International Journal of Mathematics And its Applications
Volume 3, Issue 3–A (2015), 47–51.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

$g^{\star}$-closed Sets with Respect to an Ideal

Research Article

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Abstract: An ideal on a set X is a non empty collection of subsets of X with heredity property which is also closed under finite unions. The concept of generalized closed (g-closed) sets was introduced by Levine [10]. Quite Recently, Jafari and Rajesh [7] have introduced and studied the notion of generalized closed (g-closed) sets with respect to an ideal. Many generalizations of g-closed sets are being introduced and investigated by modern researchers. One among them is $g^{\star}$-closed sets which were introduced by Veerakumar [17]. In this paper, we introduce and investigate the concept of $g^{\star}$-closed sets with respect to an ideal.

MSC: 54C10.

Keywords: Topological space, open set, $g^{\star}$closed set, g-closed set, $\mathcal{I}_g$-closed set, $\mathcal{I}_{\pi g}$-closed set, ideal.

1. Introduction and Preliminaries

The notion of closed set is fundamental in the study of topological spaces. In 1970, Levine [10] introduced the concept of generalized closed sets in a topological space by comparing the closure of a subset with its open supersets. He defined a subset A of a topological space X to be generalized closed (briefly, g-closed) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open. This notion has been studied extensively in recent years by many topologists. After advent of g-closed sets, many generalizations of g-closed sets are being introduced and investigated by modern topologists. One among them is $g^{\star}$-closed sets which were introduced by Veerakumar [17]. Indeed ideals are very important tools in General Topology. It was the works of Newcomb [11], Rancin [12], Samuels [14] and Hamlett and Jankovic (see [3–6, 8]) which motivated the research in applying topological ideals to generalize the most basic properties in General Topology. A nonempty collection $\mathcal{I}$ of subsets on a topological space (X, τ) is called a topological ideal [9] if it satisfies the following two conditions:

1. If A $\in \mathcal{I}$ and B $\subseteq$ A implies B $\in \mathcal{I}$ (heredity)
2. If A $\in \mathcal{I}$ and B $\in \mathcal{I}$, then A $\cup$ B $\in \mathcal{I}$ (finite additivity)

If A is a subset of a topological space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A, respectively. Let A $\subseteq$ B $\subseteq$ X. Then cl_B(A) (resp. int_B(A)) denotes closure of A (resp. interior of A) with respect to B. In this paper, we introduce and study the concept of $g^{\star}$-closed sets with respect to an ideal, which is the extension of the concept of $g^{\star}$-closed sets. The following Definitions and Remarks are useful in the sequel.

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Definition 1.1. *A subset A of a topological space X is regular open [15] if* $A = int(cl(A))$.

Definition 1.2. *The finite union of regular open sets is called* π*-open [18]. The complement of* π*-open set is* π*-closed [18].*

Definition 1.3. *A subset A of a topological space X is called* π*-generalized closed (briefly,* πg*-closed) [2] if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* π*-open.*

Definition 1.4. *A subset A of a topological space X is called generalized closed (briefly, g-closed) [10] if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is open. The complement of g-closed set is g-open.*

Definition 1.5. *A subset A of a topological space X is called* $g^\star$*-closed [17] or strongly g-closed [16] if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is g-open.*

Definition 1.6. *Let* (X, τ) *be a topological space and* $\mathcal{I}$ *be an ideal on X. A subset A of X is said to be generalized closed with respect to an ideal (briefly* $\mathcal{I}_g$*-closed) [7] if and only if* $cl(A) - B \in \mathcal{I}$*, whenever* $A \subseteq B$ *and B is open.*

Remark 1.7 ([17])**.** *For a subset of a topological space, the following properties hold:*

1. *Every closed set is* $g^\star$*-closed but not conversely.*
2. *Every* $g^\star$*-closed set is g-closed but not conversely.*

Remark 1.8 ([7])**.** *Every g-closed set is* $\mathcal{I}_g$*-closed but not conversely.*

Definition 1.9 ([13])**.** *Let* (X, τ) *be a topological space and* $\mathcal{I}$ *be an ideal on X. A subset A of X is said to be* π*-generalized closed with respect to an ideal (briefly* $\mathcal{I}_{\pi g}$*-closed) if and only if* $cl(A) - B \in \mathcal{I}$*, whenever* $A \subseteq B$ *and B is* π*-open.*

Remark 1.10 ([13])**.** *For several subsets defined above, we have the following implications.*

$$\begin{array}{ccccc} & & \mathcal{I}_g\text{-closed set} & \longrightarrow & \mathcal{I}_{\pi g}\text{-closed set} \\ & & \uparrow & & \uparrow \\ \text{closed set} & \longrightarrow & g\text{-closed set} & \longrightarrow & \pi g\text{-closed set} \end{array}$$

The reverse implications are not true.

Remark 1.11 ([10])**.** *The intersection of a g-closed set and a closed set is g-closed.*

Definition 1.12 ([1])**.** *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called gc-irresolute if the inverse image of g-closed set of Y is g-closed in X.*

2. $g^\star$-Closed Sets with Respect to an Ideal

Definition 2.1. *Let* (X, τ) *be a topological space and* $\mathcal{I}$ *be an ideal on X. A subset A of X is said to be* $g^\star$*-closed with respect to an ideal (briefly* $\mathcal{I}_{g^\star}$*-closed) if and only if* $cl(A) - B \in \mathcal{I}$*, whenever* $A \subseteq B$ *and B is g-open.*

Remark 2.2. *Every* $g^\star$*-closed set is* $\mathcal{I}_{g^\star}$*-closed, but the converse need not be true, as this may be seen from the following Example.*

Example 2.3. *Let* $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}\}$ *and* $\mathcal{I} = \{\emptyset, \{c\}\}$*. Then* $\{b\}$ *is* $\mathcal{I}_{g^\star}$*-closed but not* $g^\star$*-closed.*

The following theorem gives a characterization of $\mathcal{I}_{g^\star}$-closed sets.

Theorem 2.4. *A set A is $\mathcal{I}_{g^\star}$-closed in (X, τ) if and only if F ⊆ cl(A)−A and F is g-closed in X implies F ∈ $\mathcal{I}$.*

Proof. Assume that A is $\mathcal{I}_{g^\star}$-closed. Let F ⊆ cl(A)−A. Suppose F is g-closed. Then A ⊆ X−F. By our assumption, cl(A)−(X−F) ∈ $\mathcal{I}$. But F ⊆ cl(A)−(X−F) and hence F ∈ $\mathcal{I}$.
Conversely, assume that F ⊆ cl(A)−A and F is g-closed in X implies that F ∈ $\mathcal{I}$. Suppose A ⊆ U and U is g-open. Then cl(A)−U = cl(A) ∩ (X−U) is a g-closed set in X, that is contained in cl(A)−A. By assumption, cl(A)−U ∈ $\mathcal{I}$. This implies that A is $\mathcal{I}_{g^\star}$-closed. □

Theorem 2.5. *If A and B are $\mathcal{I}_{g^\star}$-closed sets of (X, τ), then their union A ∪ B is also $\mathcal{I}_{g^\star}$-closed.*

Proof. Suppose A and B are $\mathcal{I}_{g^\star}$-closed sets in (X, τ). If A ∪ B ⊆ U and U is g-open, then A ⊆ U and B ⊆ U. By assumption, cl(A)−U ∈ $\mathcal{I}$ and cl(B)−U ∈ $\mathcal{I}$ and hence cl(A ∪ B)−U = (cl(A)−U) ∪ (cl(B)−U) ∈ $\mathcal{I}$. That is A ∪ B is $\mathcal{I}_{g^\star}$-closed. □

Remark 2.6. *The intersection of two $\mathcal{I}_{g^\star}$-closed sets need not be an $\mathcal{I}_{g^\star}$-closed as shown by the following Example.*

Example 2.7. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. Then $A = \{a, b, c\}$ and $B = \{a, b, d\}$ are $\mathcal{I}_{g^\star}$-closed but their intersection $A \cap B = \{a, b\}$ is not $\mathcal{I}_{g^\star}$-closed.*

Remark 2.8. *Every $\mathcal{I}_{g^\star}$-closed set is $\mathcal{I}_g$-closed but not conversely.*

Example 2.9. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\{b\}$ is $\mathcal{I}_g$-closed but not $\mathcal{I}_{g^\star}$-closed.*

Remark 2.10. *For several subsets defined above, we have the following implications.*

$$\begin{array}{ccccccc} & & \mathcal{I}_{g^\star}\text{-closed set} & \longrightarrow & \mathcal{I}_g\text{-closed set} & \longrightarrow & \mathcal{I}_{\pi g}\text{-closed set} \\ & & \uparrow & & \uparrow & & \uparrow \\ \text{closed set} & \longrightarrow & g^\star\text{-closed set} & \longrightarrow & g\text{-closed set} & \longrightarrow & \pi g\text{-closed set} \end{array}$$

The reverse implications are not true.

Theorem 2.11. *If A is $\mathcal{I}_{g^\star}$-closed and A ⊆ B ⊆ cl(A) in (X, τ), then B is $\mathcal{I}_{g^\star}$-closed in (X, τ).*

Proof. Suppose A is $\mathcal{I}_{g^\star}$-closed and A ⊆ B ⊆ cl(A) in (X, τ). Suppose B ⊆ U and U is g-open. Then A ⊆ U. Since A is $\mathcal{I}_{g^\star}$-closed, we have cl(A)−U ∈ $\mathcal{I}$. Now B ⊆ cl(A). This implies that cl(B)−U ⊆ cl(A)−U ∈ $\mathcal{I}$. Hence B is $\mathcal{I}_{g^\star}$-closed in (X, τ). □

Theorem 2.12. *Let A ⊆ Y ⊆ X and suppose that A is $\mathcal{I}_{g^\star}$-closed in (X, τ). Then A is $\mathcal{I}_{g^\star}$-closed relative to the subspace Y of X, with respect to the ideal $\mathcal{I}_Y = \{F \subseteq Y : F \in \mathcal{I}\}$.*

Proof. Suppose A ⊆ U ∩ Y and U is g-open in (X, τ), then A ⊆ U. Since A is $\mathcal{I}_{g^\star}$-closed in (X, τ), we have cl(A)−U ∈ $\mathcal{I}$. Now (cl(A) ∩ Y)−(U∩Y) = (cl(A)−U) ∩ Y ∈ $\mathcal{I}$, whenever A ⊆ U ∩ Y and U is g-open. Hence A is $\mathcal{I}_{g^\star}$-closed relative to the subspace Y. □

Theorem 2.13. *Let A be an $\mathcal{I}_{g^\star}$-closed set and F be a closed set in (X, τ), then A ∩ F is an $\mathcal{I}_{g^\star}$-closed set in (X, τ).*

Proof. Let A ∩ F ⊆ U and U is g-open. Then A ⊆ U ∪ (X−F). Since A is $\mathcal{I}_{g^\star}$-closed, we have cl(A)−(U ∪ (X−F)) ∈ $\mathcal{I}$. Now, cl(A ∩ F) ⊆ cl(A) ∩ F = (cl(A) ∩ F)−(X−F). Therefore, cl(A ∩ F)−U ⊆ (cl(A) ∩ F)−(U ∩ (X−F)) ⊆ cl(A)−(U ∪ (X−F)) ∈ $\mathcal{I}$. Hence A ∩ F is $\mathcal{I}_{g^\star}$-closed in (X, τ). □

Definition 2.14. *Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. A subset $A \subseteq X$ is said to be $g^{\star}$-open with respect to an ideal (briefly $\mathcal{I}_{g^{\star}}$-open) if and only if $X-A$ is $\mathcal{I}_{g^{\star}}$-closed.*

Theorem 2.15. *A set A is $\mathcal{I}_{g^{\star}}$-open in (X, τ) if and only if $F-U \subseteq int(A)$, for some $U \in \mathcal{I}$, whenever $F \subseteq A$ and F is g-closed.*

Proof. Suppose A is $\mathcal{I}_{g^{\star}}$-open. Suppose $F \subseteq A$ and F is g-closed. We have $X-A \subseteq X-F$. By assumption, $cl(X-A) \subseteq (X-F) \cup U$, for some $U \in \mathcal{I}$. This implies $X-((X-F) \cup U) \subseteq X-(cl(X-A))$ and hence $F-U \subseteq int(A)$.

Conversely, assume that $F \subseteq A$ and F is g-closed. Then $F-U \subseteq int(A)$, for some $U \in \mathcal{I}$. Consider an g-open set G such that $X-A \subseteq G$. Then $X-G \subseteq A$. By assumption, $(X-G)-U \subseteq int(A) = X-cl(X-A)$. This gives that $X-(G \cup U) \subseteq X-cl(X-A)$. Then, $cl(X-A) \subseteq G \cup U$, for some $U \in \mathcal{I}$.
This shows that $cl(X-A)-G \in \mathcal{I}$. Hence $X-A$ is $\mathcal{I}_{g^{\star}}$-closed. □

Recall that the sets A and B are said to be separated if $cl(A) \cap B = \emptyset$ and $A \cap cl(B) = \emptyset$.

Theorem 2.16. *If A and B are separated $\mathcal{I}_{g^{\star}}$-open sets in (X, τ), then $A \cup B$ is $\mathcal{I}_{g^{\star}}$-open.*

Proof. Suppose A and B are separated $\mathcal{I}_{g^{\star}}$-open sets in (X, τ) and F be a g-closed subset of $A \cup B$. Then $F \cap cl(A) \subseteq A$ and $F \cap cl(B) \subseteq B$. By assumption, $(F \cap cl(A))-U_1 \subseteq int(A)$ and $(F \cap cl(B))-U_2 \subseteq int(B)$, for some $U_1, U_2 \in \mathcal{I}$. It means that $((F \cap cl(A))-int(A)) \in \mathcal{I}$ and $((F \cap cl(B))-int(B)) \in \mathcal{I}$. Then $((F \cap cl(A))-int(A)) \cup ((F \cap cl(B))-int(B)) \in \mathcal{I}$.

Hence $(F \cap (cl(A) \cup cl(B))-(int(A) \cup int(B))) \in \mathcal{I}$. But $F = F \cap (A \cup B) \subseteq F \cap cl(A \cup B)$, and we have $F-int(A \cup B) \subseteq (F \cap cl(A \cup B))-int(A \cup B) \subseteq (F \cap cl(A \cup B))-(int(A) \cup int(B)) \in \mathcal{I}$. Hence, $F-U \subseteq int(A \cup B)$, for some $U \in \mathcal{I}$. This proves that $A \cup B$ is $\mathcal{I}_{g^{\star}}$-open. □

Corollary 2.17. *Let A and B be $\mathcal{I}_{g^{\star}}$-closed sets and suppose $X-A$ and $X-B$ are separated in (X, τ). Then $A \cap B$ is $\mathcal{I}_{g^{\star}}$-closed.*

Corollary 2.18. *If A and B are $\mathcal{I}_{g^{\star}}$-open sets in (X, τ), then $A \cap B$ is $\mathcal{I}_{g^{\star}}$-open.*

Proof. If A and B are $\mathcal{I}_{g^{\star}}$-open, then $X-A$ and $X-B$ are $\mathcal{I}_{g^{\star}}$-closed. By Theorem 2.5, $X-(A \cap B)$ is $\mathcal{I}_{g^{\star}}$-closed, which implies $A \cap B$ is $\mathcal{I}_{g^{\star}}$-open. □

Theorem 2.19. *If $int(A) \subseteq B \subseteq A$ and A is $\mathcal{I}_{g^{\star}}$-open in (X, τ), then B is $\mathcal{I}_{g^{\star}}$-open in X.*

Proof. Suppose $int(A) \subseteq B \subseteq A$ and A is $\mathcal{I}_{g^{\star}}$-open. Then $X-A \subseteq X-B \subseteq cl(X-A)$ and $X-A$ is $\mathcal{I}_{g^{\star}}$-closed. By Theorem 2.11, $X-B$ is $\mathcal{I}_{g^{\star}}$-closed and hence B is $\mathcal{I}_{g^{\star}}$-open. □

Theorem 2.20. *Let (X, τ) be a topological space. Then a set A is $\mathcal{I}_{g^{\star}}$-closed in X if and only if $cl(A)-A$ is $\mathcal{I}_{g^{\star}}$-open in X.*

Proof. **Necessity:** Suppose $F \subseteq cl(A)-A$ and F be g-closed. Then by Theorem 2.4, $F \in \mathcal{I}$. This implies that $F-U = \emptyset$, for some $U \in \mathcal{I}$. Clearly, $F-U \subseteq int(cl(A)-A)$. By Theorem 2.15, $cl(A)-A$ is $\mathcal{I}_{g^{\star}}$-open.

Sufficiency: Suppose $A \subseteq G$ and G is g-open in (X, τ). Then $cl(A) \cap (X-G) \subseteq cl(A) \cap (X-A) = cl(A)-A$. By hypothesis, $(cl(A) \cap (X-G))-U \subseteq int(cl(A)-A) = \emptyset$, for some $U \in \mathcal{I}$. This implies that $cl(A) \cap (X-G) \subseteq U \in \mathcal{I}$ and hence $cl(A)-G \in \mathcal{I}$. Thus, A is $\mathcal{I}_{g^{\star}}$-closed. □

Theorem 2.21. *Let f : (X, τ) $\rightarrow$ (Y, σ) be gc-irresolute and closed. If A $\subseteq$ X is $\mathcal{I}_{g^\star}$-closed in X, then f(A) is f($\mathcal{I}$)$_{g^\star}$-closed in (Y, σ), where f($\mathcal{I}$) = $\{f(U) : U \in \mathcal{I}\}$.*

Proof. Suppose A $\subseteq$ X and A is $\mathcal{I}_{g^\star}$-closed. Suppose f(A) $\subseteq$ G and G is g-open. Then A $\subseteq$ f^{-1}(G). By definition, cl(A)$-f^{-1}$(G) $\in \mathcal{I}$ and hence f(cl(A))$-$G $\in$ f($\mathcal{I}$). Since f is closed, cl(f(A)) $\subseteq$ cl(f(cl(A))) = f(cl(A)). Then cl(f(A))$-$G $\subseteq$ f(cl(A))$-$G $\in$ f($\mathcal{I}$) and hence f(A) is f($\mathcal{I}$)$_{g^\star}$-closed. □

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