0$ for every $\alpha \in \Lambda$. Since family $\mathcal{G}$ is separating we deduce that $u_1 = u_2$. □

From the proof of Theorem 2.3 we immediately obtain the following result guaranteeing the existence of a fixed point as limit of the successive approximation sequence which starts from a given element of the space.

Theorem 2.4. *Let $(Z, \mathcal{G})$ be a generalized gauge space with $\mathcal{G} = \{D\alpha\}\alpha \in \Lambda$ and let $T : Z \to Z$ be a mapping. Assume that there is $u_0 \in Z$, $C > 0, \varphi : \Lambda \to \Lambda$ and$M \in M_{n\times n}(\mathbb{R}_+)^\Lambda$, $M = \{M_\alpha\}_{\alpha\in\Lambda}$ such that the following conditions hold:*

$$D\alpha(T(u), T(v)) \leq M_\alpha D_{\varphi(\alpha)}(u, v) \quad \text{for all} \quad \alpha \in \Lambda \quad \text{and} \quad u, v \in Z,$$

$$\sum_{i=1}^{\infty} M_\alpha M_{\varphi(\alpha)} M_{\varphi^2(\alpha)} \ldots M_{\varphi^{i-1}(\alpha)} D_{\varphi^i(\alpha)}(u, v) < \infty \tag{8}$$

$$D\alpha(u_0, T(u_0)) \leq C \quad \text{for all} \quad \alpha \in \Lambda. \tag{9}$$

Then T has at least one fixed point which can be obtained by successive approximations starting from u_0.

Remark 2.5. *Here are some useful particular cases: If there is an integer $p \geq 2$ with $\varphi^p = \varphi$, then conditions 6 and 8 reduce to the assumption that*

$$M_{\varphi(\alpha)} \ldots M_{\varphi^{p-1}(\alpha)}$$

is convergent to zero for every $\alpha \in \Lambda$. Thus, if $p = 2$, that is $\varphi^2 = \varphi$ (Marinescu's situation), then 6 and 8 hold if

$$M_{\varphi(\alpha)} \quad \text{is convergent to zero for every} \quad \alpha \in \Lambda.$$

In particular, if $\varphi = 1_\Lambda$ (Tarafdar's situation), then 6 and 8 are satisfied provided that M_α is convergent to zero for every $\alpha \in \Lambda$.

Now we turn back to system 1. We assume that X is a complete gauge space with the family of pseudo-metrics $\mathcal{P} = \{d_\alpha\}_{\alpha\in\Lambda}$. We denote $Z := X^4$, $T := (A_1, A_2, A_3, A_4)$ and $\mathcal{G} := \{D_\alpha\}_{\alpha\in\Lambda}$, where

$$D_\alpha(u,v) = \begin{bmatrix} d_\alpha(x,x_1) \\ d_\alpha(y,y_1) \\ d_\alpha(z,z_1) \\ d_\alpha(w,w_1) \end{bmatrix} \tag{10}$$

for every $u := (x,y,z,w)$, $v := (x_1,y_1,z_1,w_1) \in X^4$ and $\alpha \in \Lambda$. Then $(Z,\mathcal{G})$ is a complete generalized gauge space. Specialized to this case, Theorems 2.3 and 2.4 yield the following results.

Theorem 2.6. *Assume that $(X,\mathcal{P})$ is a complete gauge space with $\mathcal{P} = \{d_\alpha\}_{\alpha\in\Lambda}$ and that there exists a function $\varphi : \Lambda \to \Lambda$ and nonnegative constants $a^1_\alpha, b^1_\alpha, c^1_\alpha, d^1_\alpha,\ a^2_\alpha, b^2_\alpha, c^2_\alpha, d^2_\alpha\ a^3_\alpha, b^3_\alpha, c^3_\alpha, d^3_\alpha,\ a^4_\alpha, b^4_\alpha, c^4_\alpha, d^4_\alpha$ such that*

$$\left.\begin{aligned} d_\alpha(A_1(x,y,z,w),A_1(x_1,y_1,z_1,w_1)) \le\ & a^1_\alpha d_{\varphi(\alpha)}(x,x_1) + b^1_\alpha d_{\varphi(\alpha)}(y,y_1) + c^1_\alpha d_{\varphi(\alpha)}(z,z_1) + d^1_\alpha d_{\varphi(\alpha)}(w,w_1), \\ d_\alpha(A_2(x,y,z,w),A_2(x_1,y_1,z_1,w_1)) \le\ & a^2_\alpha d_{\varphi(\alpha)}(x,x_1) + b^2_\alpha d_{\varphi(\alpha)}(y,y_1) + c^2_\alpha d_{\varphi(\alpha)}(z,z_1) + d^2_\alpha d_{\varphi(\alpha)}(w,w_1), \\ d_\alpha(A_3(x,y,z,w),A_3(x_1,y_1,z_1,w_1)) \le\ & a^3_\alpha d_{\varphi(\alpha)}(x,x_1) + b^3_\alpha d_{\varphi(\alpha)}(y,y_1) + c^3_\alpha d_{\varphi(\alpha)}(z,z_1) + d^3_\alpha d_{\varphi(\alpha)}(w,w_1), \\ d_\alpha(A_4(x,y,z,w),A_4(x_1,y_1,z_1,w_1)) \le\ & a^4_\alpha d_{\varphi(\alpha)}(x,x_1) + b^4_\alpha d_{\varphi(\alpha)}(y,y_1) + c^4_\alpha d_{\varphi(\alpha)}(z,z_1) + d^4_\alpha d_{\varphi(\alpha)}(w,w_1),, \end{aligned}\right\} \tag{11}$$

for all $x,x_1,y,y_1,z,z_1,w,w_1 \in X$ and $\alpha \in \Lambda$. Let

$$M_\alpha = \begin{pmatrix} a^1_\alpha & b^1_\alpha & c^1_\alpha & d^1_\alpha \\ a^2_\alpha & b^2_\alpha & c^2_\alpha & d^2_\alpha \\ a^3_\alpha & b^3_\alpha & c^3_\alpha & d^3_\alpha \\ a^4_\alpha & b^4_\alpha & c^4_\alpha & d^4_\alpha \end{pmatrix}$$

If

$$\sum_{i=1}^{\infty} M_\alpha M_{\varphi(\alpha)} M_{\varphi^2(\alpha)} \dots M_{\varphi^{i-1}(\alpha)} D_{\varphi^i(\alpha)}(u,v) < \infty \tag{12}$$

for all $u,v \in X^4$ and $\alpha \in \Lambda$, then system 1 has a unique solution. Moreover, the solution is the limit of the sequence of successive approximations

$$\left.\begin{aligned} u_k &= (x_k,y_k,z_k,w_k), \\ x_{k+1} &= A_1(x_k,y_k,z_k,w_k), \\ y_{k+1} &= A_2(x_k,y_k,z_k,w_k) \\ z_{k+1} &= A_3(x_k,y_k,z_k,w_k), \\ w_{k+1} &= A_4(x_k,y_k,z_k,w_k) \end{aligned}\right\} \tag{13}$$

for $k = 0,1,\dots,$ and starting from any initial pair $(x_0,y_0,z_0,w_0) \in X^4$.

Proof. Clearly inequalities 11 can be written in the vector form

$$D_\alpha(T(u),T(v)) \le M_\alpha D_{\varphi(\alpha)}(u,v).$$

The result is now a direct consequence of Theorem 2.3. □

Theorem 2.7. *Under the assumptions of Theorem 7, if there is* $u_0 = (x_0, y_0, z_0, w_0) \in X^4$ *and* $C > 0$ *such that*

$$D_\alpha(u_0, T(u_0)) \leq C \tag{14}$$

and

$$\sum_{i=1}^{\infty} M_\alpha M_{\varphi(\alpha)} M_{\varphi^2(\alpha)} \dots M_{\varphi^{i-1}(\alpha)} < \infty \tag{15}$$

for every $\alpha \in \Lambda$ *, then system 1 has at least one solution which is the limit of sequence 13 starting from* u_0.

Proof. The result is a direct consequence of Theorem 2.4. □

3. Applications to Integral Systems

Consider the system of integral equations with advanced argument

$$\left.\begin{aligned} x(t) &= \int_{t-1}^{t} f_1(s, x(s+4), y(s+4), z(s+4), w(s+4))ds \\ y(t) &= \int_{t-1}^{t} f_2(s, x(s+4), y(s+4), z(s+4), w(s+4))ds \\ z(t) &= \int_{t-1}^{t} f_3(s, x(s+4), y(s+4), z(s+4), w(s+4))ds \\ w(t) &= \int_{t-1}^{t} f_4(s, x(s+4), y(s+4), z(s+4), w(s+4))ds \end{aligned}\right\} \tag{16}$$

for $t \in [0, \infty)$. Assume that

$$\left.\begin{aligned} |f_1(t,x,y,z,w) - f_1(t,x_1,y_1,z_1,w_1)| \leq\ & k_1(t)|x-x_1| + k_2(t)|y-y_1| + k_3(t)|z-z_1| + k_4(t)|w-w_1|, \\ |f_2(t,x,y,z,w) - f_2(t,x_1,y_1,z_1,w_1)| \leq\ & k_5(t)|x-x_1| + k_6(t)|y-y_1| + k_7(t)|z-z_1| + k_8(t)|w-w_1|, \\ |f_3(t,x,y,z,w) - f_3(t,x_1,y_1,z_1,w_1)| \leq\ & k_9(t)|x-x_1| + k_{10}(t)|y-y_1| + k_{11}(t)|z-z_1| + k_{12}(t)|w-w_1|, \\ |f_4(t,x,y,z,w) - f_4(t,x_1,y_1,z_1,w_1)| \leq\ & k_{13}(t)|x-x_1| + k_{14}(t)|y-y_1| + k_{15}(t)|z-z_1| + k_{16}(t)|w-w_1|, \end{aligned}\right\} \tag{17}$$

for every $x, x_1, y, y_1, z, z_1, w, w_1 \in \mathbb{R}$, $t \in [-1, \infty)$ and some $k_i \in L^1([-1,\infty), \mathbb{R}_+)$, $i = \{1, 2, 3, 4, \dots, 16\}$. For each $n \in \mathbb{N}$, let

$$\begin{aligned} a_n^1 &= \int_{n-1}^{2n+1} k_1(t)dt, \quad b_n^1 = \int_{n-1}^{2n+1} k_2(t)dt, \quad c_n^1 = \int_{n-1}^{2n+1} k_3(t)dt, \quad d_n^1 = \int_{n-1}^{2n+1} k_4(t)dt \\ a_n^2 &= \int_{n-1}^{2n+1} k_5(t)dt, \quad b_n^2 = \int_{n-1}^{2n+1} k_6(t)dt, \quad c_n^2 = \int_{n-1}^{2n+1} k_7(t)dt, \quad d_n^2 = \int_{n-1}^{2n+1} k_8(t)dt \\ a_n^3 &= \int_{n-1}^{2n+1} k_9(t)dt, \quad b_n^3 = \int_{n-1}^{2n+1} k_{10}(t)dt, \quad c_n^3 = \int_{n-1}^{2n+1} k_{11}(t)dt, \quad d_n^3 = \int_{n-1}^{2n+1} k_{12}(t)dt \\ a_n^4 &= \int_{n-1}^{2n+1} k_{13}(t)dt, \quad b_n^4 = \int_{n-1}^{2n+1} k_{14}(t)dt, \quad c_n^4 = \int_{n-1}^{2n+1} k_{15}(t)dt, \quad d_n^4 = \int_{n-1}^{2n+1} k_{16}(t)dt \end{aligned}$$

and consider the matrix

$$M_\alpha = \begin{pmatrix} a_\alpha^1 & b_\alpha^1 & c_\alpha^1 & d_\alpha^1 \\ a_\alpha^2 & b_\alpha^2 & c_\alpha^2 & d_\alpha^2 \\ a_\alpha^3 & b_\alpha^3 & c_\alpha^3 & d_\alpha^3 \\ a_\alpha^4 & b_\alpha^4 & c_\alpha^4 & d_\alpha^4 \end{pmatrix}.$$

Also define the matrix M_∞ by

$$M_\infty = \begin{pmatrix} |k_1|_{L^1([-1,\infty))} & |k_2|_{L^1([-1,\infty))} & |k_3|_{L^1([-1,\infty))} & |k_4|_{L^1([-1,\infty))} \\ |k_5|_{L^1([-1,\infty))} & |k_6|_{L^1([-1,\infty))} & |k_7|_{L^1([-1,\infty))} & |k_8|_{L^1([-1,\infty))} \\ |k_9|_{L^1([-1,\infty))} & |k_{10}|_{L^1([-1,\infty))} & |k_{11}|_{L^1([-1,\infty))} & |k_{12}|_{L^1([-1,\infty))} \\ |k_{13}|_{L^1([-1,\infty))} & |k_{14}|_{L^1([-1,\infty))} & |k_{15}|_{L^1([-1,\infty))} & |k_{16}|_{L^1([-1,\infty))} \end{pmatrix}.$$

Our main result on system 16 is the following theorem.

Theorem 3.1. *Let $f_1, f_2, f_3, f_4 : [-1,\infty)\times\mathbb{R}^4 \to \mathbb{R}$ be four continuous functions and assume that inequalities 17 hold for some $k_i \in L^1([-1,\infty),\mathbb{R}_+)$, $i = \{1,2,3,4,\dots,16\}$. In addition assume that there is $u_0 = (x_0, y_0, z_0, w_0) \in C([0,\infty),\mathbb{R}^4)$ and $C > 0$ such that*

$$|T(u_0)(t) - u_0(t)| \le C \quad \text{for all} \quad t \in [0,\infty), \tag{18}$$

where $T = (A_1, A_2, A_3, A_4)$ is given bellow. If the matrix

$$M_\infty \to 0, \tag{19}$$

then system 16 has at least one solution $(x, y, z, w) \in C([0,\infty),\mathbb{R}^4)$

Proof. We shall use Theorem 2.7. Here $X = C[0,\infty), \Lambda = \mathbb{N}$ and for $n \in \mathbb{N}$, $d_n : X\times X \to \mathbb{R}_+$ is given by

$$d_n(x,y) = \max_{t\in[n,2n+1]} |x(t) - y(t)|.$$

Let $A_1, A_2, A_3 : C[0,\infty),\mathbb{R}^3 \to C[0,\infty)$ be defined by

$$\left.\begin{aligned} A_1(x,y,z,w)(t) &= \int_{t-1}^{t} f_1(s, x(s+4), y(s+4), z(s+4), w(s+4))ds \\ A_2(x,y,z,w)(t) &= \int_{t-1}^{t} f_2(s, x(s+4), y(s+4), z(s+4), w(s+4))ds \\ A_3(x,y,z,w)(t)) &= \int_{t-1}^{t} f_3(s, x(s+4), y(s+4), z(s+4), w(s+4))ds \\ A_4(x,y,z,w)(t)) &= \int_{t-1}^{t} f_4(s, x(s+4), y(s+4), z(s+4), w(s+4))ds \end{aligned}\right\}$$

First we prove the Lipschitz condition 11 with $\varphi : \mathbb{N} \to \mathbb{N}$ given by $\varphi(n) = n+1$. Let $t \in [n, 2n+1]$. We have $t-1 \in [n-1, 2n]$, and when $s \in [t-1, t]$, then $s+2 \in [n+1, 2n+3]$. It follows that

$$\begin{aligned} |A(x,y,z,w)(t) - A(x_1,y_1,z_1,w_1)(t)| &\le \int_{n-1}^{2n+1} |f_1(s, x(s+4), y(s+4), z(s+4), w(s+4)) \\ &- \int_{n-1}^{2n+1} f_1(s, x_1(s+4), y_1(s+4), z_1(s+4), w_1(s+4))|ds \\ &\le \int_{n-1}^{2n+1} k_1|x(s+4) - x_1(s+4)|ds + \int_{n-1}^{2n+1} k_2|y(s+4) - y_1(s+4)|ds \\ &+ \int_{n-1}^{2n+1} k_3|z(s+4) - z_1(s+4)|ds + \int_{n-1}^{2n+1} k_4|w(s+4) - w_1(s+4)|ds \\ &\le \max_{s\in[n+1,2n+3]} |x(s+4) - x_1(s+4)| \int_{n-1}^{2n+1} k_1 ds + \max_{s\in[n+1,2n+3]} |y(s+4) - y_1(s+4)| \int_{n-1}^{2n+1} k_2 ds \end{aligned}$$

$$
\begin{aligned}
&+ \max_{s\in[n+1,2n+3]} |z(s+4)-z_1(s+4)| \int_{n-1}^{2n+1} k_3 ds + \max_{s\in[n+1,2n+3]} |w(s+4)-w_1(s+4)| \int_{n-1}^{2n+1} k_4 d\\
\leq\ & \max_{\tau\in[n+1,2n+3]} |x(\tau)-x_1(\tau)| \int_{n-1}^{2n+1} k_1 ds + \max_{\tau\in[n+1,2n+3]} |y(\tau)-y_1(\tau)| \int_{n-1}^{2n+1} k_2 ds\\
&+ \max_{\tau\in[n+1,2n+3]} |z(\tau)-z_1(\tau)| \int_{n-1}^{2n+1} k_3 ds + \max_{\tau\in[n+1,2n+3]} |w(\tau)-w_1(\tau)| \int_{n-1}^{2n+1} k_4 ds\\
=\ & a_n^1 d_{n+1}(x,x_1) + b_n^1 d_{n+1}(y,y_1) + c_n^1 d_{n+1}(z,z_1) + d_n^1 d_{n+1}(w,w_1).
\end{aligned}
$$

Taking the maximum over $[n, 2n+1]$ yields

$$
\begin{aligned}
d_n(A_1(x,y,z,w),A_1(x_1,y_1,z_1,w_1)) &\leq a_n^1 d_{n+1}(x,x_1) + b_n^1 d_{n+1}(y,y_1) + c_n^1 d_{n+1}(z,z_1) + d_n^1 d_{n+1}(w,w_1)\\
d_n(A_1(x,y,z,w),A_1(x_1,y_1,z_1,w_1)) &\leq a_n^1 d_{\varphi(n)}(x,x_1) + b_n^1 d_{\varphi(n)}(y,y_1) + c_n^1 d_{\varphi(n)}(z,z_1) + d_n^1 d_{\varphi(n)}(w,w_1)
\end{aligned}
$$

for every $(x,y,z,w),(x_1,y_1,z_1,w_1) \in X^4$. Similarly, for A_2, A_3, A_4 we have,

$$
\begin{aligned}
d_n(A_2(x,y,z,w),A_2(x_1,y_1,z_1,w_1)) &\leq a_n^2 d_{\varphi(n)}(x,x_1) + b_n^2 d_{\varphi(n)}(y,y_1) + c_n^2 d_{\varphi(n)}(z,z_1) + d_n^2 d_{\varphi(n)}(w,w_1)\\
d_n(A_3(x,y,z,w),A_3(x_1,y_1,z_1,w_1)) &\leq a_n^3 d_{\varphi(n)}(x,x_1) + b_n^3 d_{\varphi(n)}(y,y_1) + c_n^3 d_{\varphi(n)}(z,z_1) + d_n^3 d_{\varphi(n)}(w,w_1)\\
d_n(A_4(x,y,z,w),A_4(x_1,y_1,z_1,w_1)) &\leq a_n^4 d_{\varphi(n)}(x,x_1) + b_n^4 d_{\varphi(n)}(y,y_1) + c_n^4 d_{\varphi(n)}(z,z_1) + d_n^4 d_{\varphi(n)}(w,w_1)
\end{aligned}
$$

for every $(x,y,z,w),(x_1,y_1,z_1,w_1) \in X^4$. Hence 11 holds. Furthermore, condition 14 is guaranteed by assumption 18. Also, for every $n \in \mathbb{N}$, $M_n \leq M_\infty$ and thus series 15 is dominated by

$$\sum_{k=0}^{\infty} M_\infty^k$$

which is convergent in view of assumption 19. Hence 15 is satisfied. Therefore Theorem 2.7 can be applied. □

4. An Integral System Without Modification of the Argument

Consider the system of integral equations

$$
\left.
\begin{aligned}
x(t) &= \int_{t-1}^{t} f_1(s,x(s+4),y(s+4),z(s+4),w(s+4))ds\\
y(t) &= \int_{t-1}^{t} f_2(s,x(s+4),y(s+4),z(s+4),w(s+4))ds\\
z(t) &= \int_{t-1}^{t} f_3(s,x(s+4),y(s+4),z(s+4),w(s+4))ds\\
w(t) &= \int_{t-1}^{t} f_4(s,x(s+4),y(s+4),z(s+4),w(s+4))ds
\end{aligned}
\right\} \tag{20}
$$

for $t \in [0,\infty)$, where $x(t) = \psi_1(t)$, $y(t) = \psi_2(t)$, $z(t) = \psi_3(t)$ and $w(t) = \psi_4(t)$ for $t \in [-1,0]$ and $\psi_1, \psi_2, \psi_3, \psi_4$ are given functions. We assume that inequalities 17 hold for every $x, x_1, y, y_1, z, z_1, w, w_1 \in \mathbb{R}$, $t \in [0,\infty)$ and some $k_i \in L^1_{loc}([0,\infty),\mathbb{R}^+)$, $i = \{1,2,3,4,\ldots,16\}$. For $n \in \mathbb{N}\setminus\{0\}$, we denote

$$
\begin{aligned}
a_n^1 &= \int_{n-1}^{2n+1} k_1(t)dt, \quad b_n^1 = \int_{n-1}^{2n+1} k_2(t)dt, \quad c_n^1 = \int_{n-1}^{2n+1} k_3(t)dt, \quad d_n^1 = \int_{n-1}^{2n+1} k_4(t)dt\\
a_n^2 &= \int_{n-1}^{2n+1} k_5(t)dt, \quad b_n^2 = \int_{n-1}^{2n+1} k_6(t)dt, \quad c_n^2 = \int_{n-1}^{2n+1} k_7(t)dt, \quad d_n^2 = \int_{n-1}^{2n+1} k_8(t)dt\\
a_n^3 &= \int_{n-1}^{2n+1} k_9(t)dt, \quad b_n^3 = \int_{n-1}^{2n+1} k_{10}(t)dt, \quad c_n^3 = \int_{n-1}^{2n+1} k_{11}(t)dt, \quad d_n^3 = \int_{n-1}^{2n+1} k_{12}(t)dt
\end{aligned}
$$

$$a_n^4 = \int_{n-1}^{2n+1} k_{13}(t)dt, \quad b_n^4 = \int_{n-1}^{2n+1} k_{14}(t)dt, \quad c_n^4 = \int_{n-1}^{2n+1} k_{15}(t)dt, \quad d_n^4 = \int_{n-1}^{2n+1} k_{16}(t)dt$$

and consider the matrix.

$$M_\alpha = \begin{pmatrix} a_\alpha^1 & b_\alpha^1 & c_\alpha^1 & d_\alpha^1 \\ a_\alpha^2 & b_\alpha^2 & c_\alpha^2 & d_\alpha^2 \\ a_\alpha^3 & b_\alpha^3 & c_\alpha^3 & d_\alpha^3 \\ a_\alpha^4 & b_\alpha^4 & c_\alpha^4 & d_\alpha^4 \end{pmatrix}.$$

Theorem 4.1. *Let $f_1, f_2, f_3, f_4 : [-1, \infty) \times \mathbb{R}^4 \to \mathbb{R}$ be four continuous functions, $\psi_1, \psi_2, \psi_3, \psi_4 \in C[-1, 0]$ and assume that inequalities 17 hold for some $k_i \in L^1_{loc}([0, \infty), \mathbb{R}^+)$, $i = \{1, 2, 3, 4, \ldots, 16\}$. For $n \in \mathbb{N} \setminus \{0\}$, matrix*

$$M_n \to 0 \tag{21}$$

then system 20 has a unique solution $(x, y, z, w) \in C[0, \infty), \mathbb{R}^3$.

Proof. The result follows from Theorem 2.6 if we take into account Remark 2.5 about Tarafdar's situation. Here $X = C[0, \infty)$, $\Lambda = \mathbb{N} \setminus \{0\}$, for each $n \in \mathbb{N} \setminus \{0\}$,

$$d_n : X \times X \to \mathbb{R}_+$$

is given by

$$d_n(x, y) = \max_{t \in [0,n]} |x(t) - y(t)|,$$

$\varphi : \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$, $\varphi(n) = n$, and $A_1, A_2, A_3, A_4 : C[0, \infty), \mathbb{R}^4 \to C[0, \infty)$ are defined by

$$A_1(x, y, z, w)(t) = \int_{t-1}^{t} f_1(s, \tilde{x}(s), \tilde{y}(s), \tilde{z}(s), \tilde{w}(s))ds,$$
$$A_2(x, y, z, w)(t) = \int_{t-1}^{t} f_2(s, \tilde{x}(s), \tilde{y}(s), \tilde{z}(s), \tilde{w}(s))ds,$$
$$A_3(x, y, z, w)(t) = \int_{t-1}^{t} f_3(s, \tilde{x}(s), \tilde{y}(s), \tilde{z}(s), \tilde{w}(s))ds,$$
$$A_4(x, y, z, w)(t) = \int_{t-1}^{t} f_4(s, \tilde{x}(s), \tilde{y}(s), \tilde{z}(s), \tilde{w}(s))ds,$$

where

$$\tilde{x}(t) = \begin{cases} \psi_1(t) \quad for \quad -1 \leq t < 0 \\ x(t) \quad for \quad t \geq 0 \end{cases}$$

$$\tilde{y}(t) = \begin{cases} \psi_2(t) \quad for \quad -1 \leq t < 0 \\ y(t) \quad for \quad t \geq 0 \end{cases}$$

$$\tilde{z}(t) = \begin{cases} \psi_3(t) \quad for \quad -1 \leq t < 0 \\ z(t) \quad for \quad t \geq 0 \end{cases}$$

and

$$\tilde{w}(t) = \begin{cases} \psi_4(t) \quad for \quad -1 \leq t < 0 \\ w(t) \quad for \quad t \geq 0 \end{cases}$$

□

Remark 4.2. *When $k_i \in L^1([0,\infty),\mathbb{R}_+)$, $i=\{1,2,3,4,\ldots,16\}$ then a sufficient condition for 21 to hold for every $n \in \mathbb{N}\backslash\{0\}$ is that the matrix*

$$M_\infty = \begin{pmatrix} |k_1|_{L^1([-1,\infty))} & |k_2|_{L^1([-1,\infty))} & |k_3|_{L^1([-1,\infty))} & |k_4|_{L^1([-1,\infty))} \\ |k_5|_{L^1([-1,\infty))} & |k_6|_{L^1([-1,\infty))} & |k_7|_{L^1([-1,\infty))} & |k_8|_{L^1([-1,\infty))} \\ |k_9|_{L^1([-1,\infty))} & |k_{10}|_{L^1([-1,\infty))} & |k_{11}|_{L^1([-1,\infty))} & |k_{12}|_{L^1([-1,\infty))} \\ |k_{13}|_{L^1([-1,\infty))} & |k_{14}|_{L^1([-1,\infty))} & |k_{15}|_{L^1([-1,\infty))} & |k_{16}|_{L^1([-1,\infty))} \end{pmatrix}. \tag{22}$$

Indeed, for each $n \in \mathbb{N}\setminus\{0\}$, one has $M_n \leq M_\infty$, whence, since the entries of all matrices are non-negative, $M_n^k \leq M_\infty^k$ for all $k \in \mathbb{N}$. Consequently, if $M_\infty^k \to 0$ as $k \to \infty$, then $M_n^k \to 0$ as $k \to \infty$, too. However, 22 is not a necessary condition for 21 as shows the following contre-example:

$$k_1(t) = (t+1)^{-2}, \quad k_i = 0 \quad for \quad i = \{2,3,4,\ldots,16\}.$$

In this case

$$M_n = \begin{pmatrix} \frac{n}{n+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$M_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly, for every $n \in \mathbb{N}\setminus\{0\}$, M_n converges to zero, but M_∞ does not.

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