Int. J. Math. And Appl., 10(1)(2022), 43–49
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

A Review on Graphs arising from Finite Groups

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Abstract: There has been a strong relationship between group (finite) and graph theories for more than a century. Arthur Cayley in 1878 introduced Cayley graphs which geometrically display the action of finite groups. In this paper, we will give a brief description of some specific graphs with its standard results, arising from finite groups.

MSC: 05C25, 05C48.

Keywords: Finite group, Graphs, Vertex-transitive.

1. Introduction

Graph theory is concerned with the study of 'graphs', which are mathematical representations of objects and their relationships. Group is a set of objects with a rule of combination such that given any two elements of the group; the rule yields another group element depending upon the elements chosen. The combinatorial properties of graphs have been employed to investigate the theoretic algebraic properties of groups and vice versa. The relationship between a graph and a group(finite) was first introduced by Arthur Cayley in 1878 [1] in which a graph represents a finite group. Cayley graphs geometrically display the actions of finite groups.

The phenomenon of vertex and edge expansion in graphs forms an expander family of graphs. Expander graphs are highly connected sparse finite graphs. Expander graphs are a class of graph that holds the seemingly contradictory properties of being both sparse and well-connected. Some other well-known graphs arising from finite groups are also discussed.

The main objective of this paper is to give a brief description on structures and standard results of some specific graphs arising from finite groups. The paper is organized as follows: In Section 2, we describe some basic concepts of Finite Groups and Graphs (undirected, finite, connected). Section 3 onwards we will focus on specific graphs arising from finite groups such as: Cayley graphs, Expander graphs, Order graphs and Power graphs.

2. Preliminaries

In this section, we describe the basic concepts of finite Groups denoted by G and Graphs (undirected, finite, connected) Γ; which are essential for further discussions.

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2.1. Groups

Throughout this paper G denotes a finite group which is defined in a standard setting of binary operation, a multiplication; the group G is called abelian, if it is commutative under multiplication. The subset H (non-empty) of G is a subgroup of G, if $H$is closed under products and inverses. The smallest subgroup generated by S denoted as $\langle S\rangle$, is the smallest subgroup of G containing every element of S. The group G is called cyclic, if it is generated by a single element. More specifically, there is some element $x \in G$ such that $G = \{x^n \,| n \in Z\}$; $G = \langle x\rangle$. Let $o(G)$ denotes the order of G, the number of elements in a group. A group G is said to be a p-group, p-prime, if order of every element of the group G is the power of prime p. The symmetric group of degree n, S_n is a group under function composition. A cycle is a string of integers which represents the elements of S_n which cyclically permute these integers and every permutation is a product of disjoint cycles. For more algebraic structures of groups and fundamental properties one can refer to [14, 15, 19, 20, 22].

2.2. Graph Theory

Definition 2.1. *A graph Γ is a pair (V, E) consisting of a non-empty set $V(\Gamma)$ of vertices, a set $E(\Gamma)$ of edges.*

The edges are often denoted by either $\{x, y\}$ or just xy. The degree of x, denoted by $\deg(x)$, is the number of edges incident with it. A path between two elements $x, y \in V(\Gamma)$ is a sequence of edges which connect a sequence of distinct vertices. The number of edges in a path is called the length of a path. The distance between $x, y \in V(\Gamma)$ denoted by $d(x, y)$, is the length of a shortest path between x and y. The diameter of a graph Γ is denoted by $diam(\Gamma)$ and is defined to be $diam(\Gamma) = \max\{d(x, y) : x, y \in V(\Gamma)\}$. A cycle is a non-trivial path in a graph from a vertex to itself. A graph $\Gamma = (V, E)$ is said to be Hamiltonian, if there exists a cycle $(x_1 x_2 .... x_n)$ so that every vertex of Γ appears exactly once in the sequence. For any two graphs Γ_1 and Γ_2 with vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$; the vertex set of the product $\Gamma_1 \times \Gamma_2$ is defined as $V(\Gamma_1) \times V(\Gamma_2)$. For more details about graphs (undirected, finite, connected) refer to [8, 11, 15].

3. Cayley Graphs

The relationship between a graph (finite, connected) and a group (finite) was first introduced by Arthur Cayley in 1878 [1] in which a graph represents a finite group. Cayley graphs are the graphs associated to a group (finite) and a set of generators for that group. Cayley graphs geometrically display the actions of a finite group [1, 16, 24, 30].

Definition 3.1. *Given a group G and a subset S of G, the Cayley graph $Cay(G : S)$ is the undirected graph with vertex set G and edge set containing an edge from g to sg and from g to $s^{-1}g$ whenever $g \in G$ and $s \in S$.*

If $|g| = 2$, the edge from $g \to sg$ and $g \to s^{-1}g$ are the same, which results in one edge. An arbitrary graph Γ is said to be a Cayley graph if there exists a group G and a generating set S such that Γ is isomorphic to the Cayley graph for G and S.

Definition 3.2. *Let Γ be a graph. Then an isomorphism from Γ onto itself is called an automorphism of Γ. The set of all such automorphisms of a graph Γ denoted by $Aut(\Gamma)$ forms a group under composition called the automorphism group of the graph Γ.*

Definition 3.3. *Let Γ be a graph, then Γ is called vertex-transitive if each pair of vertices $\{x, y\} \subset V(\Gamma)$ there exist $f \in Aut(\Gamma)$ such that $f(x) = y$.*

All vertex-transitive graphs are regular, but the converse is not necessarily true.

Theorem 3.4. *Let S be a set of generators of a finite group G. The Cayley graph $Cay(G : S)$ is* *(i). Connected.*

(ii). $|S|$-regular, where $|S|$ is the cardinality of S.

(iii). Vertex-transitive.

Remark 3.5. *Every Cayley graph is a vertex-transitive but not every vertex-transitive graph is Cayley graph.*

Theorem 3.6. *Let G be an abelian group generated by two elements, that is $|S| = 2$. If k is the diameter of $Cay\,(G : S)$ and m be its size, then $m \leq 2k^2 + 2k + 1$.*

Remark 3.7. *A group G is planar provided its Cayley graph $Cay\,(G : S)$ is planar and S is called planar generating set.*

Theorem 3.8. *Every subgroup of a planar group is planar.*

The Hamiltonian graphs in Cayley graphs are described as follows:

Theorem 3.9. *Let $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_n$ be Hamiltonian graphs, then $\Gamma_1 \times \Gamma_2, ... \times \Gamma_n$ is Hamiltonian.*

The proof follows by the method of induction. For $n = 1$, the case is trivial. For $n = k$ and let $\Gamma_1, \Gamma_2, ..., \Gamma_k$ each have Hamiltonian cycle, then we assume that $\Gamma_1 \times \Gamma_2, ... \times \Gamma_k$ has a Hamiltonian cycle and finally for $n = k+1$, $\Gamma_1 \times \Gamma_2, ... \times \Gamma_{k+1}$ also has a Hamiltonian cycle.

Theorem 3.10. *There is a Hamiltonian path on every Cayley digraph on an abelian group.*

3.1. Conjecture

Any connected Cayley graph is Hamiltonian [2]. Every connected Cayley graph of an abelian group of order at least three is Hamiltonian [24].

Theorem 3.11. *$Cay\,(D_{2n} : \{r, s\})$ is Hamiltonian*

For any dihedral group with generator r, there will be an $n - gon$ generated by r. Further, the generators $\{r, s\}$ extends the vertices of $n - gon$ by multiplying each vertex by s. The vertices are connected to form Hamiltonian cycle.

Theorem 3.12. $(Cay\,(G : S) \times Cay\,(H : T)) = Cay\,(G \times H : [(S \times \{1\}) \bigcup (\{1\} \times T)])$.

For more results on Cayley graphs one can consider the surveys [10, 16, 24, 30, 31]. In the next section, we discuss the expansion properties of graphs which lead to expander family of graphs and special case as a Ramanujan graphs.

4. Expander Graphs

Expander graphs [3, 4, 12, 17, 25, 29] are highly connected sparse finite graphs. Expander graphs are a class of graph that holds the seemingly contradictory properties of being both sparse and well-connected. An expander graph is a graph in which every subset S of vertices is connected to many vertices in the complementary set $\overline{S}$ of vertices. Expander graphs are sparse graphs that have many useful properties such as low diameter, high connectivity and a high chromatic number. Suppose a graph $\Gamma = (V, E)$ has n-vertices. Consider a subset S of the vertices in $V(\Gamma)$ and its complement $\overline{S}$. The edge boundary of S denoted by ∂S and is defined as the set of edges $\{x, y\} \in E$ such that $x \in S$ and $y \in \overline{S}$.

Definition 4.1. *The edge expansion ratio of a graph $\Gamma = (V, E)$ on n-vertices is given by $h(\Gamma) = \min_{S \subset V : 1 \leq |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}$. A higher value of $h(\Gamma)$ means a higher minimum value for the ratio of the edge boundary of a vertex subset S and the size of* *the subset* $|S|$. *Since*$|S| \leq \frac{n}{2}$, *this implies that subsets of vertices that comprise less than half of the total number of vertices will be well-connected to larger subsets of vertices. A complete graph* K_n *is well connected but not sparse, the edge boundary* $|\partial S| = |S| \, . \left|\overline{S}\right|$ *and* $h\left(K_n\right) = \min_{S \subset V : 1 \leq |S| \leq \frac{n}{2}} (n - |S|) = \left\lceil \frac{n}{2} \right\rceil$.

Proposition 4.2 (Expansion & diameter). *Let* Γ *be a finite connected graph, then we have diam* $(\Gamma) \leq 2 \frac{\log \frac{|\Gamma|}{2}}{\log\left(1 + \frac{h(\Gamma)}{v}\right)} + 3$, *where* $v = \max_{c \in V} val\,(x)$ *is the maximal valency.*

Lemma 4.3. *Let* Γ *be a finite connected graph and* $x \in V$. *For any* $n \geq 0$, *let* $B_x\,(n)$ *be the ball of radius* n *around* x. *That is,* $B_x\,(n) = \{y \in V \,|d_\Gamma\,(x, y) \leq n\}$. *Then, by using maximal valency* v *for* Γ, *we have* $|B_x\,(n)| \geq \min \left(\frac{|\Gamma|}{2}, \left(1 + \frac{h(\Gamma)}{v}\right)^n\right)$.

Definition 4.4 (Expander graphs). *A family* $(\Gamma_i)_{i \in I}$ *of finite connected graphs* $\Gamma_i = (V_i, E_i)$ *is an expander family or a family of expanders, if there exist constants* $v \geq 1$ *and* $h > 0$ *independent of* i, *such that:*

1. *The number of vertices* $|V_i|$ *"tends to infinity", for any* $N \geq 1$, *there are only finitely many such that* Γ_i *has at most* N *vertices.*
2. *For each* $i \in I$, *we have* $\max_{x \in V_i} val\,(x) \leq v$, *the maximal valency of the graphs is bounded independently of* i.
3. *For each* $i \in I$, *the expansion constant satisfies* $h\,(\Gamma_i) \geq h > 0$.

The pair (h, v) *are called the expansion parameters of the family.*

Corollary 4.5 (Diameter of expanders). *Let* (Γ_i) *be an expander family of graphs. Then we have, daim* $(\Gamma_i) \leq \log\left(3\left|(\Gamma_i)\right|\right)$ *for all* i, *where the implied constant depends on the expansion parameters* (h, v) *of the family.*

We now talk about k-regular expander graphs or family of k-regular graphs that satisfy the properties of definition as given follows:

Definition 4.6. *A sequence of* k*-regular graphs* $(\Gamma_i)_{i \in N}$ *of size increasing with* i, *is a family of Expanders graphs if there exist* $\varepsilon > 0$ *such that* $h\,(\Gamma_i) \geq \varepsilon$, *for all* i.

Now, by using the concept of Spectral graph theory, we will see some results on expander graphs whichs leads to Ramanujan graphs. For more details of Spectral graph theory, refer to [8, 13].

Definition 4.7. *Let* $\Gamma = (V, E)$ *is an undirected graph with eigenvalues* $\lambda_1 \geq \lambda_2 \geq .... \geq \lambda_n$ *then* $\Delta\,(\Gamma) \equiv \lambda_1 - \lambda_2$ *is the Spectral gap of* Γ.

Theorem 4.8 (Cheeger's Inequality). *The edge expansion ratio* $h\,(\Gamma)$ *for a* k*-regular graph is related to the spectral gap* $\Delta\,(\Gamma)$ *by* $\frac{\Delta(\Gamma)}{2} \leq h\,(\Gamma) \leq \sqrt{2k\Delta\,(\Gamma)}$.

Since $\lambda_1 = k$ for a k-regular graph, we can also write above inequality as $\frac{k - \lambda_2}{2} \leq h\,(\Gamma) \leq \sqrt{2k\,(k - \lambda_2)}$.
For a k-regular, n-vertex graph Γ, let $k = \lambda_1 \geq \lambda_2 \geq .... \geq \lambda_n \geq -k$ be the real eigenvalues, then $\lambda\,(\Gamma) = \max_{|\lambda_i| < k} |\lambda_i|$ is the maximum absolute value of all eigenvalues. The bound for eigenvalues is given by the Alon - Boppana result:

Theorem 4.9 (Alon - Boppana). *Let* $(\Gamma_i)_{i \in N}$ *be an infinite family of* k*-regular connected graph on* n *vertices, where* k *is fixed and* n *increases with* i. *Then for all* i, $\lambda\,(\Gamma_i) \geq 2\sqrt{k - 1} - o\,(1)$, *where* $o\,(1)$ *tends to zero for every fixed* k *as* $n \to \infty$.

Definition 4.10 (Ramanujan Graph). *A* k*-regular finite graph* Γ *is called a Ramanujan graph, if* $\lambda\,(\Gamma) \leq 2\sqrt{k - 1}$.

Lubotzky, Phillips, and Sarnak [4] proved that the explicit construction of infinite families of k-regular Ramanujan graphs is possible when $k - 1$ is prime. From both the Alon - Boppana bound and the definition of Ramanujan graphs, we observe that Ramanujan graphs define a classification of graph where the spectral gap is almost as large as possible. For further details, refer to [4, 25, 29].

5. Order Graphs

The order graph [5, 9, 28] of a finite group is the undirected graph whose vertices are the elements of the group (finite) and for any two distinct vertices there is an edge if and only if, one divides the other.

Definition 5.1. *Let G be a group. The order graph of G is the graph (undirected) $\Gamma(G)$, whose vertices are non-trivial sub-groups of G and for two distinct vertices $H$and K there is an edge from H to$K$, if and only if either $o(H)/o(K)$ or $o(K)/o(H)$.*

Lemma 5.2. *For any finite group G, there exist a vertex of $\Gamma(G)$ which is adjacent to every other vertices.*

Lemma 5.3. *Let G be a finite group. Then $\Gamma(G)$ is a regular graph, if and only if, it is complete.*

Lemma 5.4. *Let G be a finite group. Then $\Gamma(G)$ is connected graph with $diam\Gamma(G) \leq 2$.*

By Lemma 5.1., $\Gamma(G)$ has a vertex adjacent to every other vertices. So that, $\Gamma(G)$ is connected graph and hence $diam\Gamma(G) \leq 2$.

Theorem 5.5. *Let G be a finite group. Then $\Gamma(G)$ is complete graph, if and only if, G is a p-group, where p is a prime number.*

Definition 5.6 (Sub graph of a order graph)**.** *Let G be a finite group. The sub-graph of $\Gamma(G)$ denoted by $\Omega(G)$ with vertex set given by $V(\Omega(G)) = V(\Gamma(G)) - \{G\}$; so that $\Omega(G)$ is a simple graph with vertices $V(\Omega(G)) = \{H\}$, where H is a non-trivial proper subgroup of G and two distinct vertices H and K are adjacent, if and only if either $o(H)/o(K)$ or $o(K)/o(H)$.*

Theorem 5.7. *Let G be a finite group. Then $\Omega(G)$ is a connected graph with diameter at most four.*

Corollary 5.8. *Let G be a finite group and H be a subgroup of G with $o(H) = p^n m$, where p is a prime number, and then $gr(\Omega(G)) = 3$.*

Theorem 5.9. *Let G be a finite abelian group such that $\Omega(G) \neq \phi$; ($\Omega(G)$ being the sub-graph of a order graph) Then $\Omega(G)$ is a complete graph if and only if, G is a p-group.*

Lemma 5.10. *Let G be a finite group with $o(G) = pq$, $(p < q)$ where p, q are distinct prime numbers, then either $\Omega(G) = \phi$ or $\Omega(G) = K_n$, complete graph with n-vertices.*

6. Power Graphs

In the literature, the power graphs have been defined and considered for groups and semi-groups with the understanding that the undirected power graphs [18, 21, 23, 26, 27] is the underlying the undirected graph of the directed power graph. P. J. Cameron et. al [26, 27] introduced the 'power graph' in the meaning of an undirected power graph.

Definition 6.1. *Let G is a finite group. The power graph $P(G)$ is a graph in which $V(P(G)) = G$ and two distinct elements x and y are adjacent, if and only if one of them is a power of other.*

The power graph $P(G)$ is connected graph of diameter 2, if G is finite group. Also, a finite group G is a cyclic group of order 1 or p^m for some prime number p and positive integer m, if and only if $P(G)$ is complete.

Corollary 6.2. *Let G be a finite group. Then power graph $P(G)$ is planar, if and only if $\pi(G) \subseteq \{1, 2, 3, 4\}$.*

Corollary 6.3. *The power graph of a cyclic group of order n is planar, if and only if $n = 2, 3, 4$.*

Theorem 6.4. *Let G be a finite group. The power graph $P(G)$ is bipartite, if and only if G is an elementary abelian group of even order.*

Theorem 6.5. *The power graph $P(Z_pn)$ has the maximum number of edges among all power graphs of p-groups of order p^n.*

Lemma 6.6. *The power graph $P(G)$ is Eulerian, if and only if $|G|$ is odd.*

Lemma 6.7. *The power graph of a finite group G is a tree, if and only if G is an elementary abelian 2 -group.*

Theorem 6.8. *Let G is a p-group. The power graph $P(G)$ is 2-connected, if and only if G is cyclic or generalized quaternion group.*

Let G is a p-group and $P(G)$ is 2-connected. We first prove that for each pair of distinct elements, there exists a cycle of length 3 or 4 containing x and y. By an inductive argument, we can find a non-trivial subgroup of G contained in each non-trivial subgroup of G. So, G is cyclic or generalized quaternion.

Corollary 6.9. *Let G be a finite p-group. Then $P(G)$ has a Hamiltonian cycle if and only if $|G| \neq 2$ and G is cyclic.*

Corollary 6.10. *If p is an odd prime, then the power graph of a p-group is 2-connected if and only if it is Hamiltonian.*

Theorem 6.11. *Let G be a nilpotent group. If G is not a p-group, then the power graph $P(G)$ is 2-connected.*

In [23], the authors conjectured that the power graph $P(Z_n)$ has the maximum number of edges among all power graphs of groups of order n.

Lemma 6.12. *Let G be a finite group that is not of prime order. If $\max \omega(G) = p$, where p is a prime and $\omega(G)$ is the set of all element orders of G, then $P(G)$ is not 2-connected.*

Theorem 6.13. *If G is a finite simple group of order n, then $|E(P(G))| \leq |E(P(z_n))|$.*

Lemma 6.14. *Let G be a finite group. If $P(G)$ is a union of complete graphs that share the identity then, the power graph of each sylow subgroup of G has the same structure.*

Theorem 6.15. *Let G be a finite p-group. Then the power graph $P(G)$ is a union of complete subgraphs which share the identity element of G if and only if, G is isomorphic to a cyclic group, p-group of exponent p or a dihedral group.*

7. Concluding Remark

In this paper, we have made an attempt to highlight structures and standard results of some well-known graphs arising from finite groups such as Cayley graphs, Expander graphs etc. which has numerous applications in cryptography (hash functions), coding theory(expander codes, LDPC codes etc.). This article will be helpful as a repository for the researchers.

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