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Conformal Ricci Soliton in Generalized Sasakian-Space-Forms

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Abstract: In this paper we consider a generalized Sasakian-space-forms with M-projective curvature tensor, Pseudo projective curvature tensor admitting Conformal Ricci soliton. We have found that M-projective Ricci symmetric generalized Sasakian-space-forms is a quadratic equation. ξ-M-projectively flat generalized Sasakian-space-forms. We have proved that a pseudo projective semi symmetric generalized Sasakian-space-forms is η-Einstein manifold.

MSC: 53C15, 53C25, 53C44, 53D15.

Keywords: Generalized Sasakian-space-forms, Conformal Ricci soliton, M-projective curvature tensor, Pseudo projective curvature tensor.

1. Introduction

The concept of Ricci soliton was introduced by Hamilton in mid 80's. Ricci solitons are natural generalizations of Einstein metrices and also correspond to self-similar solutions of Hamilton's Ricci flow [8]. It often arises as limits of dilations of singularities in the Ricci flow. Many geometers ([5, 22]) studies Ricci soliton in Contact manifolds. The Ricci soliton equation is given by

$$\pounds_X + 2S + 2\lambda g = 0, \tag{1}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ a scalar. In 2005 A.E. Fischer [10] has introduced a new concept called conformal Ricci flow, which a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named the conformal Ricci flow equations because of the role that conformal geometry plays in constraining the scalar curvature and because these equations are the vector field sum of a conformal flow equation and a Ricci flow equation. These new equations are given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg, \tag{2}$$
$$R(g) = -1.$$

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Where $R(g)$ is the scalar curvature of the manifold and p is scalar non-dynamical field.

N. Basu and A. Bhattacharyya [4] brought the notion of Conformal Ricci soliton it assures the Conformal Ricci flow equation as follows

$$\pounds_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g. \tag{3}$$

In this paper, we study M-projective curvature tensor and Pseudo projective curvature tensor of generalized Sasakian-space-forms. In Section 3, we have proved $W^*(\xi, X).S = 0$ is a quadratic equation, where W^* is M-projective curvature tensor. In Section 4, we study ξ-M-projectively flat generalized Sasakian-space-forms. In Section 5, We have found that $R(\xi, X).\tilde{P} = 0$ is an η-Einstein manifold, where $\tilde{P}$ is pseudo projective curvature tensor.

2. Preliminaries

An n-dimensional smooth manifold (M, g) is almost contact metric structure (ϕ, ξ, η, g) if it satisfies the following relations:

$$\phi^2 X = -X + \eta(X)\xi, \ \ \phi(\xi) = 0, \tag{4}$$

$$\eta(\xi) = 1, \ \ g(X, \xi) = \eta(X), \ \ \eta(\phi X) = 0, \tag{5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{6}$$

for all vector fields X, Y on M. In the view of relations, we have

$$g(\phi X, Y) = -g(X, \phi Y), \ \ g(\phi X, X) = 0, \tag{7}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y).$$

An n-dimensional generalized Sasakian-space-form is given by [1]

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{8}$$

for all vector fields X, Y, Z on M, where f_1, f_2, f_3 are functions on M, R denotes curvature tensor, and $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$. In a generalized Sasakian-space-form the following relations hold

$$S(X,Y) = [(n-1)f_1 + 3f_2 - f_3]g(X,Y) - [3f_2 + (n-2)f_3]\eta(X)\eta(Y), \tag{9}$$

$$QX = [(n-1)f_1 + 3f_2 - f_3]X - [3f_2 + (n-2)f_3]\eta(X)\xi, \tag{10}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{11}$$

$$R(\xi, X)Y = -R(X,\xi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{12}$$

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X), \tag{13}$$

$$Q\xi = (n-1)(f_1 - f_3)\xi, \tag{14}$$

$$g(QX, Y) = S(X,Y), \tag{15}$$

where S is Ricci tensor and Q is Ricci operator. By the above results, we prove the following sections.

3. Generalized Sasakian-space-forms Admitting Conformal Ricci Soliton $W^*(\xi, X).S = 0$

G. P. Pokhariyal and R. S. Mishra [13] defined M-projective curvature tensor W^* on a Riemannian manifold as

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{16}$$

From (16), we can write

$$W^*(\xi,X)Y = R(\xi,X)Y - \frac{1}{2(n-1)}[S(X,Y)\xi - S(\xi,Y)X + g(X,Y)Q\xi - g(\xi,Y)QX]. \tag{17}$$

Using (12) and (13) in (17), we have

$$\begin{aligned} W^*(\xi,X)Y &= (f_1-f_3)[g(X,Y)\xi - \eta(Y)X] - \frac{1}{2(n-1)}[S(X,Y)\xi - (n-1)(f_1-f_3)\eta(Y)X \\ &+ (n-1)(f_1-f_3)g(X,Y)\xi - \eta(Y)QX], \end{aligned} \tag{18}$$

similarly, we have

$$\begin{aligned} W^*(\xi,X)Z &= (f_1-f_3)[g(X,Z)\xi - \eta(Z)X] - \frac{1}{2(n-1)}[S(X,Z)\xi - (n-1)(f_1-f_3)\eta(Z)X \\ &+ (n-1)(f_1-f_3)g(X,Z)\xi - \eta(Z)QX], \end{aligned} \tag{19}$$

We consider that the tensor derivative of S by $W^*(\xi,X)$ is zero i.e., $W^*(\xi,X).S = 0$. Then the generalized Sasakian-space-forms admitting Conformal Ricci soliton is M-projective Ricci symmetric. It gives

$$S(W^*(\xi,X)Y,Z) + S(Y,W^*(\xi,X)Z) = 0, \tag{20}$$

Using (18) and (19) in (20), we have

$$\begin{aligned} S((f_1-f_3)[g(X,Y)\xi - \eta(Y)X] - \frac{1}{2(n-1)}[S(X,Y)\xi - (n-1)(f_1-f_3)\eta(Y)X \\ +(n-1)(f_1-f_3)g(X,Y)\xi - \eta(Y)QX],Z) + S(Y,(f_1-f_3)[g(X,Z)\xi - \eta(Z)X] \\ -\frac{1}{2(n-1)}[S(X,Z)\xi - (n-1)(f_1-f_3)\eta(Z)X + (n-1)(f_1-f_3)g(X,Z)\xi - \eta(Z)QX]) = 0. \end{aligned} \tag{21}$$

Put Z=ξ and using (13) in (21), we get

$$(f_1-f_3)S(X,Y) = \frac{1}{2}(n-1)(f_1-f_3)^2 g(X,Y) + \frac{1}{2(n-1)}S(Y,QX), \tag{22}$$

which implies

$$S(X,Y) = a_1 S(QX,Y) + b_1 g(X,Y), \tag{23}$$

where $a_1 = \frac{1}{2(n-1)(f_1-f_3)}$ and $b_1 = \frac{1}{2}(n-1)(f_1-f_3)$ which implies

$$\begin{aligned} QX &= a_1 Q^2 X + b_1 X \\ i.e., \quad & a_1 Q^2 - Q + b_1 = 0. \end{aligned} \tag{24}$$

Thus we can state the following:

Theorem 3.1. *If a generalized Sasakian-space-sorms admitting conformal Ricci soliton and the manifold is M-projective Ricci symmetric $W^*(\xi,X).S = 0$ then Ricci operator Q satisfies the quadratic equation.*

4. ξ-M-projectively Flat Generalized Sasakian-space-forms

If we consider that a generalized Sasakian-space-forms is ξ-M-projectively flat, that is

$$W^*(X,Y)\xi = 0. \tag{25}$$

Then from (16), we get

$$R(X,Y)\xi - \frac{1}{2(n-1)}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] = 0. \tag{26}$$

Using (11) and (13) in (26), we get

$$(n-1)(f_1 - f_3)[\eta(Y)X - \eta(X)Y] = \eta(Y)QX - \eta(X)QY, \tag{27}$$

put $Y = \xi$ in (27), we get

$$QX = (n-1)(f_1 - f_3)X. \tag{28}$$

Now taking inner product with U, we get

$$S(X,U) = (n-1)(f_1 - f_3)g(X,U). \tag{29}$$

Conversely, if the relation (29) is satisfied then from (26) and (28), we get

$$W^*(X,Y)\xi = 0. \tag{30}$$

Thus we can state:

Theorem 4.1. *generalized Sasakian-space-forms is ξ-M-projectively flat if and only if it is Einstein manifold.*

Again, put $Z = \xi$ and using (11) and (13) in (16), we get

$$W^*(X,Y)\xi = R(X,Y)\xi - \frac{1}{2(n-1)}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY], \tag{31}$$

which implies

$$W^*(X,Y)\xi = \frac{1}{2}(f_1 - f_3)[\eta(Y)X - \eta(X)Y] - \frac{1}{2(n-1)}[\eta(Y)QX - \eta(X)QY]. \tag{32}$$

If we consider X, Y orthogonal to ξ, we get

$$W^*(X,Y)\xi = 0. \tag{33}$$

Thus we can state, generalized Sasakian-space-forms is horizontal ξ-M-projectively flat.

5. Generalized Sasakian-space-forms Admitting Conformal Ricci Soliton $R(\xi, X).\tilde{P} = 0$

The Pseudo-projective curvature tensor $\tilde{P}$ in a generalized Sasakian-space-forms is defined by

$$\tilde{P}(X,Y)Z \;=\; aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}[\frac{a}{n-1} + b][g(Y,Z)X - g(X,Z)Y], \tag{34}$$

put $Z = \xi$ and using (11) and (13) in (34), we get

$$\tilde{P}(X,Y)\xi = [(f_1 - f_3)a + (n-1)(f_1 - f_3)b - \frac{r}{n}(\frac{a}{n-1} + b)][\eta(Y)X - \eta(X)Y]. \tag{35}$$

It can be written as

$$\tilde{P}(X,Y)\xi = \gamma[\eta(Y)X - \eta(X)Y], \tag{36}$$

where $\gamma = [(f_1 - f_3)a + (n-1)(f_1 - f_3)b - \frac{r}{n}(\frac{a}{n-1} + b)]$, which implies

$$g(\tilde{P}(X,Y)\xi, Z) = \gamma[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{37}$$

this can be written as

$$\eta(\tilde{P}(X,Y)Z) = \gamma[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]. \tag{38}$$

Now we consider that generalized Sasakian-space-forms admits conformal Ricci soliton and is pseudo projective semi symmetric i.e., $R(\xi, X).\tilde{P} = 0$ holds in M, which implies

$$R(\xi,X)(\tilde{P}(Y,Z)W) - \tilde{P}(R(\xi,X)Y,Z)W - \tilde{P}(Y,R(\xi,X)Z)W - \tilde{P}(Y,Z)R(\xi,X)W = 0. \tag{39}$$

Using (12) in (39) and putting $W = \xi$, we get

$$\begin{aligned}(f_1 - f_3)[g(X, \tilde{P}(Y,Z)\xi)\xi - \eta(\tilde{P}(Y,Z)\xi)X] - \tilde{P}\{(f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], Z\}\xi \\ - \tilde{P}\{Y, (f_1 - f_3)[g(X,Z)\xi - \eta(Z)X]\}\xi - (f_1 - f_3)\tilde{P}(Y,Z)[\eta(X)\xi - X] = 0,\end{aligned} \tag{40}$$

using (37), above equation becomes

$$\begin{aligned}(f_1 - f_3)\{\gamma[\eta(Z)g(X,Y)\xi - \eta(Y)g(X,Z)\xi] + \eta(Y)\tilde{P}(X,Z)\xi - g(X,Y)\tilde{P}(\xi,Z)\xi + \eta(Z)\tilde{P}(Y,X)\xi \\ - g(X,Z)\tilde{P}(Y,\xi)\xi - \eta(X)\tilde{P}(Y,Z)\xi + \tilde{P}(Y,Z)X\} = 0.\end{aligned} \tag{41}$$

Taking inner product with ξ in (41), we get

$$\begin{aligned}(f_1 - f_3)\{\gamma[\eta(Z)g(X,Y) - \eta(Y)g(X,Z)] + \eta(Y)\eta(\tilde{P}(X,Z)\xi) - g(X,Y)\eta(\tilde{P}(\xi,Z)\xi) + \eta(Z)\eta(\tilde{P}(Y,X)\xi) \\ - g(X,Z)\eta(\tilde{P}(Y,\xi)\xi) - \eta(X)\eta(\tilde{P}(Y,Z)\xi) + \eta(\tilde{P}(Y,Z)X)\} = 0,\end{aligned} \tag{42}$$

using (36) in (42), we get

$$(f_1 - f_3)\{\gamma\eta(Z)g(X,Y) - \gamma\eta(Y)g(X,Z) + \eta(\tilde{P}(Y,Z)X)\} = 0. \tag{43}$$

Put $Z = \xi$ in (43), we get

$$(f_1 - f_3)\{\gamma g(X,Y) - \gamma\eta(X)\eta(Y) + \eta(\tilde{P}(Y,\xi)X)\} = 0. \tag{44}$$

From (34), we get

$$\begin{aligned}\eta(\tilde{P}(Y,\xi)X) &= a(f_1 - f_3)[\eta(X)\eta(Y) - g(Y,X)] + b[(n-1)(f_1 - f_3)\eta(X)\eta(Y) - S(Y,X)] \\ &\quad - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[\eta(X)\eta(Y) - g(Y,X)].\end{aligned} \tag{45}$$

Using (45) in (44) and simplifying we get

$$\begin{aligned}S(X,Y) &= \frac{1}{b}\left[\gamma - a(f_1 - f_3) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right] g(X,Y) \\ &\quad - \frac{1}{b}\left[\gamma - a(f_1 - f_3) - b(n-1)(f_1 - f_3) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right]\eta(X)\eta(Y).\end{aligned} \tag{46}$$

This can be written as

$$S(X,Y) = a_2 g(X,Y) + b_2\eta(X)\eta(Y), \tag{47}$$

where

$$a_2 = \frac{1}{b}\left[\gamma - a(f_1 - f_3) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right],$$

and

$$b_2 = -\frac{1}{b}\left[\gamma - a(f_1 - f_3) - b(n-1)(f_1 - f_3) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right].$$

Hence we state:

Theorem 5.1. *If a generalized Sasakian-space-forms admits conformal Ricci soliton and is pseudo projective semi symmetric $R(\xi, X).\tilde{P} = 0$, then the manifold is η-Einstein manifold.*

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