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The Relatively Prime Nature of Consecutive Integers

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Abstract: The purpose of this article is to demonstrate that consecutive integers are relatively prime and that the converse is true for a special case. This paper introduces two methods/proofs of showing this. Firstly, a proof that depends on the Division Algorithm, Euclidean Algorithm, and Bezout's Lemma has been discussed. The uniqueness of this proof is its dependence of on other theorems. It can be thought to be an application of the aforementioned algorithms and lemma. In this article, the postulate that consecutive integers are relatively prime has been referred to as The Relative Prime Nature of Consecutive Integers. Secondly, a proof by mathematical induction has been also used to show that two consecutive integers are relatively prime. Since mathematical induction is only applicable when working with positive integers, this proof applies only to positive integers.

MSC: 05E10.

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1. Introduction

The properties of integers play an important role in number theory and algebra. Their divisibility structure and other properties have been of great importance in mathematics. As we know, the detection of an error on identification numbers depends on modular arithmetic which applies the divisibility of integers [3]. Every integer has divisors; two integers can have multiple common divisors with the greatest divisor among them. Sometimes the greatest common divisor of two integers maybe a 1. If that is the case, then we say that the particular integers are relatively prime. The term relatively prime, sometimes referred to as coprime, which implies that the greatest common divisor between integers is 1. Integers have the property that if they are consecutive, then they are relatively prime. This sounds so true but the important question is, can we prove it? The answer is, yes. The next question would be, how? Currently, there has been one way of doing this which depends on a single property. The property is that if p is a prime that divides integers a and b, then p must divide their difference, that is $p|(a-b)$, this implies that a and b are not relatively prime. But for consecutive integers, it is impossible to find p that divides a, $a+1$ and their difference, $(a+1)-a=1$, since $p>1$ [1]. Based on that fact, then two consecutive integers are always relatively prime. As stated in the abstract, this article provides us with two more ways of proving this.

1.1. Preliminaries

Definition 1.1 (Well Ordering Principle)**.** *Two integers a and b with $b>a$ are said to be consecutive if for every given integer n, we have $a=n$ and $b=n+1$.*

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The Well-Ordering Principle: *Every non-empty set of positive integers contains the smallest member* [3].

Theorem 1.2 (The Division Algorithm [4])**.** *For any integers a and b with $b \neq 0$, there exist unique integers q and r such that $a = bq + r$; $0 \leq r < b$.*

Proof. We first prove that r and q actually exist. Let $S = \{\ a - bk : k \in Z \text{ and } a - bk \geq 0\}$ if $0 \in S$, we can let $r = 0$ and $q = \frac{a}{b}$ and obtain the desired results.
Now assume that $0 \in S$. We need to show that S is non-empty and apply the well-ordering principle. If $a > 0$, then $a - b.0 \in S$. If $a < 0$, we have $a - b(2a) = a(1 - 2b) \in S$. Hence $S \neq \emptyset$. Then by the well-ordering principle, S must have a smallest member, say $r = a - bq$. Then $a = bq + r$ and $r > 0$. We now show that $r < b$. Suppose $r > b$, then $a - b(q + 1) = a - bq - b = r - > 0$. If this is the case, then $a - b(q + 1) \in S$. But $a - b(q + 1) < a - bq$, which contradicts the fact that $r = a - bq$ is the smallest member of S. Then $r \leq b$. Since $0 \in S$, $r \neq b$ therefore, $r < b$. We now show that show that r and q are unique. We suppose that there are integer $q, q', r \text{ and } r'$ such that,

$$a = bq + r, \quad 0 \leq r < b$$
$$a = bq' + r', \quad 0 \leq r' < b'$$

then $bq + r = bq' + r'$. Assume that $r' \geq r$, then $b(q - q') = r' - r$. Then b divides $r' - r$ and $0 \leq r' - r \leq r' < b$. Thus, the only way this is possible is when; $r' - r = 0 \Rightarrow r = r' \Rightarrow q = q'$. □

Definition 1.3. *The greatest common divisor of two non-zero integers a and b is the largest of all common divisors of a and b [1, 4].*

Lemma 1.4 (Bezout's lemma)**.** *Let a and b be non-zero integers.Then there exist integers r and s such that greatest common divisor of a and b is a linear combination. That is $gcd(a, b) = ar + bs$. Furthermore, the greatest common divisor of a and b is unique.*

Proof. Let $S = \{am + bn : m, n \in Z \text{ and } am + bn > 0\}$. Clearly S is non-empty; hence, by the well-ordering principle S contains a smallest member, say $d = ar + bs$. Suppose $d = gcd(a, b)$. then $a = dq + r$ where $0 \leq r < d$. if $r > 0$ then;

$$\begin{aligned} r &= a - dq \\ &= a - (ar + bs)q \\ &= a - arq - bsq \\ &= a(1 - rq) + b(-sq) \in S \end{aligned}$$

But $r \in S$ contradicts the fact that d is the smallest member of S. Hence $r = 0$ and d divides a. Similarly d divides b. This prove that d is the greatest common divisor of a and b. We now show the uniqueness of$d$. Suppose d' is another greatest common divisor of a and b, write $a = d'h$ and $b = d'k$, Then

$$\begin{aligned} d &= ar + bs \\ &= d'hr + d'ks \\ &= d'(hr + ks) \\ d &= d'(hr + ks) \end{aligned}$$

Then d' must divide d. Hence, d must be a unique greatest common divisor. □

Theorem 1.5 (Euclidean Algorithm). *For any pair of positive integers a and b, we may find the greatest common divisor, $gcd = (a, b)$ by the repeated use of division to produce a decreasing sequence of integers $r_1 < r_2 < r_3 ...$ as follows;*

$$\begin{aligned}
a &= bq_1 + r_1, & 0 < r_1 < b \\
b &= r_1q_2 + r_2, & 0 < r_2 < r_1 \\
r_1 &= r_2q_3 + r_3, & 0 < r_3 < r_2 \\
&\vdots \\
r_{k-3} &= r_{k-2}q_{k-1} + r_{k-1}, & 0 < r_{k-1} < r_{k-2} \\
r_{k-2} &= r_{k-1}q_k + r_k, & 0 < r_k < r_{k-1} \\
r_{k-1} &= r_kq_{k+1} + 0,
\end{aligned}$$

then, r_k is the last non-zero remainder, hence it is the $gcd(a, b)$ [2].

2. The Relative Prime Nature of Consecutive Integers

This section introduces the theorem which talks about how consecutive integers are relative prime and the **main results** (proofs).

Definition 2.1. *Two Integers a and b are said to be relatively prime if their greatest common divisor is 1. That is if $gcd(a, b) = 1$.*

Theorem 2.2 (The Relative Prime Nature of Consecutive Integers). *If a and b are two consecutive integers such that $a = n$ and $b = n + 1$, where n is an integer, then a and b are relatively prime.*

2.1. First Results (proof)

Proof. Suppose a and b are consecutive integers with $b > a$, then

$$b - a = 1 \tag{1}$$

$$b = a + 1 \tag{2}$$

From the division algorithm, (2) is in the form $b = aq + r$, where $q = 1$ and $r = 1$. Then by the Euclidean algorithm, (2) can be reduced to $a = 1(a) + 0$. Since 1 is the last non-zero remainder, then 1 is the greatest common divisor of a and b. Thus $gcd(a, b) = 1$. By Definition 2.1, this implies that a and b are relatively prime.
Conversely *(Spacial case)*: Suppose the $gcd(a, b) = 1$, then by *Bezout's Lemma*, there exist integers t and s such that; $bs + at = 1$. Letting $s = 1$ and $t = -1$, then $b(1) + a(-1) = 1$. Then we have $b - a = 1$. This is true if and only if $b = n + 1$ and $a = n$ for any integer n. Therefore a and b are consecutive with $b > a$, by Definition 1.1. □

Note 2.1. *Note that the converse only works when $s = 1$ and $t = -1$.*

For us to prove Theorem 2.2 by mathematical induction, we need the mathematical induction theorem of the first form.

Theorem 2.3 (Mathematical Induction of the first form). *Suppose $P(n)$ is a statement about positive integers, and we know two things:*

1. $P(1)$ *is true,*

2. *For every positive integer* m*, if* $P(m)$ *is true, then* $P(m+1)$ *is true.*

Proof. Suppose $P(n)$ is false for some positive integer n. Then S=$\{n|n \in Z^+$ and $P(n)$ is false$\}$ is non-empty subset of Z^+. By the *well-ordering principle*, S has a smallest element say n_0. Clearly, $n_0 \neq 1$, because $P(1)$ is true by (i). Therefore, $n_0 - 1$ is a positive integer, and $P(n_0 - 1)$ is true because $n_0 - 1$ is smaller than n_0. By (ii), this means that $P(n_0 - 1 + 1)$ is true; that is, $P(n_0)$ is true, and contradicts the fact that $P(n_0)$ is false.
Since the supposition that $P(n)$ is false for some n has led us to a contradiction, we conclude that $P(n)$ holds for $n \in Z^+$. □

With Theorem 2.3 in mind, we now prove Theorem 2.2 using mathematical induction.

2.2. Second Results (proof)

Proof. Let $a = n$ and $b = n+1$ be consecutive integers and let $gcd(b, a) = 1$. Then $gcd(n+1, n) = 1$. Suppose $n = 1$ then $gcd(2, 1) = 1$, thus the statement is true. Now let $n = m$ then

$$gcd(m + 1, m) = 1 \tag{3}$$

is assumed to be true where m is a positive integer. Let $n = m + 1$, we need to show that

$$gcd(m + 1 + 1, m + 1) = 1 \tag{4}$$

Firstly, we need to generate (4) from (3). Note that (3) can be written in the form $m+1 = m(1)+1$ as in $b = aq+r$ of the division algorithm, then adding 1 on both sides we have;

$$m + 1 + 1 = (m + 1)(1) + 1 \tag{5}$$

which is in the form $b = aq + r$ where $b = m + 1 + 1$, $a = m + 1$, $q = 1$ and $r = 1$. Hence (5) can be expressed as;

$$gcd(m + 2, m + 1) = 1. \tag{6}$$

Therefore, by Theorem 2.3 and Definition 2.1, Theorem 2.2 is true for all positive integers. □

3. Conclusion

From what has been discussed in section 2 of this article, we have observed that the two new proofs of showing that consecutive integers are relatively prime are mathematical sound. Therefore, they are adequate alternatives for proving that consecutive integers are relatively prime. And they are a form of application of the Division Algorithm, Euclidean Algorithm, Bezout's Lemma and Mathematical Induction.

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