Int. J. Math. And Appl., 10(1)(2022), 119–124
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Recursive Form of B-spline Based Collocation Solution to Homogeneous Differential Equations with Neumann's Boundary Value Problems

Y. Rajashekhar Reddy[1,*]

1 Department of Mathematics, JNT University College of Engineering, Jagitial, Karimnagar, Telangana, India.

Abstract: The area of differential equations is a very broad field of study. The versatility of differential equations allows the area to be applied to a variety of topics from physics to population growth to the stock market. They are a useful tool for modeling and studying naturally occurring phenomena such as determining when beams may break as well as predicting future outcomes such as the spread of disease or the changes in populations of different species over time. Anytime an unknown phenomenon is changing with respect to time or space, a differential equation is involved Such differential equations have been solved by using many numerical methods. One numerical method which is developed by using simplified form of Recursive B-spline function in collocation method as basis function. The present method is used to solve second order and third order homogeneous differential equations with Neumann's boundary conditions. Solutions of Numerical examples for non -uniform length of the points show the efficiency of the method and easiness. Stability of B-spline based collocation method and accuracy of numerical solution is constantly improved by decreasing the nodal space.

Keywords: Neumann's boundary conditions, Collocation method, Homogeneous differential equations, B-splines.

1. Introduction

In recent years, many numerical methods are developed to solve Homogeneous differential equations with Neumann-Dirchlet's boundary conditions. The methods include like B-spline collocation method [13], finite difference method [4], kernel space [5, 6] sinc collocation method [7] and predictor and corrector method [8] and many more. The B-spline based collocation method is used to evaluate boundary value problems including singular boundary value problems [9].

However, it is observed from the recent literature that B-spline basis functions are derived using fixed equidistant space for a particular degree only. If the recursive formulation given by Carl. De boor [12] is applied, the basis function evaluation can be generalized and without fixing of degree of the basis function can be used in collocation method for uniform or non-uniform mesh sizes.

In this paper, after defining the B-spline basis function recursively, the B-spline collocation method is described and formulated. The efficiency of the method is demonstrated using the third order Homogeneous differential equations with Neumann's boundary conditions. Considering third order homogeneous linear differential equations

$$k_1 x^3 \frac{d^3V}{dx^3} + k_2 x^2 \frac{d^2V}{dx^2} + k_3 x \frac{dV}{dx} + k_4 V = R(x), \quad a \le x \le b \tag{1}$$

* *E-mail: yrsreddy4@gmail.com*

with Neumann's boundary conditions $V'(b) = d_1, V(b) = d_2$ and $V''(b) = d_3$ where $a, b, d_1, d_2, d_3, k_1, k_2, k_3$ and k_4 are constants $R(x)$ is a functions of x. Let

$$V^h(x) = \sum_{i=-2}^{n-1} C_i M_{i,p}(x) \tag{2}$$

where C_i's are constants to be determined and $M_{i,p}(x)$ are B-spline basis functions, be the approximate global solution to the exact solution $V(x)$ of the considered third order Homogeneous differential equation (1).

2. B-splines

In this section, definition and properties of B-spline basis functions [1, 2] are given in detail. A zero degree and other than zero degree B-spline basis functions are defined at x_i recursively over the knot vector space $X = \{x_1, x_2, x_3, \ldots, x_{n-1}, x_n\}$ as

(i) If $p = 0$

$$M_{i,p}(x) = \begin{cases} 1, & \text{if } x \in (x_i, x_{i+i}); \\ 0, & \text{if } x \notin (x_i, x_{i+i}). \end{cases}$$

(ii) If $p \geq 1$

$$M_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} M_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} M_{i+1,p-1}(x) \tag{3}$$

where p is the degree of the B-spline basis function and $x$is the parameter belongs to X. When evaluating these functions, ratios of the form 0/0 are defined as zero.

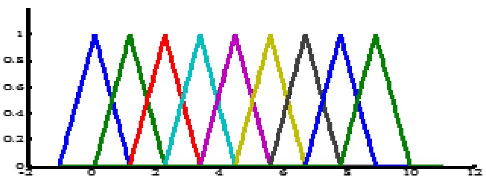

Figure 1. **First degree B-spline basis function with uniform Knot vector** $X = \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

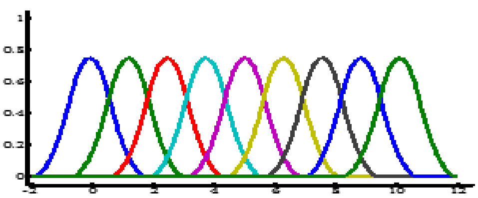

Figure 2. **Second degree B-spline basis function with uniform Knot vector** $X = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

2.1. Derivatives of B-splines

If $p = 2$, we have

$$M'_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} M'_{i,p-1}(x) + \frac{M_{i,p-1}(x)}{x_{i+p} - x_i} + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} M'_{i+1,p-1}(x) - \frac{M_{i+1,p-1}(x)}{x_{i+p+1} - x_{i+1}}$$

$$M''_{i,p}(x) = 2\frac{M'_{i,p-1}(x)}{x_{i+p} - x_i} - 2\frac{N'_{i+1,p-1}(x)}{x_{i+p+1} - x_{i+1}} \tag{4}$$

In the above equations, previous degree functions are used as recursively to define next level degree basis function. Knot vectors are based to define the B-spline based functions. Knots are real quantities. Knot vector is a non-decreasing set of Real numbers. Knot vectors are classified as non-uniform knot vectors, uniform knot vector and open uniform knot vectors. Uniform knot vector in which difference of any two consecutive knots is constant is used for test problems in this paper. Two knots are required to define the zero degree basis function In a similar way, a p^{th} degree B-spline basis function at a knot have a domain of influence of $(p+2)$ knots. B-spline basis functions of degree one and degree two over uniform knot vector are shown graphically below in figures 1 and 2.

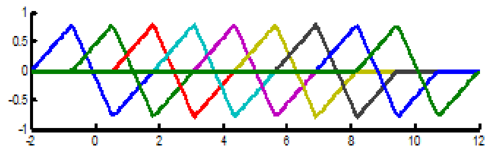

Figure 3. First derivative of second degree B-spline basis function with uniform Knot vector $X = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

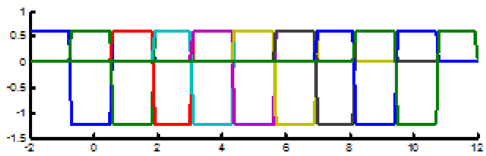

Figure 4. Second derivative of second degree B-spline basis function with uniform Knot vector $X = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

2.2. B-spline collocation method

Collocation method is widely used in approximation theory particularly to solve differential equations. In collocation method, the assumed approximate solution is made it exact at some nodal points by equating residue zero at that particular node. B-spline basis functions are used as the basis in B-spline collocation method whereas the base functions which are used in normal collocation method are the polynomials vanishes at the boundary values. Residue which is obtained by substituting equation (2) in equation (1) is made equal to zero at nodes in the given domain to determine unknowns in (2). Let $[a, b]$ be the domain of the governing differential equation and is partitioned as $X = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$ with unequal length of n sub domains. The x_i's are known as nodes, the nodes are treated as knots in collocation B-spline method where B-spline basis functions are defined and these nodes are used to make the residue equal to zero to determine unknowns C_i's in (2). Two extra knot vectors are taken into consideration beside the domain of problem both side when evaluating the second degree B-spline basis functions at the nodes.

Substituting, the approximate solution (2) and its derivatives in (1).

$$\frac{d^2V^h}{dx^2} + k_1 P(x)\frac{dV^h}{dx} + k_2 Q(x)V^h = R(x)$$

$$\sum_{i=-2}^{n-1} C_i M''_{i,p}(x) + k_1 P(x) \sum_{i=-2}^{n-1} C_i M'_{i,p}(x) + k_2 Q(x) \sum_{i=-2}^{n-1} C_i M_{i,p}(x) = R(x) \tag{5}$$

Equation (5) which is valuated at x_i's, $i = 0, 1, 2, \ldots, n-1$ gives the system of $(n-1) \times (n+1)$ equations in which $(n+1)$ arbitrary constants are involved. Two more equations are needed to have $(n+1) \times (n+1)$ square matrix which helps to determine the $(n+1)$ arbitrary constants. The remaining two equations are obtained using

$$\sum_{i=-2}^{n-1} C_i M_{i,p}(a) = d_1, \tag{6}$$

$$\sum_{i=-2}^{n-1} C_i M_{i,p}(b) = d_2, \tag{7}$$

Now using all the above equations (5), (6), (7) i.e. $(n+1)$ a square matrix is obtained which is diagonally dominated matrix because every second degree basis function has values other than zeros only in three intervals and zeros in the remaining intervals, it is a continuing process like when one function is ending its effect in its surrounding region than other function starts its effectiveness as parameter value changing. In other words, every parameter has at most under the three $(p = 2)$ basis functions. The systems of equations are easily solved for arbitrary constants C_i's. Substituting these constants in (2), the approximation solution is obtained and used to estimate the values at domain points. Absolute Relative error is evaluated by using the following relationship in exact and approximate solutions

$$\text{Absolute Relative Error} = \left| \frac{M_{exact} - M_{appro}}{M_{exact}} \right|$$

3. Numerical Experiments

The effectiveness of the present method is demonstrated by considering the various example

Example 3.1. *The exact solution second order homogeneous boundary value problem [14] considered below*

$$3x^2 \frac{d^2V}{dx^2} - 4x\frac{dV}{dx} + 2V = 0; \quad V'(1) = 1, \quad V(1) = 2$$

The exact solution is $V(x) = \left(\frac{9}{5}\right) x^{1/3} + \frac{1}{5}x^2$. *Table 1 gives comparison of the solutions of second degree B-spline based collocation method and Exact solution at various points. These nodes are taken at non uniform length. From Table 1 approximating solution values are good fit with the values of exact solution values. This shows the effectiveness of this method in evaluating second order homogeneous differential equations with Neumann's boundary conditions.*

Nodes	1	1.00250	1.1230	1.1510	1.2225	1.2590	1.3023	1.3580	1.4020	1.4551
BCS*	2.0000	2.0025	2.1230	2.1510	2.2229	2.2597	2.3037	2.3605	2.4057	2.4607
Exact Solution	2.0000	2.0225	2.1232	2.1513	2.2236	2606	2.3049	2.3621	2.4077	2.4632

Table 1. **Computed value and exact value at different nodes with non-uniform lengths**

Nodes	1.5500	1.5560	1.6210	1.6550	1.7000	1.7540	1.8200	1.8534	1.9210	1.9570	2
BCS	2.5602	2.5665	2.6357	2.6722	2.7210	2.7800	2.8531	2.8905	2.9671	3.0083	3.0579
Exact Solution	2.5636	2.5700	2.6400	2.6769	2.7263	2.7861	2.8602	2.8981	2.9756	3.0175	3.0679

Overall behavior of B-spline collocation solution for Example 3.2 throughout the domain is displayed in figure **??**.

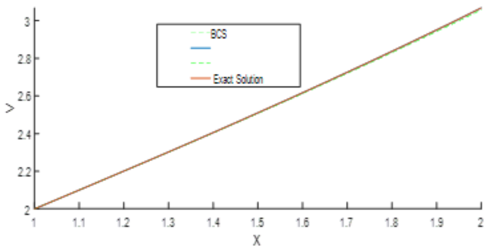

Figure 5. **Comparison of approximate and Exact solutions for the Example 3.2 for mesh size** $h = 0.01$

Example 3.2. *Consider a Homogeneous third order boundary value problem [14] given as*

$$x^3\frac{d^3V}{dx^3}+10x^2\frac{d^2V}{dx^2}-20x\frac{dV}{dx}+20V=0;\quad V'(1)=-1,\quad V(1)=0\ \ and\ \ V''(b)=1$$

The exact solution is $V(x)=\left(-\frac{3}{4}\right)x^2+\frac{1}{4}x^{-10}+)\frac{8}{11}x$. *The domain is divided into equal intervals and non-uniform length and associated with knot vector space. Overall behavior of B-spline collocation for Example 3.2 solution throughout the domain with the mesh size 0.020 is displayed in figure 6.*

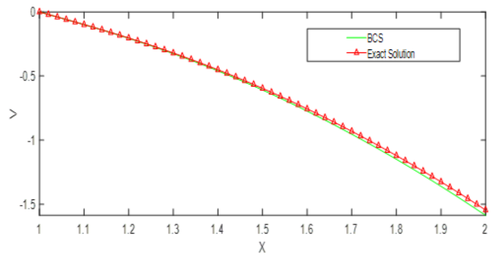

Figure 6. Comparison of the B-spline collocation and Exact solutions

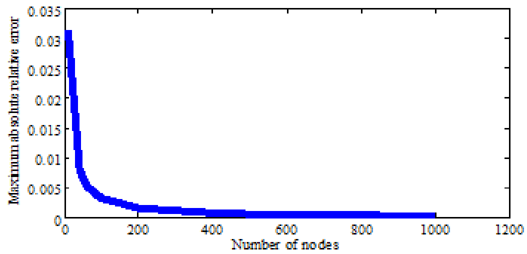

Figure 7. Trend of maximum absolute relative error for different sizes of nodes

Maximum absolute relative error is approximating the x-axis when moving along the x-axis, i.e Result of increasing in interpolating points gives the falling in values of maximum absolute relative error.

4. Conclusion

The third degree and second degree B-spline basis functions defined recursively are incorporated in the collocation method and applied the same to the homogeneous boundary value problems for non -uniform lengths. The effectiveness of the proposed method is examined by considering two numerical examples. The present numerical solution is compared with exact solution and found to be in good approximation. This method may be applied to different types of homogeneous differential equations with different types boundary value problems for its efficiency.

References

[1] T. J. R. Hughes, J. A. Cottrell and Y. Bazilevs, *Isgeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement*, Comput. Methods Appl. Mech. Engg., 194(39-41)(2005), 4135-4195.

[2] David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., Tata McGraw-Hill Edition, New Delh.

[3] C. de Boor and K. Hollig, *B-splines from parallelepipeds*, J. Analyse Math., 42(1982), 99-115.

[4] R. K. Pandey and Arvind K. Singh, *On the convergence of a finite difference method for a class of singular boundary value problems arising in physiology*, J. Comput. Appl. Math., 166(2004), 553-564.

[5] F. Z. Geng and M. G. Cui, *Solving Singular Nonlinear Second-Order Periodic Boundary Value Problems in the Reproducing Kernel Space*, Applied Mathematics and Computation, 192(2007), 389-398.

[6] Z. Y. Li, Y. L. Wang, F. G. Tan, X. H. Wan and T. F. Nie, *The Solution of a Class of Singularly Perturbed Two-Point Boundary Value Problems by the Iterative Reproducing Kernel Method*, Abstract and Applied Analysis, (2012), 1-7.

[7] A. Mohsen and M. El-Gamel, *On the Galerkin and Collocation Methods for Two Point Boundary Value Problems Using Sinc Bases*, Computers and Mathematics with Allications, 56(2008), 930-941.

[8] Abdalkaleg Hamad, M. Tadi and Miloje Radenkovic, *A Numerical Method for Singular Boundary-Value Problems*, Journal of Applied Mathematics and Physics, 2(2014), 882-887.

[9] S. Joan Goh, Ahmad Abd. Majid and Ahmad Izani Md. Ismail, *Extended cubic uniform B-spline for a class of singular boundary value problems*, Science Asia, 37(2011), 79-82.

[10] I. J. Schoenberg, *Contributions to the problem of approximation of equidistant data by analytic functions*, Quart. Appl. Math., 4(1946), 45-99.

[11] H. B. Currya and I. J. Schoenberg, *On Polya frequency functions IV: The fundamental spline functions and their limits*, J. Anal. Math., 17(1966), 71-107.

[12] Carl De Boor, *On Calculating with B-plines*, Journal of Approximation Theory, 6(1972).

[13] Hikmet Caglar, Nazan Caglar and Khaled Elfaituri, *B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problems*, Applied Mathematics and Computation, 175(2006), 72-79.

[14] Y. Rajashekhar Reddy, *Numerical Solution to Euler–Cauchy Equation with Neumann Boundary Conditions Using B-Spline Collocation Method*, Annals of Pure and Applied Mathematics, 11(2)(2016), 1-6.