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Study on Modified Sieve of Eratosthenes Using Discrete Fourier Transform

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Abstract: The Sieve of eratosthenes (SOE) is one of the simplest ways to obtain prime numbers on a smaller scale, by eliminating all the multiples of prime numbers up to a given value. The algorithm was slightly modified to eliminate the multiples of prime numbers, for all-natural numbers, rather than to a limit n. An algorithm was proposed using some basic properties of the sieve. Using the algorithm many arithmetic properties of the sequence obtained by modified SOE was discussed and generalized using Discrete fourier transform (DFT) and the results are helpful in strengthening the twin prime conjecture.

MSC: 65T50, 11A99, 11N05.

Keywords: Sieve of eratosthenes, Periodic sequence, Prime number, Discrete fourier transform.

1. Introduction

The Sieve of Eratosthenes is one of the simplest ways to generate prime numbers and it also helps to understand how they are arranged in between the integers. In Sieve of Eratosthenes, the following steps are involved to obtain prime numbers up to a positive integer n.

- List all positive numbers up to n
- Delete all multiples of the primes less than or equal to $\sqrt{n}$
- List all remaining numbers

In Sieve of eratosthenes, only the multiples of the prime numbers are eliminated in each step. But in this work, we have modified SOE, thereby eliminating one prime number at a time and also their corresponding multiples [1]. At each sieve, the consecutive difference of the result is a periodic sequence. The arithmetic properties of the sequence of numbers generated using modified SOE is studied using an algorithm proposed in this paper. Even though Theorem 2.3, 2.4, were already proved by Aiazzi [1], different approach is made to prove them again so that it is easy to approach the other theorems. Aiazzi [1] conjectured an argument (see Equation (38) in [1]) that deals with the distribution of twin primes. That conjecture is proved in this paper (Theorem 2.6). The sequences are generalized using DFT, since the difference of consecutive terms of sequence follows a periodic pattern. Currently many article work was done within the certain interval of integer sequence for finding regularities in the distribution of prime numbers [2–4].

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2. Observations

Definition 2.1. m^{th} order odd number sequence, $\big(f_m(n)\big)_{m,n\geq 0}$.

The sequence obtained by eliminating all primes from 2 to p_m and their multiples from the set of natural numbers is defined as m^{th} order odd number sequence *and is denoted as $\big(f_m(n)\big)_{m,n\geq 0}$. Here p_m is the m^{th} prime and n denotes the n^{th} term. When 2 and its multiples are eliminated from the set of natural numbers, we get the set of 1^{st} order odd numbers sequence $\{1,3,5,7,\dots\}$ and it is denoted as $f_1(n)$.*

When 3 and its multiples are eliminated from $f_1(n)$, we get the set of 2^{nd} order odd numbers sequence $\{1,5,7,11,\dots\}$ and it is denoted as $f_2(n)$.

Assuming that natural numbers are 0^{th} order odd numbers sequence and denoted as $f_0(n)$.

$$f_0(n) = n+1 \tag{1}$$

$$f_1(n) = 2n+1 \tag{2}$$

$$f_2(n) = 3n - \frac{(-1)^n}{2} + \frac{3}{2} \tag{3}$$

In order to generalize the sequence for all m, their arithmetic properties need to be studied.

Definition 2.2. *The m^{th}* order difference sequence, $\big(d_m(n)\big)_{m,n\geq 0}$, *is defined as*

$$d_m(n) = f_m(n+1) - f_m(n) \tag{4}$$

The 1^{st} order difference sequence is obtained from $f_1(n) = 1,3,5,7,9,\dots$ as $d_1(n) = 2,2,2,2,2,2,2,\dots$.

The 2^{nd} order difference sequence is obtained from $f_2(n) = 1,5,7,11,\dots$ as $d_2(n) = 4,2,4,2,4,2,4,2,4,2,4,2,4,\dots$.

The 3^{rd} order difference sequence is obtained from $f_3(n) = 1,7,11,13,\dots$ as $d_3(n) = 6,4,2,4,2,4,6,2,6,4,2,4,2,4,6,2,6,4,\dots$.

Theorem 2.3. *The m^{th} order difference sequence $d_m(n)$ is a periodic sequence with the fundamental period, $\alpha(m)$, as*

$$\alpha(m) = (p_1-1)(p_2-1)(p_3-1)\cdots(p_m-1) \tag{5}$$

Proof. The $(m+1)^{th}$ order odd number sequence is obtained by eliminating p_{m+1} and its multiples from m^{th} order odd number sequence. It is also achieved by multiplying p_{m+1} with all the numbers in $f_m(n)$ and eliminate those resulting numbers in the same $f_m(n)$ sequence, since the composite numbers in $f_m(n)$ are only numbers whose prime factor is greater than p_m. In other words $\{(p_{m+1})f_m(n)\}$ is a subset of $\{f_m(n)\}$.

Assuming that $d_m(n)$ is a periodic sequence with the fundamental period as $\alpha(m)$. When $f_m(n)$ is multiplied by p_{m+1} then their consecutive difference, $d_m(n)$ also gets multiplied by p_{m+1}. As both the sequence $\{(p_{m+1})f_m(n)\}, \{f_m(n)\}$ having the same fundamental period and also $\{(p_{m+1})f_m(n)\}$ is a subset of $\{f_m(n)\}$, the elimination also occurs in a periodic way. We have, $d_m(n) = d_m\big(n+\alpha(m)\big)$.

Let us consider $\{d_m(0), d_m(1), d_m(2), \dots, d_m\big(\alpha(m)-1\big)\}$ as a single pair which repeats periodically in $d_m(n)$ and for the multiplied sequence, the consecutive difference $(p_{m+1})d_m(n)$ is also periodic and the single pair which repeats periodically is $\{(p_{m+1})d_m(0), (p_{m+1})d_m(1), (p_{m+1})d_m(2), \dots, (p_{m+1})d_m(\alpha(m)-1)\}$.

$$\sum_{n=0}^{\alpha(m)-1} p_{m+1} d_m(n) = p_{m+1} \sum_{n=0}^{\alpha(m)-1} d_m(n)$$

Hence the way of elimination repeats periodically after each p_{m+1} pair of $d_m(n)$. The fundamental period of the $(m+1)^{th}$ order difference sequence, $\alpha(m+1)$.

$$= \alpha(m)(p_{m+1}) - (no.\ of\ elimination\ in\ p_{m+1}\ pairs\ of\ d_m(n))$$
$$= \alpha(m)(p_{m+1}) - \big(period\ of\ a\ multiplied\ difference\ sequence, (p_{m+1})d_m(n)\big)$$
$$= \alpha(m)(p_{m+1}) - \big(\alpha(m)\big)$$
$$= \alpha(m)(p_{m+1} - 1)$$

The fundamental period of $d_1(n)$ is $(p_1 - 1) = 1$, which implies that the fundamental period of $d_2(n)$ is $(p_1 - 1)(p_2 - 1) = 2$. Hence by induction it is true for all m.

In general, the fundamental period of $m^(th)$ order difference sequence $d_m(n)$ is

$$\alpha(m) = (p_1 - 1)(p_2 - 1)(p_3 - 1) \cdots (p_m - 1)$$

Represented as $R(k)$ by Aiazzi [1].

As $p_1, p_2, \ldots, p_m$ are relatively prime, by the multiplicative property of Euler totient function

$$\phi(p_1 p_2 \cdots p_m) = (p_1 - 1)(p_2 - 1) \cdots (p_m - 1)$$
$$\phi(p_1 p_2 \cdots p_m) = \alpha(m)$$

m	1	2	3	4	5	6	7	8	9
$\alpha(m)$	1	2	8	48	480	5760	92160	1658880	36495360

Table I. **List of fundamental period upto 9^{th} order difference sequence**

□

Theorem 2.4. *For an m^{th} order difference sequence, $d_m(n)$*

$$\sum_{n=0}^{\alpha(m)} d_m(n) = p_1 p_2 p_3 \cdots p_m \tag{6}$$

Proof. Whenever an elimination occurs in $f_m(n)$, the consecutive difference $d_m(n)$ gets added. Thus, the elimination occurs periodically, in other words the way of elimination repeats after each p_{m+1} pair of $d_m(n)$. Assuming, $d_m(0) + d_m(1) + d_m(2) + \cdots + d_m(\alpha(m) - 1) = x$. For the $(m+1)^{th}$ order difference sequence $d_{m+1}(n)$,

$$\sum_{n=0}^{\alpha(m+1)-1} d_{m+1}(n) = p_{m+1}d_m(0) + p_{m+1}d_m(1) + \cdots + p_{m+1}d_m(\alpha(m) - 1)$$
$$= p_{m+1}\big(d_m(0) + d_m(1) + \cdots + d_m(\alpha(m) - 1)\big)$$
$$= p_{m+1}(x)$$

For 1^{st} order difference sequence, $\alpha(1) = 1$, thus $d_1(0) = 2 = p_1$. For 2^{nd} order difference sequence, $\alpha(2) = 2$, thus $d_2(0) + d_2(1) = 4 + 2 = 6 = p_1 p_2$. By induction, it is true for all m. □

2.1. Algorithm

An algorithm was proposed to obtain $d_{m+1}(n)$ from $d_m(n)$.

$$d_{m+1}(0) = d_m(0) + d_m(1)$$

$$Find\ k_1,\ such\ that\ p_{m+1}d_m(0) = \sum_{j=1}^{k_1} d_m(j)$$

$$d_{m+1}(i) = d_m(i+1)\ for\ i = 1, 2, 3, \ldots, k_1 - 2$$

$$d_{m+1}(k_1 - 1) = d_m(k_1) + d_m(k_1 + 1)$$

$$Find\ k_2,\ such\ that\ p_{m+1}d_m(1) = \sum_{j=k_1+1}^{k_2} d_m(j)$$

$$d_{m+1}(i) = d_m(i+2)\ for\ i = k_1, k_1 + 1, k_1 + 2, \ldots, k_2 - 3$$

$$d_{m+1}(k_2 - 2) = d_m(k_2) + d_m(k_2 + 1)$$

$$Find\ k_3,\ such\ that\ p_{m+1}d_m(2) = \sum_{k_2+1}^{k_3} d_m(j)$$

$$d_{m+1}(i) = d_m(i+3)\ for\ i = k_2 - 1, k_2, k_2 + 1, \ldots, k_3 - 4$$

$$d_{m+1}(k_3 - 3) = d_m(k_2) + d_m(k_2 + 1)$$

$$In\ general,\ find\ k_v,\ such\ that\ p_{m+1}d_m(v-1) = \sum_{k_{v-1}}^{k_v} d_m(j)$$

$$d_{m+1}(i) = d_m(i+v)\ for\ i = k_v - 1 - (v-2), k_v - 1 - (v-1), k_v - 1 - v, \ldots, k_v - (v+1)$$

$$d_{m+1}(k_v - v) = d_m(k_2) + d_m(k_2 + 1)$$

In order to generate sequence $d_{m+1}(n)$, it is enough to find up to $v = \alpha(m)$.

Theorem 2.5. *If we consider the possible set of consecutive differences that gets added during elimination process of obtaining $f_{m+1}(n)$ from $f_m(n)$ as*

$$\{d_m(0) + d_m(1), d_m(2) + d_m(3), \ldots, d_m(\alpha(m) - 2) + d_m(\alpha(m) - 1), d_m(\alpha(m) - 1) + d_m(0)\}$$

then each element in the above set gets added exactly one time for every p_{m+1} pairs of $d_m(n)$.

Proof. Let us consider $\{d_m(0), d_m(1), d_m(2), \ldots, d_m\big(\alpha(m) - 1\big)\}$ as a single pair which repeats periodically in $d_m(n)$. If $f_m(n+1)$ is divisible by p_{m+1}, then $d_m(n), d_m(n+1)$ gets added during elimination. Assuming that $f_m(n+1)$ is divisible by p_{m+1} and the consecutive differences $d_m(n), d_m(n+1)$ were present in the y^{th} pair of $d_m(n)$, where $y \leq p_{m+1}$, then from Theorem 2.3, 2.4, the below equation gives a positive integer value.

$$\frac{1 + d_m(0) + d_m(1) + \cdots + d_m(n)}{p_{m+1}} \tag{7}$$

By Theorem 2.3, we know that the way of elimination repeats periodically after p_{m+1} pairs of $d_m(n)$. Hence the same condition must be satisfied by the consecutive differences that gets added which are present in the other pairs of $d_m(n)$, greater than y.

$$\frac{1 + d_m(0) + d_m(1) + \cdots + d_m(n) + y'(p_1p_2p_3\cdots p_m)}{p_{m+1}} = \frac{f_m(n)}{p_{m+1}} + \frac{y'(p_1p_2p_3\cdots p_m)}{p_{m+1}} \tag{8}$$

where $y' = 0, 1, 2, \ldots$. If (8) gives the integer value for y', then it denotes the consecutive differences $d_m(n), d_m(n+1)$ in the $(y+y')^{th}$ pair of $d_m(n)$. It is obvious that (8) gives an integer value only when y' is a multiple of p_{m+1}. Hence a consecutive differences $d_m(n), d_m(n+1)$ will gets added only after every p_{m+1} pairs of $d_m(n)$ and the number of eliminations within p_{m+1} pairs of $d_m(n)$ is $\alpha(m)$. From which it can be concluded that the only possible set of consecutive differences that gets added within p_{m+1} pairs of $d_m(n)$ is

$$\{d_m(0) - d_m(1), d_m(2) - d_m(3), \ldots, d_m(\alpha(m) - 2) - d_m(\alpha(m) - 1), d_m(\alpha(m) - 1) - d_m(0)\}$$

□

Justification For example if we consider of obtaining $d_4(n)$ from $d_3(n)$ by using the above algorithm. The set of $p_4 = 7$ pair of $d_3(n)$ was given as

$$\{6, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4,$$
$$2, 4, 6, 2, 6, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2, 4, 6, 2\}$$

After elimination of multiples of 7, by the above algorithm we get a single pair of $d_4(n)$ from the above set as

$$\{\mathbf{6} + \mathbf{4}, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, \mathbf{2} + \mathbf{4}, 6, 2, 6, 4, 2, \mathbf{4} + \mathbf{2}, 4, 6, \mathbf{2} + \mathbf{6}, 4, 2, 4,$$
$$2, 4, \mathbf{6} + \mathbf{2}, 6, 4, \mathbf{2} + \mathbf{4}, 2, 4, 6, 2, 6, \mathbf{4} + \mathbf{2}, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2, \mathbf{4} + \mathbf{6}, 2\}$$

where the set consecutive differences that gets added are given as

$$\{6 + 4, 4 + 2, 2 + 4, 4 + 2, 2 + 4, 4 + 6, 6 + 2, 2 + 6\}$$

Theorem 2.6. *The number of 2 or 4 in a single pair of $d_m(n)$ is*

$$\prod_{i=2}^{m}(p_i - 2), \ \textit{where } m \geq 2$$

Proof. Assuming the number of 2 or 4 in a single pair of $d_m(n)$ is q. Then the number of 2 or 4 in p_{m+1} pairs of $d_m(n)$ is qp_{m+1}. From Theorem 2.5, it is obvious that each $d_m(n)$ gets eliminated twice within p_{m+1} pairs of $d_m(n)$ by getting added to their two consecutive differences. Hence the number of 2 or 4 in a single pair of $d_{m+1}(n)$ is given as

$$= \textit{Number of } 2 \textit{ or } 4 \textit{ in } p_{m+1} \textit{ pairs of } d_m(n) - \textit{Number of } 2 \textit{ or } 4 \textit{ gets added}$$
$$= qp_{m+1} - 2q$$
$$= q(p_{m+1} - 2)$$

For $m = 2$, the number of 2 or 4 in a single pair of $d_2(n) = \{4, 2\}$ is $q = p_2 - 2 = 1$. Hence by induction it is true for all m. □

The same argument is not valid for the values of $d_m(n)$ other than 2 and 4, because 2 and 4 are the minimum values of $d_m(n)$ and when the consecutive differences gets added, they further contribute to the number of other values of $d_m(n)$ other than 2 and 4.

Theorem 2.7. *For m^{th} order difference sequence, $d_m(nX-1)=2$, where X is the fundamental period of $d_m(n)$, just to make the equations simpler.*

Proof. From the algorithm, it is obvious that no two elimination occur at successive terms. Here single pair of $d_m(n)$ denotes $d_m(0), d_m(1), d_m(2), \dots, d_m(X-2), d_m(X-1)$. According to algorithm,
Find k_1 such that $p_{m+1}d_m(0) = \sum_{j=1}^{k_1} d_m(j)$. Hence, $d_{m+1}(k_1) = d_m(k_1) + d_m(k_1+1)$.
In the elimination process, the terms $f_m(np_{m+1}X+1)$ for $n = 1, 2, \dots$ get eliminated for all m and X is a period. i.e., first term in each p_{m+1} pairs of $d_m(n)$.
Now, going in a reverse way, we can find the last elimination in a first p_{m+1} pairs of $d_m(n)$ with the knowledge that the first elimination occurs at first term in second p_{m+1} pairs of $d_m(n)$.
Considering that for a m^{th} order odd number sequence $d_m(X-1)=2$, and let $d_m(X-2)=a$. Hence to obtain $d_{m+1}(n)$, the algorithm was discussed,

$$\begin{aligned} p_{m+1}d_m(X-1) &= d_m(X) + d_m(X-1) + d_m(X-2) + \dots + d_m(X-j) \quad (9) \\ p_{m+1}2 &= (p_{m+1}-1) + 2 + a + \dots + d_m(X-j) \\ p_{m+1}2 &> (p_{m+1}-1) + 2 \end{aligned}$$

Hence $j > 1$, which shows that after the eliminations, we obtain $d_{m+1}(X-1) = 2$. For $m = 1$, $d_1(n) = 2$ which implies $d_2(1) = 2$. Hence it is true for all m, and it shows that 2 will never get eliminated from $d_m(n)$. □

Theorem 2.8. *Single pair of $d_m(n)$ is symmetrical about $d_m(\frac{\alpha(m)}{2}-1) = 4$, neglecting the last term $d_m(\alpha(m)-1) = 2$, where $m \geq 3$. For example, single pair of $d_3(n) = \{6,4,2,4,2,4,6,2\}$ is symmetrical about $d_3(3) = 4$, neglecting the last term $d_3(7) = 2$.*

Proof. Single pair of $d_m(n)$ is given as

$$\left\{d_m(0), \dots, d_m(\frac{\alpha(m)}{2}-2), d_m(\frac{\alpha(m)}{2}-1), d_m(\frac{\alpha(m)}{2}), \dots, d_m(\alpha(m)-2), d_m(\alpha(m)-1)\right\}$$

Assuming that the above theorem holds for $d_m(n)$, then we have

$$\left\{d_m(0), \dots, d_m(\frac{\alpha(m)}{2}-2), d_m(\frac{\alpha(m)}{2}-1) = 4, d_m(\frac{\alpha(m)}{2}-2), \dots, d_m(0), 2\right\}$$

By algorithm to obtain single pair of $d_{m+1}(n)$ from $d_m(n)$, we need p_{m+1} pairs of $d_m(n)$, which itself has the same symmetrical property with symmetrical about $d_m(\frac{p_{m+1}\alpha(m)}{2}-1) = 4$, since all prime numbers $p_m \geq 3$ are odd.
Also by the same argument made in the Theorem 2.7, the last consecutive differences (elimination) in the p_{m+1} pairs of $d_m(n)$ can be calculated. It is already assumed that p_{m+1} pairs of $d_m(n)$ is symmetrical, neglecting the last term 2. Hence, when $a = p_{m+1}-1$, (9) becomes

$$\begin{aligned} p_{m+1}d_m(X-1) &= d_m(X) + d_m(X-1) + d_m(X-2) + \dots + d_m(X-j) \\ p_{m+1}2 &= (p_{m+1}-1) + 2 + a \\ p_{m+1}2 &= (p_{m+1}-1) + 2 + (p_{m+1}-1) \end{aligned}$$

We have $j = 2$. Within the p_{m+1} pairs of $d_m(n)$ the algorithm starts with the first two consecutive differences and ends with their symmetrical consecutive pairs. Hence by proceeding the algorithm either from 0 to $\alpha(m)-1$ or in the opposite way, the addition of consecutive differences will be symmetrical. Which implies that $d_{m+1}(n)$ is also symmetrical

Number of eliminations(addition of consecutive integers) within the p_{m+1} pairs of $d_m(n)$, $\alpha(m)$, is even and no two eliminations occur consecutively, the symmetrical point, $d_m(\frac{p_{m+1}\alpha(m)}{2} - 1) = 4$ will never change. It can also be verified in justification for Theorem 2.5. As it is already mentioned that $d_3(n)$ is symmetrical, by induction it is true for all m ☐

3. Generalization Using Discrete Fourier Transform

Every periodic sequence can be represented using DFT. The m^{th} order difference sequence, $d_m(n)$ is represented as

$$\big(d_m(n)\big)_{m,n\geq 0} = \sum_{k=0}^{\alpha(m)-1} C_m(k) \exp\left(\frac{2\pi ikn}{\alpha(m)}\right) \tag{10}$$

where $C_m(k)$ is given by Inverse DFT (IDFT)

$$\big(C_m(k)\big)_{m,k\geq 0} = \frac{1}{\alpha(m)} \sum_{n=0}^{\alpha(m)-1} d_m(n) \exp\left(\frac{-2\pi ikn}{\alpha(m)}\right) \tag{11}$$

From [6] when substituting $k = 0$ in [11], we have

$$C_m(0) = \frac{1}{\alpha(m)} \sum_{n=0}^{\alpha(m)-1} d_m(n)$$

$$C_m(0) = \frac{p_1 p_2 \cdots p_m}{\alpha(m)} \tag{12}$$

From (4) the recursive relation for the m^{th} order odd number sequence was given as $f_m(n+1) - f_m(n) = d_m(n)$, where $f_m(0) = 1$ which implies,

$$\big(f_m(n)\big)_{m,n\geq 0} = 1 + \sum_{j=0}^{n-1} \sum_{k=0}^{\alpha(m)-1} C_m(k) \exp\left(\frac{2\pi ikj}{\alpha(m)}\right) \tag{13}$$

Definition 3.1. *For an m^{th} order odd number sequence, $\big(N_m(x)\big)_{m,x\geq 0}$ is defined as a sequence of numbers that satisfy the below equation*

$$f_m\big(N_m(x)\big) = p_{m+1} f_m(x), \tag{14}$$

where $x = 0, 1, 2, 3, \ldots$. As the difference of consecutive terms of $f_m(n)$ is periodic, the same property holds for $N_m(x)$.

Definition 3.2. *For an m^{th} order odd number sequence, $\big(B_m(y)\big)_{m,y\geq 0}$ is defined as*

$$B_m(y) = \begin{cases} 1, & \text{when } y = N_m(x) - x \text{ ;} \\ 0, & \text{otherwise.} \end{cases}$$

where $x = 0, 1, 2, \ldots$. Since the difference of consecutive terms of $N_m(x)$ is periodic, then $B_m(y)$ is periodic with only 0 and 1 with the fundamental period as $\alpha(m+1)$. For example, $B_1(y) = 0, 1, 0, 1, 0, 1, 0, 1, \ldots$. A periodic binary number sequence (0 and 1) can be represented in a mathematical formula using DFT if the place of occurrence of 1 is known [5]. For $B_m(y)$, the place at which 1 occurs is given by $N_m(x) - x$. $B_m(y)$ is given as

$$B_m(y) = \frac{1}{\alpha(m+1)} \sum_{k=0}^{\alpha(m)-1} \sum_{j=0}^{\alpha(m+1)-1} \exp\left(\frac{2\pi ij(y + N_m(k) - k)}{\alpha(m+1)}\right) \tag{15}$$

Theorem 3.3. *Whenever $n = N_m(x)$ in $f_m(n)$, $f_m(n)$ succeeds $f_{m+1}(n)$ by one term, while eliminating p_{m+1} and its multiples from $f_m(n)$. Hence $f_{m+1}(n)$ is obtained from $f_m(n)$ as*

$$f_{m+1}(n) = f_m\Big(n + \sum_{y=0}^{n} B_m(y)\Big) \tag{16}$$

where $f_0(n) = n + 1$.

Justification For $m = 1$, (13) becomes

$$\begin{aligned} f_1(n) &= nC_1(0) + 1 \\ &= \frac{np_1}{p_1 - 1} + 1 \\ f_1(n) &= 2n + 1 \end{aligned}$$

The 2^{nd} order odd number sequence, $f_2(n)$ was obtained from $f_1(n)$ as follows

$$\begin{aligned} f_1\big(N_1(x)\big) &= p_2 f_1(n) \\ N_1(x) &= xp_2 + \frac{(p_1-1)(p_2-1)}{p_1} \\ N_1(x) &= 3x + 1 \\ B_1(y) &= \frac{1}{\alpha(2)} \sum_{k=0}^{\alpha(1)-1} \sum_{j=0}^{\alpha(2)-1} \exp\left(\frac{2\pi i j(y + (3k+1) - k)}{\alpha(2)}\right) \\ B_1(y) &= \frac{(-1)^{y+1} + 1}{2} \\ f_2(n) &= f_1\Big(n + \sum_{y=0}^{n} B_1(y)\Big) \\ f_2(n) &= f_1\left(n + \sum_{y=0}^{n} \frac{(-1)^{n+1} + 1}{2}\right) \\ f_2(n) &= 3n - \frac{(-1)^n}{2} + \frac{3}{2} \end{aligned}$$

Hence (3) was obtained from (2) by using above defined sequences (14), (15), (16). For $m = 3$, by solving IDFT for $d_m(0), d_m(1), d_m(2), d_m(3), \ldots, d_m(7)$ using (11), we have

$$f_3(n) = 1 + \sum_{j=0}^{n-1} \sum_{k=0}^{7} C_3(k) \exp\left(\frac{2\pi i k j}{8}\right)$$

k	0	1	2	3	4	5	6	7
$C_3(k)$	$\frac{15}{4}$	$(\frac{4-\sqrt{2}}{8})(1+i)$	$(\frac{-1}{4})i$	$(\frac{4+\sqrt{2}}{8})(1-i)$	$\frac{1}{4}$	$(\frac{4+\sqrt{2}}{8})(1+i)$	$(\frac{1}{4})i$	$(\frac{4-\sqrt{2}}{8})(1-i)$

Table II. **Values of the constant $C_3(k)$**

For an m^{th} order odd number sequence we have $f_m(0) = 1, f_m(1) = p_{m+1}$. The sequence $f_m(n)$ is obtained with the knowledge up to the m^{th} prime number and by substituting $n = 1$ in $f_m(n)$, we will get $(m+1)^{th}$ prime number. In other words, p_{m+1} can be calculated mathematically, if $p_1, p_2, p_3, \ldots, p_m$ is known. Eventhough this method is tedious one compared to the computer programs to generate prime numbers, it can be further developed to study the distribution prime numbers.

4. Conclusion

In this paper, many properties of the sequence obtained from modified Sieve of Eratosthenes were discussed. Theorem 2.7, and the symmetrical property (Theorem 2.8) of $d_m(n)$ shows some insight into the distribution of gap size in the sequence. More knowledge about gap size in a modified SOE sequence and in turn prime gaps can be gained by studying the m^{th} order difference sequence, $d_m(n)$ in more detail. While studying with the Discrete Fourier Transform, the sequences, $[C_m(k), N_m(x), B_m(y)]$ are further studied to get their relations in a more general way as because in this paper they are discussed only with the help of given algorithm.

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