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On Certain Properties of the β-Flexible Q-Fuzzy Groups and β-Flexible Normal Q-Fuzzy Subgroups

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Abstract: The main purpose of this study is to introduce the new concept of fuzzy set and β-flexible fuzzy set. Based on this, the concept of β-flexible Q-fuzzy group and β-flexible normal Q-fuzzy subgroups are given. The necessary properties related to these two concepts are discussed and proved by using these definitions. We shall also extend some new results related to this subject.

Keywords: Fuzzy set, β-flexible fuzzy subset, β-flexible Q-fuzzy subset, β-flexible Q-fuzzy group and β-flexible Q-fuzzy normal subgroups.

1. Introduction

Zadeh [9] first introduced the fuzzy set concepts. Then Rosenfeld [1] introduced the elementary concepts of fuzzy subgroups, which was the first fuzzification of any algebraic structures. Motivated by this subject, many mathematicians started to review the various concepts and the notion of different fuzzy algebraic structures such as fuzzy ideals in both ring and semi ring etc. Zadeh [9] introduced the concepts of interval-valued fuzzy sets, where the values are members instead of the real points. Zadeh's definition has been generalized by Anthony and Sherwood [2]. They introduced the concept of fuzzy normal subgroups. Mukherjee and Bhattacharya [4] studied the normal fuzzy groups and fuzzy cosets. On the other hand, the notion of a fuzzy subgroup of abelian group was introduced by Murali and Makamba [6], who counted the number of fuzzy subgroups in an abelian group of order $p^n q$, where p, n, q are positive integers, whereas Solairaju and Nagarajan [10] have introduced the notion of Q-fuzzy subgroups and upper Q-fuzzy order. This study was based on the references of Sarangapani and Muruganantham [11], Geethalakshmi Manickam and Solairaju [17] have introduced the concept of β-flexible Q-fuzzy groups and β-flexible normal Q-fuzzy groups and some of its elementary properties are discussed and derived.

2. Preliminaries and Definitions

The basic definitions and results are presented now in a β-flexible Q-fuzzy group.

Definition 2.1. *Let X be a set. Then the mapping $X : G \to [0,1]$ is called a fuzzy subset of X.*

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Definition 2.2. *For any Q-fuzzy set A in X and $t \in [0,1]$, the set $U(A:t) = \{x \in X : A(x,q) \geq t, \forall \ \ q \in Q\}$ which is a cut-set of A.*

Definition 2.3. *Let X be any group. A mapping $\Omega : X_\beta \to [0,1]$ is a β-fuzzy group of X if*

(i). $\Omega(x\beta y\beta) \leq \max\{\Omega(x\beta), \Omega(y\beta)\}$.

(ii). $\Omega(x^{-1}\beta) \leq \Omega(x\beta)$ *for all $x, y \in X_\beta$ and for any β in X_β.*

Definition 2.4. *A β-Q-fuzzy set A is called β-Q-fuzzy group of X_β if*

(β-QFG1): $A(x\beta y\beta, q) \geq \min\{A(x\beta, q), A(y\beta, q)\}$.

(β-QFG2): $A(x^{-1}\beta, q) = A(x\beta, q)$.

(β-QFG3): $A(x\beta, q) = 1$ *for all $x, y \in X_\beta$ and $q \in Q$.*

Definition 2.5. *If Ω is a β-Q-fuzzy group of a group X_β having identity e, then*

(i). $\Omega(x^{-1}\beta, q) = \Omega(x\beta, q)$.

(ii). $\Omega(e\beta, q) \leq \Omega(x\beta, q)$ *for all $x \in X_\beta$.*

Definition 2.6. *Let Ω be a β-Q-fuzzy group of X_β. Then Ω is called a β-Q-fuzzy normal group of X_β if $\Omega(x\beta y\beta, q) = \mu(y\beta x\beta, q)$ for all $x, y \in X_\beta$.*

Definition 2.7. *The T-norm is a function that maps the pair of number in the unit interval into the unit interval, which is given as $T : [0,1] \times [0,1] \to [0,1]$. The following four conditions are hold hood for all a_1, a_2, a_3 and $a_4 \in [0,1]$.*

(i). $T(a_1, a_2) = T(a_2, a_1)$.

(ii). $T(a_1, T(a_2, a_3)) = T(T(a_1, a_2), a_3)$.

(iii). $T(a_1, 1) = T(1, a_1) = 1$.

(iv). *If $a_1 \leq a_3$ and $a_2 \leq a_4$ then $T(a_1, a_2) \leq T(a_3, a_4)$.*

Note: The T-norm is a minimum norm.

Definition 2.8. *Let X_β be a set. Then a mapping $\Omega : X_\beta \times Q \to Q^*([0,1])$ is called β-Q-flexible subset of X_β, where $Q^*([0,1])$ denotes the set of all non-empty subsets in the interval [0, 1].*

Definition 2.9. *Let X_β be a non-empty set and Ω, λ be two β-flexible Q-fuzzy subsets of X_β. Then the intersection of Ω and λ is denoted by $(\Omega \cap \lambda)$ is defined by $\Omega \cap \lambda = \{\min\{a,b\}/a \in \Omega(x\beta), b \in \lambda(x\beta)\}$ for all $x \in X_\beta$.*
The union of Ω and λ is denoted by $(\Omega \cap \lambda)$ is defined by $\Omega \cap \lambda = \{\max\{a,b\}/a \in \Omega(x\beta), b \in \lambda(y\beta)\}$ for all $x \in X_\beta$.

Definition 2.10. *Let X_β be a groupoid (it is a set which is closed under a binary relation, namely multiplicatively). Then the mapping is called β-flexible Q-fuzzy groupoid if the following conditions are hold hood:*

(i). $\inf \Omega(x\beta y\beta, q) \geq T\{\inf \Omega(x\beta, q), \inf \Omega(y\beta, q)\}$.

(ii). $\sup \Omega(x\beta y\beta, q) \geq T\{\sup \Omega(x\beta, q), \sup \Omega(y\beta, q)\}$ *for all $x, y \in X_\beta$ and $q \in Q$.*

Definition 2.11. *Let X_β be a group. A mapping $\Omega : X_\beta \times Q \to Q^*([0,1])$ is called a β-flexible Q-fuzzy group of X_β if the following conditions are hold hood:*

(i). $\inf \Omega(x\beta y\beta, q) \geq T\{\inf \Omega(x\beta, q), \inf \Omega(y\beta, q)\}$.

(ii). $\sup \Omega(x\beta y\beta, q) \geq T\{\sup \Omega(x\beta, q), \sup \Omega(y\beta, q)\}$.

(iii). $\inf \Omega(x^{-1}\beta, q) \geq \inf \Omega(x\beta, q)$.

(iv). $\sup \Omega(x^{-1}\beta, q) \geq \sup \Omega(x\beta, q)$ *for all* $x, y \in X_\beta$ *and* $q \in Q$.

Example 2.12. *Let* $G = \{e, p, q, r\}$ *be Klein's four groups. We define the multiplication in a group G as follows*

•	e	p	q	r
e	e	e	e	e
p	p	p	p	p
q	e	e	e	q
r	p	p	p	e

Then $(G, \bullet)$ *is a group. Define a flexible fuzzy subset* $\mu : G \to Q^*([0,1])$ *by* $\mu(e) = 0.75, \mu(p) = 0.25, \mu(q) = 0.0.25, \mu(r) = 0.75$. *Then* μ *is a flexible fuzzy subgroup of G.*

Note 2.13. *In definition* $*$ *if* $\Omega : X_\beta \times Q \to Q^*([0,1])$, *then* $\Omega(x\beta, q)$ $\forall$ $x \in X_\beta$ *are real points in [0, 1] and also* $\inf(\Omega(x\beta, q)) = \sup(\Omega(x\beta, q)) = \Omega(x\beta, q)$, $x \in X_\beta$ *and* $q \in Q$. *Thus definition* $*$ *reduces to definition of Rosenfeld's fuzzy group. So a* β*-flexible Q-fuzzy group is a generalization of Rosenfeld's fuzzy group.*

3. Properties of β-Flexible Q-Fuzzy Group

Theorem 3.1. *A* β*-flexible Q-fuzzy subset* Ω *of a group* X_β *is a* β*-flexible Q-fuzzy group if and only if the following conditions are hold.*

(i). $\inf \Omega(x\beta y^{-1}\beta, q) \geq T\{\inf \Omega(x\beta, q), \inf \Omega(y\beta, q)\}$.

(ii). $\sup \Omega(x\beta y^{-1}\beta, q) \geq T\{\sup \Omega(x\beta, q), \sup \Omega(y\beta, q)\}$.

for all $x, y \in X_\beta$.

Proof. At first let Ω be a β-flexible Q-fuzzy group of X_β and $x, y \in X_\beta$. Then

$$\begin{aligned}
\inf \Omega(x\beta y^{-1}\beta, q) &\geq T\{\inf \Omega(x\beta, q), \inf \Omega(y^{-1}\beta, q)\} \\
&= T\{\inf \Omega(x\beta, q), \inf \Omega(y\beta, q)\} \\
\sup \Omega(x\beta y^{-1}\beta, q) &\geq T\{\sup \Omega(x\beta, q), \sup \Omega(y\beta, q)\} \\
&= T\{\sup \Omega(x\beta, q), \sup \Omega(y\beta, q)\}.
\end{aligned}$$

Conversely, let Ω be a β-flexible Q-fuzzy subset of X_β and given conditions hold. Then it follows that

$$\begin{aligned}
\inf \Omega(e\beta, q) &= \inf \Omega(x\beta x^{-1}\beta, q) \\
&\geq T\{\inf \Omega(x\beta, q), \inf \mu(x\beta, q)\} \\
&= \inf \Omega(x\beta, q) \qquad (1) \\
\sup \Omega(e\beta, q) &= \sup \Omega(x\beta x^{-1}\beta, q) \\
&\geq T\{\sup \Omega(x\beta, q), \sup \Omega(x\beta, q)\}
\end{aligned}$$

$$= \sup \Omega(x\beta, q) \tag{2}$$

for all $x \in X_\beta$. It implies that

$$\begin{aligned}
\inf \Omega(x^{-1}\beta, q) &= \inf \Omega(e\beta x^{-1}\beta, q) \\
&\geq T\{\inf \Omega(e\beta, q), \inf \Omega(x\beta, q)\} \\
&= \inf \Omega(x\beta, q) \quad \text{by using } (1) \\
\sup \Omega(x^{-1}\beta, q) &= \sup \Omega(e\beta x^{-1}\beta, q) \\
&\geq T\{\sup \Omega(e\beta, q), \sup \Omega(x\beta, q)\} \\
&= \sup \Omega(x\beta, q) \quad \text{by using } (2)
\end{aligned}$$

Again,

$$\begin{aligned}
\inf \Omega(x\beta y\beta, q) &\geq T\{\inf \Omega(x\beta, q), \inf \Omega(y^{-1}\beta, q)\} \\
&\geq T\{\inf \Omega(x\beta, q), \inf \Omega(y\beta, q)\}. \\
\sup \Omega(x\beta y\beta, q) &\geq T\{\sup \Omega(x\beta, q), \sup \Omega(y\beta, q)\} \\
&\geq T\{\sup \Omega(x\beta, q), \sup \Omega(y\beta, q)\}.
\end{aligned}$$

Hence Ω is a β-flexible Q-fuzzy group of X_β. □

Theorem 3.2. *If Ω is a β-flexible Q-fuzzy groupoid of an infinite group X_β, then Ω is a β-flexible Q-fuzzy group of X_β.*

Proof. Let $x \in X_\beta$. Since X_β is finite, x has finite order, say p. Then $x^p = e$, where e is the identity element of X_β. Thus $x^{-1}\beta = \mu^{p-1}\beta$ using the definition of β-flexible Q-fuzzy groupoid, it gets that $\inf \Omega(x^{-1}\beta, q) = \inf \Omega(x^{p-1}\beta, q) = \inf \Omega(x^{p-2}\beta, q) \geq T\{\inf \Omega(x^{p-2}\beta, q), \Omega(x\beta, q)\}$. Again, $\inf \Omega(x^{p-2}\beta, q) = \inf \Omega(x^{p-3}\beta, x, q) \geq T\{\inf \Omega(x^{p-3}\beta, q), \Omega(x\beta, q)\}$. Then we have $\inf \Omega(x^{-1}\beta, q) \geq T\{\inf \Omega(x^{p-3}\beta, q), \inf \Omega(x\beta, q)\}$. So applying the definition of β-flexible Q-fuzzy groupoid repeatedly, we have now $\inf \Omega(x^{-1}\beta, q) \leq \inf \Omega(x\beta, q)$. Similarly it gives that $\sup \Omega(x^{-1}\beta, q) \leq \sup \Omega(x\beta, q)$. Therefore Ω is a β-flexible Q-fuzzy group of X_β. □

Theorem 3.3. *The intersection of any two β-flexible Q-fuzzy groups is also a β-flexible Q-fuzzy group of X_β.*

Proof. Let A and B be any two β-flexible Q-fuzzy groups of X_β, both x and y are elements of X_β. Then

$$\begin{aligned}
\inf(A \cap B)(x\beta y^{-1}\beta, q) &= T\{\inf A(x\beta y^{-1}\beta, q), \inf B(x\beta y^{-1}\beta, q)\} \\
&\geq T\{T\{\inf A(x\beta, q), \inf A(x\beta, q)\}, \{\inf B(x\beta, q), \inf B(y\beta, q)\}\} \\
&= T\{T\{\inf A(x\beta, q), \inf B(x\beta, q)\}, T\{\inf A(x\beta, q), \inf B(y\beta, q)\}\} \\
&= T\{\inf A \cap B(x\beta, q), \inf A \cap B(y\beta, q)\} \qquad (3)
\end{aligned}$$

Again

$$\begin{aligned}
\sup(A \cap B)(x\beta y^{-1}\beta, q) &= T\{\sup A(x\beta y^{-1}\beta, q), \sup B(x\beta y^{-1}\beta, q)\} \\
&\geq T\{T\{\{\sup A(x\beta, q), \sup A(x\beta, q)\}, \{\sup B(x\beta, q), \sup B(y\beta, q)\}\}\}
\end{aligned}$$

$$= T\{T\{\sup A(x\beta, q), \sup B(x\beta, q)\}, T\{\sup A(x\beta, q), \sup B(y\beta, q)\}\}$$
$$= T\{\sup A \cap B(x\beta, q), \sup A \cap B(y\beta, q)\} \tag{4}$$

By (3) and (4) and using results, then it gets that $(A \cap B)$ is a β-flexible Q-fuzzy group of X_β. □

Theorem 3.4. *If A is a β-flexible Q-fuzzy group of a group X_β having identity e, then*

(i). $\inf A(x^{-1}\beta, q) = \inf A(x\beta, q)$ *and* $\sup A(x^{-1}\beta, q) = \sup A(x\beta, q)$.

(ii). $\inf A(e\beta, q) = \inf A(x\beta, q)$ *and* $\sup A(e\beta, q) = \sup A(x\beta, q)$

for all $x \in X_\beta$.

Proof.

(i). As A is a β-flexible Q-fuzzy group of a group X_β, then it gets that $\inf A(x^{-1}\beta, q) \leq \inf A(x\beta, q)$. Again, $\inf A(x\beta, q) = \inf A((x^{-1})^{-1}\beta, q) \leq \inf A(x^{-1}\beta, q)$. So $\inf A(x^{-1}\beta, q) = \inf A(x\beta, q)$. Similarly it can prove that $\sup A(x^{-1}\beta, q) = \sup A(x\beta, q)$

$$\inf A(e\beta, q) = \inf A(x\beta x^{-1}\beta, q) \geq T\{\inf A(x\beta, q), \inf A(x^{-1}\beta, q)\}$$
$$\sup A(e\beta, q) = \sup A(x\beta x^{-1}\beta, q) \geq T\{\sup A(x\beta, q), \sup A(x^{-1}\beta, q)\}.$$

□

Theorem 3.5. *Let Ω and λ be two β-flexible Q-fuzzy groups of X_{β_1} and X_{β_2} respectively, and f be a homomorphism from X_{β_1} to X_{β_2}. Then*

(i). $f(\Omega\beta, q)$ *is a β-flexible Q-fuzzy group of X_{β_2}.*

(ii). $f(\Omega\beta, q)$ *is a β-flexible Q-fuzzy group of X_{β_1}.*

Proof. It is straight forward. □

Remark 3.6. *If Ω is a β-flexible Q-fuzzy group of X_β and K is a subgroup of X_β, then the restriction of Ω to $K(\Omega/K)$ is a β-flexible Q-fuzzy group of K.*

4. Normal β-Flexible Q-Fuzzy Group

Definition 4.1. *If Ω is a β-flexible Q-fuzzy group of a group X_β, then Ω is called a normal β-flexible Q-fuzzy group of X_β if*

$$\inf \Omega(x\beta y\beta) = \inf \Omega(y\beta x\beta)$$
$$\sup \Omega(x\beta y\beta) = \sup \Omega(y\beta x\beta)$$

for all $x, y \in X_\beta$.

Theorem 4.2. *The intersection of any two normal β-flexible Q-fuzzy groups of X_β is also a normal β-flexible Q-fuzzy group of X_β.*

Proof. Let A and B be any two normal β-flexible Q-fuzzy groups of X_β. Then $A \cap B$ is a β-flexible Q-fuzzy group of X_β. Let $x, y \in X_\beta$. Then by definition

$$\begin{aligned}
\inf(A \cap B)(x\beta y\beta, q) &= T\{\inf A(x\beta y\beta, q), \inf B(x\beta y\beta, q)\} \\
&= T\{\inf A(y\beta x\beta, q), \inf B(y\beta x\beta, q)\} \\
&= \inf A \cap B(y\beta x\beta, q).
\end{aligned}$$

Similarly $\sup(A\cap B)(x\beta y\beta, q) = \sup(A\cap B)(y\beta x\beta, q)$. This shows that $A\cap B$ is a normal β-flexible Q-fuzzy group of X_β. □

Theorem 4.3. *The intersection of any arbitrary collection of normal β-flexible Q-fuzzy groups of a group X_β is also a normal β-flexible Q-fuzzy group of X_β.*

Proof. Let $x, y \in X_\beta$ and $\alpha \in X_\beta$. Now it finds that

$$\begin{aligned}
\inf A(x\beta y^{-1}\beta, q) &= \inf A(\alpha^{-1}x\beta y^{-1}\beta\alpha, q) \\
&= \inf A(\alpha^{-1}x\beta\alpha\alpha^{-1}y^{-1}\beta\alpha, q) \\
&= \inf(A(\alpha^{-1}x\beta\alpha, q), A((\alpha^{-1}y\beta\alpha)^{-1}, q)) \\
&\geq T\{\inf(A(\alpha^{-1}x\beta\beta\alpha, q), \inf A((\alpha^{-1}y\alpha), q))\} \\
&= T\{\inf(A(x\beta, q), A((y\beta, q))\}.
\end{aligned}$$

Again,

$$\begin{aligned}
\sup A(x\beta y^{-1}\beta, q) &= \sup A(\alpha^{-1}x\beta y^{-1}\beta\alpha, q) \\
&= \sup A(\alpha^{-1}x\beta\alpha\alpha^{-1}y^{-1}\beta\alpha, q) \\
&= \sup(A(\alpha^{-1}x\beta\alpha, q), A((\alpha^{-1}y\beta\alpha)^{-1}, q)) \\
&\geq T\{\sup(A(\alpha^{-1}x\beta\alpha, q), \sup(A(\alpha^{-1}y\beta\alpha), q))\} \\
&= T\{\sup(A(x\beta, q), A(y\beta, q))\}.
\end{aligned}$$

It follows that A is a normal β-flexible Q-fuzzy group of a group X_β. □

5. Conclusion

In this study we have introduced the concept of a β-flexible fuzzy set and a β-flexible Q-fuzzy group. Based on this β-flexible Q-fuzzy normal groups are given and some elementary properties are discussed. Some of these properties are proved.

References

[1] A. Rosenfield, *Fuzzy groups*, J. Math. Anal. Appl., 35(1965), 521-517.

[2] J. H. Anthony and H. Sherwood, *Fuzzy groups redefined*, J. Hath. Anal. Appl., 69(1979), 124-130.

[3] N. P. Mukherjee and P. Bhattacharya, *Fuzzy normal subgroups and fuzzy cosets*, Information Sciences, 34(1984), 225-239.

[4] N. P. Mukherjee and P. Bhattacharya, *Fuzzy groups: Some group theoretic analogs*, Information Sciences, 39(1986), 247-269.

[5] V. Murali and B. B. Makamba, *Fuzzy subgroups of finite abelian groups*, Far East journal of Mathematical Science, 14(2004), 360-371.

[6] V. Murali and B. B. Makamba, *Counting the number of fuzzy subgroups of on abelian group of order* $p^n q$, Fuzzy sets and systems, 44(2006), 459-470.

[7] A. Solairaju and R. Nagarajan, *Q-fuzzy left R-subgroup of near rings with respect to T-norms*, Antarctica Journal of Mathematics, 5(1-2)(2008), 59-63.

[8] A. Solairaju and R. Nagarajan, *A new structure and construction of Q-fuzzy groups*, Advances in Fuzzy Mathematics, 4(1)(2009), 23-29.

[9] A. Zadeh, *Fuzzy Sets*, Information and Control, 8(1965), 338-353.

[10] A. Solairaju and R. Nagarajan, *Some structure properties of upper Q-fuzzy index order with upper Q-fuzzy subgroups*, Int. Journal of open Problem Math. Appl., 1(2011), 21-29.

[11] P. Sarangapani and P. Muruganantham, *New structures on upper flexible Q-fuzzy groups*, International Journal of Mathematics Research, 8(2)(2016), 107-112.

[12] W. H. Wu, *Normal fuzzy subgroups*, Fuzzy Math., 1(1981), 21-30.

[13] M. Massadeh, *Properties of fuzzy subgroups in particular the normal subgroups*, Doctorate Thesis, Damacus University-Syrian Arab Republic.

[14] V. Vanitha and G. Subbiah, *Certain thresholds on flexible fuzzy subgroups with flexible fuzzy order*, Int. Rearch Journal of Pure Algebra, 6(11)(2016), 436-442.

[15] H. J. Zimmerman, *Fuzzy set theory and its applications*, Third Edition, Kluwer Academic publishers London, (1997).

[16] Muhammad Gulzar and Ghazanfar Abbas, *Algebraic Properties of ω-Q-fuzzy subgroups*, Int. Journal of Mathematics and Computer Science, 15(1)(2020), 265-274.

[17] Geethalakshmi Manickam and A. Solairaju, *New structure properties of flexible Q-fuzzy groups and flexible normal Q-fuzzy subgroups*, Int. Research Journal of Pure Algebra, 10(7)(2020), 14-18.