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On the Transcendental Equation With Three Unknowns $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \pm \sqrt{x-\sqrt{C}y}$ for Different Values of C by Using the Continued Fraction Method

S. Sriram[1] and P. Veeramallan[1,2,*]

1 Department of Mathematics, National College (Affiliated to Bharathidasan University), Tiruchirappalli, Tamilnadu, India.

2 P.G Assistant in Mathematics, GHSS, Yethapur, Salem, Tamilnadu, India.

Abstract: The Transcendental equation with three unknowns is given by $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \pm \sqrt{x-\sqrt{C}y}$ is considered and analyzed for finding a different set of integer solutions utilizingthe continued fraction method, under numerous patterns with some numerical examples.

MSC: 11D04, 11D09, 11J70, 11J81.

Keywords: Transcendental equation, Integer solutions, Continued fraction, Pell equation, Recurrence relation.

1. Motivation

Diophantine equations have a versatile field of research. Many Diophantine equations are in algebraic form. In this paper, we try to find the integer solutions for the given transcendental equation by using the continued fraction method. Here we consider the Pell equation of $X^2 - CY^2 = 1$ for different values of C where $(C \neq 1)$ is a positive non-square integer. Also, we enumerate some of the most salient qualities of the simple continued fraction. I referred to the journal [5] in the detailed but slightly flawed book [3] as the main source of inspiration for numerous experiments I have made on this problem. Following definitions are needed in our paper.

(i). A simple continued fraction expansion of order n is an expression of the form $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots \cdots + \frac{1}{a_n}}}}$ and it can be abbreviated as $[a_0; a_1, a_2, \dots, a_n]$ and a_0 maybe positive or negative or zero.

(ii). $\sqrt{C} = [a_0; a_1, a_2, \dots, a_l]$ denote the Continued fraction expansion of period length l. Set $A_{-2} = 0, A_{-1} = 1, A_k = a_k A_{k-1} + A_{k-2}$ and $B_{-2} = 1, B_{-1} = 2, B_k = a_k B_{k-1} + B_{k-2}$ for $k > 0$. $P_k = \frac{A_k}{B_k}$ is the k^{th} convergent of $\sqrt{C}$.

The fundamental solution of $X^2 - DY^2 = 1$ is $(X_1, Y_1) = \begin{cases} (A_{l-1}, B_{l-1}) & \text{if l is even} \\ (A_{2l-1}, B_{2l-1}) & \text{if l is odd} \end{cases}$ and the fundamental solution of $X^2 - DY^2 = -1$ is$(X_1, Y_1) = \{(A_{l-1}, B_{l-1})$ if l is odd.

* E-mail: veeramallan1986@gmail.com (Part-time Research Scholar)

2. Description of Method

The Transcendental equation $\sqrt{2z-4}=\sqrt{x+\sqrt{C}y}\pm\sqrt{x-\sqrt{C}y}$.

Consider the Transcendental equation

$$\sqrt{2z-4}=\sqrt{x+\sqrt{C}y}\pm\sqrt{x-\sqrt{C}y} \tag{1}$$

Squaring on both sides, we get

$$z-2=x\pm\sqrt{x^2-Cy^2} \tag{2}$$

Take $x^2-Cy^2=\alpha^2$, so that $z=x+2\pm\alpha$. Take $\alpha=1$ and therefore

$$x^2=Cy^2+1 \tag{3}$$

$$z=x+2\pm 1 \tag{4}$$

That is $z=x+3$ or $Z=x+1$. With the choices of C, we get all the solutions for (3). The choices of C (Non-square integer) are m^2+1, m^2-1, m^2+2, m^2-2, m^2+m, and m^2-m. Our main results can be stated as the following theorems.

3. Main Theorems

Theorem 3.1. *Let $m\geq 1$ be any integer, and let $C=m^2+1$.*

(1). The continued fraction expansion of $\sqrt{C}$ is $\sqrt{C}=\begin{cases}[1;\overline{2}] & \text{if } m=1\\ [m;\overline{2m}] & \text{if } m>1\end{cases}$

(2). $(x_1,y_1)=\left(2m^2+1,2m\right)$ be the fundamental solution of (3). Set $\{(x_n,y_n)\}$, where

$$\frac{x_n}{y_n}=\left[m;\ \underbrace{2m,\dots 2m}_{2n-1\ times}\right] \tag{5}$$

for $n\geq 2$. Then (x_n,y_n) is a solution of $x^2-\left(m^2+1\right)y^2=1$. The consecutive solutions (x_n,y_n,z_n) and $(x_{n+1},y_{n+1},z_{n+1})$ satisfy

$$x_{n+1}=\left(2m^2+1\right)x_n+\left(2m^3+2m\right)y_n$$
$$y_{n+1}=2mx_n+\left(2m^2+1\right)y_n$$
$$z_{n+1}=\left(2m^2+1\right)x_n+\left(2m^3+2m\right)y_n+2\pm 1 \quad for \quad n\geq 1.$$

(3). The solutions satisfy the following recurrence relations

$$x_n=\left(4m^2+1\right)(x_{n-1}+x_{n-2})-x_{n-3}$$
$$y_n=\left(4m^2+1\right)(y_{n-1}+y_{n-2})-y_{n-3}$$
$$z_n=\left(4m^2+1\right)[z_{n-1}+z_{n-2}-2(2\pm 1)]-[z_1-2(2\pm 1)], \quad for \quad n\geq 4.$$

Proof. (1). Let $C=m^2+1$.

Case (i): If $m = 1$, then it is easily seen that $\sqrt{C} = [1; \overline{2}]$.

Case (ii): Let $m > 1$, then

$$\begin{aligned}
\sqrt{m^2+1} &= m + \sqrt{m^2+1} - m \; = m + \cfrac{1}{\frac{1}{\sqrt{m^2+1}-m}} \\
&= m + \cfrac{1}{\frac{\sqrt{m^2+1}+m}{\left(\sqrt{m^2+1}-m\right)\left(\sqrt{m^2+1}+m\right)}} \\
&= m + \frac{1}{\sqrt{m^2+1}+m} = m + \frac{1}{2m + \sqrt{m^2+1} - m} \\
&= m + \cfrac{1}{2m + \cfrac{1}{\frac{1}{\sqrt{m^2+1}-m}}}
\end{aligned}$$

So $\sqrt{C} = [m; \overline{2m}]$.

(2). We see as above that $\sqrt{C} = [m; \overline{2m}]$. By the definition (ii), we get $A_0 = m$, $A_1 = 2m^2+1$, $B_0 = 1$ and $B_1 = 2m$. Therefore, $(x_1, y_1) = (A_1, B_1) = \left(2m^2+1, 2m\right)$ is the fundamental solution. Indeed $\left(2m^2+1\right)^2 - \left(m^2+1\right)(2m)^2 = 1$. Now we assume that (x_n, y_n) is a solution of $x^2 - \left(m^2+1\right)y^2 = 1$. Applying (5), we get

$$\begin{aligned}
\frac{x_{n+1}}{y_{n+1}} &= m + \cfrac{1}{2m + \cfrac{1}{2m + \cfrac{1}{2m + \cfrac{1}{2m + \dots \atop \dots 2m + \frac{1}{2m}}}}} \\
&= m + \cfrac{1}{2m + \cfrac{1}{m + m + \cfrac{1}{2m + \cfrac{1}{2m + \dots \atop \dots 2m + \frac{1}{2m}}}}} \\
&= m + \cfrac{1}{2m + \cfrac{1}{m + \frac{x_n}{y_n}}} = m + \frac{my_n + x_n}{2m^2 y_n + 2m x_n + y_n}
\end{aligned}$$

Thus,

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\left(2m^2+1\right)x_n + \left(2m^3+2m\right)y_n}{2mx_n + \left(2m^2+1\right)y_n} \tag{6}$$

Applying (6), we find that

$$\begin{aligned}
x_{n+1}^2 - \left(m^2+1\right)y_{n+1}^2 &= \left(\left(2m^2+1\right)x_n + \left(2m^3+2m\right)y_n\right)^2 - \left(m^2+1\right)\left(2mx_n + \left(2m^2+1\right)y_n\right)^2 \\
&= \left(2m^2+1\right)^2 x_n^2 + 2\left(2m^2+1\right)\left(2m^3+2m\right)x_n y_n + \left(2m^3+2m\right)^2 y_n^2 \\
&- \left(m^2+1\right)\left[4m^2 x_n^2 + 4m\left(2m^2+1\right)x_n y_n + \left(2m^2+1\right)^2 y_n^2\right] \\
&= \left(2m^2+1\right)^2\left(x_n^2 - \left(m^2+1\right)y_n^2\right) + 2x_n y_n\left(\left(2m^2+1\right)\left(2m^3+2m\right)\right. \\
&\left. - 2m\left(m^2+1\right)\left(2m^2+1\right)\right) - 4m^2\left(m^2+1\right)\left(x_n^2 - \left(m^2+1\right)y_n^2\right) \\
&= \left(2m^2+1\right)^2(1) - 2x_n y_n(0) - 4m^2\left(m^2+1\right)(1) \\
&= 1
\end{aligned}$$

Therefore (x_{n+1}, y_{n+1}) is a solution of $x^2 - \left(m^2+1\right)y^2 = 1$. From the equation (4), $z_{n+1} = x_{n+1} + 2 \pm 1$. That is, $z_{n+1} = \left(2m^2+1\right)x_n + \left(2m^3+2m\right)y_n + 2 \pm 1$. Thus, $(x_{n+1}, y_{n+1}, z_{n+1})$ is the solution set of (1) in which $x^2 - Cy^2 = x^2 - \left(m^2+1\right)y^2 = 1$.

(3). We prove this recurrence relation only for

$$x_n = \left(4m^2+1\right)\left(x_{n-1}+x_{n-2}\right) - x_{n-3} \tag{7}$$

by induction on n. Applying (4), we get $x_1 = 2m^2+1, x_2 = 8m^4+8m^2+1$, $x_3 = 32m^6+48m^4+18m^2+1$ and $x_4 = 128m^8+256m^6+160m^4+32m^2+1$. The recurrence relation in (7) is true for $n=4$. Since,

$$\begin{aligned} x_4 &= 128m^8+256m^6+160m^4+32m^2+1 \\ &= 128m^8+224m^6+104m^4+8m^2+32m^6+56m^4+26m^2+2-2m^2-1 \\ &= \left(4m^2+1\right)\left(32m^6+56m^4+26m^2+2\right) - \left(2m^2+1\right) \\ &= \left(4m^2+1\right)\left(32m^6+48m^4+18m^2+1+8m^4+8m^2+1\right) - \left(2m^2+1\right) \\ x_4 &= \left(4m^2+1\right)\left(x_3+x_2\right) - x_1. \end{aligned}$$

Assume that the equality, $x_n = \left(4m^2+1\right)\left(x_{n-1}+x_{n-2}\right) - x_{n-3}$ is satisfied for $n-1$, that is,

$$x_{n-1} = \left(4m^2+1\right)\left(x_{n-2}+x_{n-3}\right) - x_{n-4} \tag{8}$$

Since, $x_{n+1} = \left(2m^2+1\right)x_n + \left(2m^3+2m\right)y_n$. Hence,

$$x_{n-1} = \left(2m^2+1\right)x_{n-2} + \left(2m^3+2m\right)y_{n-2} \tag{9}$$

$$x_{n-2} = \left(2m^2+1\right)x_{n-3} + \left(2m^3+2m\right)y_{n-3} \tag{10}$$

$$x_{n-3} = \left(2m^2+1\right)x_{n-3} + \left(2m^3+2m\right)y_{n-3} \tag{11}$$

(8), (9), (10) and (11) yield that $x_n = \left(4m^2+1\right)\left(x_{n-1}+x_{n-2}\right) - x_{n-3}$.

□

Example 3.2. *Let $m=3$. So that $C = m^2+1 = 10$. Then $\sqrt{10} = \left[3;\overline{6}\right]$. Further, the fundamental solution of $x^2-10y^2=1$ is $(x_1,y_1) = (19,6)$ and so $z_1 = 21 \pm 1$ and other solutions are listed from the following table.*

n	$\frac{x_n}{y_n} = [3;\overline{6}]$	(x_n, y_n)	(x_n, y_n, z_n)
1	$[3;6]$	$(19,6)$	$(19,6,21\pm1)$
2	$[3;6,6,6]$	$(721,228)$	$(721,228,723\pm1)$
3	$[3;6,6,6,6,6]$	$(27379,8658)$	$(27379,8658,27381\pm1)$
4	$[3;6,6,6,6,6,6,6]$	$(1039681,328776)$	$(1039681,328776,1039683\pm1)$
5	$[3;6,6,6,6,6,6,6,6,6]$	$(39480499,12484830)$	$(39480499,12484830,39480501\pm1)$
6	$[3;6,6,6,6,6,6,6,6,6,6,6]$	$(1499219281,474094764)$	$(1499219281,474094764,1499219283\pm1)$

Example 3.3. *Let $m=4$. So that $C = m^2+1 = 17$. Then $\sqrt{17} = \left[4;\overline{8}\right]$. Further, the fundamental solution of $x^2-17y^2=1$ is $(x_1,y_1) = (33,8)$ and so $z_1 = 35 \pm 1$ and other solutions are listed in the following table.*

n	$\frac{x_n}{y_n} = [4;\overline{8}]$	(x_n, y_n)	(x_n, y_n, z_n)
1	$[4;8]$	$(33,8)$	$(33,8,35\pm1)$
2	$[4;8,8,8]$	$(2177,528)$	$(2177,528,2179\pm1)$
3	$[4;8,8,8,8,8]$	$(143649,34840)$	$(143649,34840,143651\pm1)$
4	$[4;8,8,8,8,8,8,8]$	$(9478657,2298912)$	$(9478657,2298912,9478659\pm1)$
5	$[4;8,8,8,8,8,8,8,8,8]$	$(625447713,151693352)$	$(625447713,151693352,625447715\pm1)$

Now, we consider the other cases of C without giving their proof since they can be proved as similar to that of Theorem 3.1 was proved.

Theorem 3.4. *Let $m \geq 1$ be any integer and let $C = m^2 - 1$.*

1. *The continued fraction expansion of $\sqrt{C}$ is $\sqrt{C} = \left[m-1; \overline{1, 2m-2}\right]$.*

2. *$(x_1, y_1) = (m, 1)$ be the fundamental solution of (3). Set $\{(x_n, y_n)\}$, where $\frac{x_n}{y_n} = \left[m-1; \underbrace{1, 2m-2, \ldots, 1, 2m-2}_{n-2 \quad times}, 1, 2m-1\right]$ for $n \geq 2$. Then (x_n, y_n) is a solution of $x^2 - \left(m^2-1\right)y^2 = 1$. The consecutive solutions (x_n, y_n, z_n) and $(x_{n+1}, y_{n+1}, z_{n+1})$ satisfy*

$$x_{n+1} = mx_n + \left(m^2-1\right)y_n$$
$$y_{n+1} = x_n + my_n$$
$$z_{n+1} = mx_n + \left(m^2-1\right)y_n + 2 \pm 1, \quad for \quad n \geq 1.$$

3. *The solutions satisfy the following recurrence relations*

$$x_n = (2m-1)\left(x_{n-1} + x_{n-2}\right) - x_{n-3}$$
$$y_n = (2m-1)\left(y_{n-1} + y_{n-2}\right) - y_{n-3}$$
$$z_n = (2m-1)\left(z_{n-1} + z_{n-2} - 2\left(2 \pm 1\right)\right) - \left[z_{n-3} - 2\left(2 \pm 1\right)\right] \quad for \quad n \geq 4.$$

Theorem 3.5. *Let $m \geq 1$ be any integer and let $C = m^2 + 2$.*

1. *The continued fraction expansion of $\sqrt{C}$ is $\sqrt{C} = \left[m; \overline{m, 2m}\right]$.*

2. *$(x_1, y_1) = \left(m^2+1, m\right)$ be the fundamental solution of (3). Set $\{(x_n, y_n)\}$, where $\frac{x_n}{y_n} = \left[m; \underbrace{m, 2m, \ldots m, 2m}_{n-1 \; times}, m\right]$ for $n \geq 2$. Then (x_n, y_n) is a solution of $x^2 - \left(m^2+2\right)y^2 = 1$. The consecutive solutions (x_n, y_n, z_n) and $(x_{n+1}, y_{n+1}, z_{n+1})$ satisfy*

$$x_{n+1} = \left(m^2+1\right)x_n + \left(m^3+2m\right)y_n$$
$$y_{n+1} = mx_n + \left(m^2+1\right)y_n$$
$$z_{n+1} = \left(m^2+1\right)x_n + \left(m^3+2m\right)y_n + 2 \pm 1 \quad for \quad n \geq 1.$$

3. *The solutions satisfy the following recurrence relations*

$$x_n = \left(2m^2+1\right)\left(x_{n-1} + x_{n-2}\right) - x_{n-3}$$
$$y_n = \left(2m^2+1\right)\left(y_{n-1} + y_{n-2}\right) - y_{n-3}$$
$$z_n = \left(2m^2+1\right)\left(z_{n-1} + z_{n-2} - \left(2 \pm 1\right)\right) - \left(z_{n-3} - 2\left(2 \pm 1\right)\right) \quad for \quad n \geq 4.$$

Theorem 3.6. *Let $m \geq 1$ be any integer and let $C = m^2 - 2$.*

1. *The continued fraction expansion of $\sqrt{C}$ is* $\sqrt{C} = \begin{cases} \left[1; \overline{2}\right] & if\ m = 2 \\ \left[m-1; \overline{1, m-2, 1, 2m-2}\right] & if\ m > 2 \end{cases}$

2. $(x_1, y_1) = (m^2-1, m)$ *be the fundamental solution of* (3). *Set* $\{(x_n, y_n)\}$, *where* $\frac{x_n}{y_n} = \left[m-1;\ \underbrace{1, m-2, 1, 2m-2, \ldots 1, m-2, 1, 2m-2}_{n-1\ times}, 1, m-1\right]$ *for* $n \geq 2$. *Then* (x_n, y_n) *is a solution of* $x^2 - (m^2-2)\,y^2 = 1$. *The consecutive solutions* (x_n, y_n, z_n) *and* $(x_{n+1}, y_{n+1}, z_{n+1})$ *satisfy*

$$x_{n+1} = (m^2-1)\,x_n + (m^3-2m)\,y_n$$

$$y_{n+1} = mx_n + (m^2-1)\,y_n$$

$$z_{n+1} = (m^2-1)\,x_n + (m^3-2m) + 2 \pm 1 \quad for \quad n \geq 1.$$

3. *The solutions satisfy the following recurrence relations*

$$x_n = (2m^2-3)\,(x_{n-1} + x_{n-2}) - x_{n-3}$$

$$y_n = (2m^2-3)\,(y_{n-1} + y_{n-2}) - y_{n-3}$$

$$z_n = (2m^2-3)\,(z_{n-1} + z_{n-2} - 2\,(2 \pm 1)) - (y_{n-3} - 2\,(2 \pm 1)) \quad for \quad n \geq 4.$$

Theorem 3.7. *Let* $m \geq 1$ *be any integer and let* $C = m^2 + m$.

1. *The continued fraction expansion of* $\sqrt{C}$ *is* $\sqrt{C} = \begin{cases} [1; \overline{2}] & if\ m = 1 \\ [m; \overline{2, 2m}] & if\ m > 1 \end{cases}$

2. $(x_1, y_1) = (2m+1,\ 2)$ *be the fundamental solution of* (3). *Set* $\{(x_n, y_n)\}$, *where* $\frac{x_n}{y_n} = \left[m;\ \underbrace{1, m, 2m, \ldots, m, 2m}_{n-1\ times}, 2\right]$ *for* $n \geq 2$. *Then* (x_n, y_n) *is a solution of* $x^2 - (m^2+m)\,y^2 = 1$. *The consecutive solutions* (x_n, y_n, z_n) *and* $(x_{n+1}, y_{n+1}, z_{n+1})$ *satisfy*

$$x_{n+1} = (2m+1)\,x_n + (2m^2+2m)\,y_n$$

$$y_{n+1} = 2x_n + (2m+1)\,y_n$$

$$z_{n+1} = (2m+1)\,x_n + (2m^2+2m)\,y_n + 2 \pm 1 \quad for \quad n \geq 1.$$

3. *The solutions satisfy the following recurrence relations*

$$x_n = (4m+1)\,(x_{n-1} + x_{n-2}) - x_{n-3}$$

$$y_n = (4m+1)\,(y_{n-1} + y_{n-2}) - y_{n-3}$$

$$z_n = (4m+1)\,(z_{n-1} + z_{n-2} - 2\,(2 \pm 1)) - (z_{n-3} - 2\,(2 \pm 1)) \quad for \quad n \geq 4.$$

Theorem 3.8. *Let* $m \geq 2$ *be any integer and let* $C = m^2 - m$.

1. *The continued fraction expansion of* $\sqrt{C}$ *is* $\sqrt{C} = \begin{cases} [1; \overline{2}] & if\ m = 2 \\ [m-1; \overline{2, 2m-2,}] & if\ m > 2 \end{cases}$

2. $(x_1, y_1) = (2m-1,\ 2)$ *be the fundamental solution of* (3). *Set* $\{(x_n, y_n)\}$, *where* $\frac{x_n}{y_n} = \left[m-1;\ \underbrace{2, 2m-2, \ldots, 2, 2m-2}_{n-1\ times}, 2\right]$ *for* $n = 2$. *Then* (x_n, y_n) *is a solution of* $x^2 - (m^2-m)\,y^2 = 1$. *The consecutive solutions* (x_n, y_n, z_n) *and* $(x_{n+1}, y_{n+1}, z_{n+1})$ *satisfy*

$$x_{n+1} = (2m-1)\,x_n + (2m^2-2m)\,y_n$$

$$y_{n+1} = 2x_n + (2m-1)\, y_n$$

$$z_{n+1} = (2m-1)\, x_n + \left(2m^2 - 2m\right) y_n + 2 \pm 1 \quad for \quad n \geq 1.$$

3. The solutions satisfy the following recurrence relations

$$x_n = (4m-3)\,(x_{n-1} + x_{n-2}) - x_{n-3}$$

$$y_n = (4m-3)\,(y_{n-1} + y_{n-2}) - y_{n-3}$$

$$z_n = (4m-3)\,(z_{n-1} + z_{n-2} - 2\,(2 \pm 1)) - (z_{n-3} - 2\,(2 \pm 1)) \quad for \quad n \geq 4.$$

4. Conclusion

In this paper, we gave all the possible non-negative integer solutions for the equation (3) and (4) by using the continued fraction method. And it is interesting to see that the researcher can also proceed for further results in this problem.

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