International Journal of Mathematics And its Applications
Volume 3, Issue 3–A (2015), 63–73.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Another Generalized Closed Sets in Ideal Topological Spaces

Research Article

O.Ravi[1]*, R.Asokan[2] and A.Thiripuram[3]

1 Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai, Tamil Nadu, India.
2 School of Mathematics, Madurai Kamaraj University, Madurai, Tamil Nadu, India.
3 Department of Mathematics, Jeppiaar Engineering College, Chennai, Tamil Nadu, India.

Abstract: Characterizations and properties of $\mathcal{I}_{g\delta}$-closed sets and $\mathcal{I}_{g\delta}$-open sets are given. A characterization of δ-$\star$-normal spaces is given in terms of $\mathcal{I}_{g\delta}$-open sets.

MSC: 54A05, Secondary 54D15, 54D30.

Keywords: $g\delta$-closed set, $\mathcal{I}_{g\delta}$-closed set, $\star$-closed set, $\mathcal{I}_{\pi g}$-closed set, $\mathcal{I}_g$-closed set.

1. Introduction and Preliminaries

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1) $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$ and

(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$.

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [10] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions [[9], Theorem 2.3] without mentioning it explicitly.

A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $\star$-topology and finer than τ, is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [27]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal space (X, τ, $\mathcal{I}$) is called $\star$-closed [9] (resp. $\star$-dense in itself [8], $\star$-perfect [9]) if $A^* \subseteq A$ (resp. $A \subseteq A^*$, $A = A^*$).

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A⊆X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and int*(A) will denote the interior of A in (X, τ^*).

* *E-mail: siingam@yahoo.com*

A subset A of a topological space (X, τ) is called an α-open [19] (resp. semi-open [11], preopen [14]) if A$\subseteq$int(cl(int(A))) (resp. A$\subseteq$cl(int(A)), $A \subset int(cl(A))$). The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X, τ^α) is denoted by cl_α(A).

A subset A of a topological space (X, τ) is called regular open [26] if A = int(cl(A)). A subset A of a topological space (X, τ) is called δ-open [28] if for each x $\in$ A, there exists a regular open set V such that x $\in$ V $\subseteq$ A and is called δ-closed if X $-$ A is δ-open. A point x $\in$ X is called a δ-cluster point of A [28] if A $\cap$ int(cl(U)) $\neq \emptyset$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by δcl(A). Finite union of regular open sets in (X, τ) is π-open [29] in (X, τ).

Definition 1.1. *A subset A of a topological space (X, τ) is said to be*

1. *g-closed [12] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is open in (X, τ),*
2. *gδ-closed [15] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is δ-open in (X, τ),*
3. *gδ-open [15] if X $-$ A is gδ-closed,*
4. *rg-closed [22] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is regular open in (X, τ),*
5. *πg-closed [4] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is π-open in (X, τ),*
6. *αg-closed [13] if cl_α(A)$\subseteq$U whenever A$\subseteq$U and U is open in (X, τ).*

The complement of αg-closed set is αg-open.

Definition 1.2. *A subset A of an ideal space (X, τ, $\mathcal{I}$) is said to be*

1. *$\mathcal{I}_g$-closed [16] if $A^*\subseteq U$ whenever A$\subseteq$U and U is open in (X, τ, $\mathcal{I}$). The complement of $\mathcal{I}_g$-closed set is $\mathcal{I}_g$-open,*
2. *$\mathcal{I}_{rg}$-closed [17] if $A^*\subseteq U$ whenever A$\subseteq$U and U is regular open in (X, τ, $\mathcal{I}$),*
3. *$\mathcal{I}_{\pi g}$-closed [23] if $A^*\subseteq U$ whenever A$\subseteq$U and U is π-open in (X, τ, $\mathcal{I}$).*

Definition 1.3. *An ideal $\mathcal{I}$ is said to be*

1. *codense [7] or τ-boundary [18] if $\tau \cap \mathcal{I}=\{\emptyset\}$,*
2. *completely codense [7] if PO(X)$\cap\mathcal{I}=\{\emptyset\}$, where PO(X) is the family of all preopen sets in (X, τ).*

Lemma 1.4. *Every completely codense ideal is codense but not conversely [7].*

The following Lemmas will be useful in the sequel.

Lemma 1.5. *Let (X, τ, $\mathcal{I}$) be an ideal space and A$\subseteq$X. If $A\subseteq A^*$, then A^*=cl(A^*)=cl(A)=cl^*(A) [[25], Theorem 5].*

Lemma 1.6. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then $\mathcal{I}$ is codense if and only if $G\subseteq G^*$ for every semi-open set G in X [[25], Theorem 3].*

Lemma 1.7. *Let (X, τ, $\mathcal{I}$) be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^*\subseteq\tau^\alpha$ [[25], Theorem 6].*

Remark 1.8. *If (X, τ) is a topological space, then every closed set is gδ-closed but not conversely [15].*

Lemma 1.9. *Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[6], Theorem 2.1].*

Remark 1.10 ([4]). *The following implications are true in any topological spaces: regular open set ⇒ π-open set ⇒ δ-open set ⇒ open set. None of the above implications is reversible.*

Remark 1.11. *The following statements are true in any topological spaces:*

1. *Every closed set is g-closed but not conversely [12].*
2. *Every g-closed set is $g\delta$-closed but not conversely [15].*
3. *Every $g\delta$-closed set is πg-closed but not conversely [15].*
4. *Every πg-closed set is rg-closed but not conversely [23].*

Remark 1.12. *The following statements are true in any ideal spaces:*

1. *Every $\star$-closed set is $\mathcal{I}_g$-closed but not conversely [16].*
2. *Every $\mathcal{I}_{\pi g}$-closed set is $\mathcal{I}_{rg}$-closed but not conversely [23].*

Remark 1.13. *The following statements are true in any ideal spaces:*

1. *Every closed set is $\star$-closed but not conversely [9].*
2. *Every πg-closed set is $\mathcal{I}_{\pi g}$-closed but not conversely [23].*
3. *Every rg-closed set is $\mathcal{I}_{rg}$-closed but not conversely [17].*

Lemma 1.14 ([9]). *Let $(X, \tau, \mathcal{I})$ be an ideal space and A, B subsets of X. Then the following properties hold:*

1. *If $A \subseteq B$ then $A^* \subseteq B^*$,*
2. *$A^* = cl(A^*) \subseteq cl(A)$,*
3. *$(A^*)^* \subseteq A^*$,*
4. *$(A \cup B)^* = A^* \cup B^*$.*

Definition 1.15. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be δ-closed [5, 20] if $f(V)$ is δ-closed in Y for every δ-closed set V of X.*

Definition 1.16. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be δ-continuous [20] if $f^{-1}(A)$ is δ-closed in (X, τ) for every closed set A of (Y, σ).*

Definition 1.17. *A topological space (X, τ) is said to be δ-normal [24] if for every pair of disjoint δ-closed subsets A, B of X, there exist disjoint open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$.*

Definition 1.18. *A topological space (X, τ) is said to be $\star$-normal [23] if for every pair of disjoint closed subsets A, B of X, there exist disjoint $\star$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$.*

Theorem 1.19 ([16]). *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$. Then A is $\mathcal{I}_g$-open if and only if $F \subseteq int^*(A)$ whenever F is closed and $F \subseteq A$.*

2. $\mathcal{I}_{g\delta}$-closed Sets

Definition 2.1. *A subset A of an ideal space (X, τ, $\mathcal{I}$) is said to be*

1. *$\mathcal{I}_{g\delta}$-closed if $A^*\subseteq U$ whenever $A\subseteq U$ and U is δ-open in (X, τ, $\mathcal{I}$),*

2. *$\mathcal{I}_{g\delta}$-open if $X - A$ is $\mathcal{I}_{g\delta}$-closed.*

Theorem 2.2. *If (X, τ, $\mathcal{I}$) is any ideal space, then every $\mathcal{I}_g$-closed set is $\mathcal{I}_{g\delta}$-closed but not conversely.*

Example 2.3. *Let X=$\{a, b, c\}$, τ=$\{\emptyset, X, \{c\}, \{a, c\}\}$ and $\mathcal{I}$=$\{\emptyset\}$. Then $\mathcal{I}_g$-closed sets are $\emptyset$, X, $\{b\}$, $\{a, b\}$, $\{b, c\}$ and $\mathcal{I}_{g\delta}$-closed sets are P(X). It is clear that $\{a\}$ is $\mathcal{I}_{g\delta}$-closed set but it is not $\mathcal{I}_g$-closed.*

Theorem 2.4. *If (X, τ, $\mathcal{I}$) is any ideal space and A$\subseteq$X, then the following are equivalent.*

1. *A is $\mathcal{I}_{g\delta}$-closed,*

2. *$cl^*(A)\subseteq U$ whenever $A\subseteq U$ and U is δ-open in X.*

Proof. (1)⇒(2) If A is $\mathcal{I}_{g\delta}$-closed, then $A^*\subseteq U$ whenever $A\subseteq U$ and U is δ-open in X and so $cl^*(A)=A\cup A^*\subseteq U$ whenever $A\subseteq U$ and U is δ-open in X. This proves (2).
(2)⇒(1) Let $cl^*(A)\subseteq U$ whenever $A\subseteq U$ and U is δ-open in X. Since $A^*\subseteq cl^*(A)\subseteq U$, $A^*\subseteq U$ whenever $A\subseteq U$ and U is δ-open in X. Therefore A is $\mathcal{I}_{g\delta}$-closed. □

Theorem 2.5. *If a subset A of (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g\delta}$-closed set, then*

1. *$cl^*(A)-A$ contains no nonempty δ-closed set,*

2. *A^*-A contains no nonempty δ-closed set.*

Proof.

(1) Suppose that A is $\mathcal{I}_{g\delta}$-closed in (X, τ, $\mathcal{I}$) and F be a δ-closed subset of $cl^*(A) - A$. Then $A \subseteq X - F$. Since $X - F$ is δ-open and A is $\mathcal{I}_{g\delta}$-closed, $cl^*(A) \subseteq X - F$. Consequently, $F \subseteq X - cl^*(A)$. We have $F \subseteq cl^*(A)$. Thus, $F \subseteq cl^*(A) \cap (X - cl^*(A)) = \emptyset$ and so $cl^*(A)-A$ contains no nonempty δ-closed set.

(2) The fact is $cl^*(A)-A=(A\cup A^*)-A=(A\cup A^*)\cap A^c=(A\cap A^c)\cup(A^*\cap A^c)=A^*\cap A^c = A^*-A$.

□

Theorem 2.6. *Every $\star$-closed set is $\mathcal{I}_{g\delta}$-closed but not conversely.*

Proof. Let A be a $\star$-closed, then $A^*\subseteq A$. Let $A\subseteq U$ where U is δ-open. Hence $A^*\subseteq U$ whenever $A\subseteq U$ and U is δ-open. Therefore A is $\mathcal{I}_{g\delta}$-closed. □

Example 2.7. *In Example 2.3, $\mathcal{I}_{g\delta}$-closed sets are P(X) and $\star$-closed sets are $\emptyset$, X, $\{b\}$, $\{a, b\}$. It is clear that $\{a\}$ is $\mathcal{I}_{g\delta}$-closed set but it is not $\star$-closed.*

Theorem 2.8. *Let (X, τ, $\mathcal{I}$) be an ideal space. For every $A\in\mathcal{I}$, A is $\mathcal{I}_{g\delta}$-closed.*

Proof. Let $A\subseteq U$ where U is δ-open set. Since $A^*=\emptyset$ for every $A\in\mathcal{I}$, then $cl^*(A)=A\cup A^* =A\subseteq U$. Therefore, by Theorem 2.4, A is $\mathcal{I}_{g\delta}$-closed. □

Theorem 2.9. *If $(X, \tau, \mathcal{I})$ is an ideal space, then A^* is always $\mathcal{I}_{g\delta}$-closed for every subset A of X.*

Proof. Let $A^*\subseteq U$ where U is δ-open. Since $(A^*)^*\subseteq A^*$, we have $(A^*)^* \subseteq U$ whenever $A^*\subseteq U$ and U is δ-open. Hence A^* is $\mathcal{I}_{g\delta}$-closed. □

Theorem 2.10. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then every $\mathcal{I}_{g\delta}$-closed, δ-open set is $\star$-closed set.*

Proof. Since A is $\mathcal{I}_{g\delta}$-closed and δ-open. Then $A^*\subseteq A$ whenever $A\subseteq A$ and A is δ-open. Hence A is $\star$-closed. □

Theorem 2.11. *Let $(X, \tau, \mathcal{I})$ be an ideal space and A be a $\mathcal{I}_{g\delta}$-closed set. Then the following are equivalent.*

1. *A is a $\star$-closed set,*
2. *$cl^*(A)-A$ is a δ-closed set,*
3. *A^*-A is a δ-closed set.*

Proof. (1)⇔(2) If A is $\star$-closed, then $A^*\subseteq A$ and so $cl^*(A)-A=(A\cup A^*)-A=\emptyset$. Hence $cl^*(A)-A$ is δ-closed set. Conversely, suppose $cl^*(A)-A$ is δ-closed set. Since A is $\mathcal{I}_{g\delta}$-closed set, by Theorem 2.5, $cl^*(A)-A = \emptyset$ and so A is $\star$-closed.
(2)⇔(3) Obvious. □

Theorem 2.12. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then every $g\delta$-closed set is $\mathcal{I}_{g\delta}$-closed set but not conversely.*

Proof. Let A be any $g\delta$-closed set in $(X, \tau, \mathcal{I})$. Then $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is δ-open. We have $A^*\subseteq cl^*(A)\subseteq cl(A)\subseteq U$ whenever $A\subseteq U$ and U is δ-open. Hence A is $\mathcal{I}_{g\delta}$-closed. □

Example 2.13. *Let $X=\{a, b, c, d\}$, $\tau=\{\emptyset, X, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $\mathcal{I}=\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $\mathcal{I}_{g\delta}$-closed sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ and $g\delta$-closed sets are $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$. It is clear that $\{b\}$ is $\mathcal{I}_{g\delta}$-closed set but it is not $g\delta$-closed.*

Theorem 2.14. *If $(X, \tau, \mathcal{I})$ is an ideal space and A is a $\star$-dense in itself, $\mathcal{I}_{g\delta}$-closed subset of X, then A is $g\delta$-closed.*

Proof. Suppose A is a $\star$-dense in itself, $\mathcal{I}_{g\delta}$-closed subset of X. Let $A\subseteq U$ where U is δ-open. Then, by Theorem 2.4, $cl^*(A)\subseteq U$ whenever $A\subseteq U$ and U is δ-open. Since A is $\star$-dense in itself, by Lemma 1.5, $cl(A)=cl^*(A)$. Therefore $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is δ-open. Hence A is $g\delta$-closed. □

Definition 2.15. *A subset A of a topological space (X, τ) is said to be $g\delta\alpha$-closed if $cl_\alpha(A)\subseteq U$ whenever $A\subseteq U$ and U is δ-open in (X, τ). The complement of $g\delta\alpha$-closed set is $g\delta\alpha$-open.*

Theorem 2.16. *If $(X, \tau, \mathcal{I})$ is any ideal space, then the following hold:*

1. *If $\mathcal{I}=\{\emptyset\}$, then A is $\mathcal{I}_{g\delta}$-closed if and only if A is $g\delta$-closed.*
2. *If $\mathcal{I}=\mathcal{N}$, then A is $\mathcal{I}_{g\delta}$-closed if and only if A is $g\delta\alpha$-closed.*

Proof.

(1) From the fact that for $\mathcal{I}=\{\emptyset\}$, $A^*=cl(A)\supseteq A$. Therefore A is $\star$-dense in itself. Since A is $\mathcal{I}_{g\delta}$-closed, by Theorem 2.14, A is $g\delta$-closed.

Conversely, by Theorem 2.12, every $g\delta$-closed set is $\mathcal{I}_{g\delta}$-closed set.

(2) If $\mathcal{I}$=$\mathcal{N}$, then A*=cl(int(cl(A))) for every subset A of X and cl_α(A) = A∪cl(int(cl(A))). Let A be a $\mathcal{I}_{g\delta}$-closed set. Then A*⊆U whenever A⊆U and U is δ-open in X. It implies that cl(int(cl(A)))⊆U whenever A⊆U and U is δ-open in X and A∪cl(int(cl(A)))⊆A∪U. It shows that cl_α(A)⊆U whenever A⊆U and U is δ-open in X. Therefore A is $g\delta\alpha$-closed. Converse is clear.

□

Corollary 2.17. *If (X, τ, $\mathcal{I}$) is any ideal space where $\mathcal{I}$ is codense and A is a semi-open, $\mathcal{I}_{g\delta}$-closed subset of X, then A is $g\delta$-closed.*

Proof. By Lemma 1.6, A is $\star$-dense in itself. By Theorem 2.14, A is $g\delta$-closed. □

Theorem 2.18. *Let (X, τ, $\mathcal{I}$) be an ideal space and A⊆X. If A⊆B⊆A*, then A*=B* and B is $\star$-dense in itself.*

Proof. Since A⊆B, then A*⊆B* and since B⊆A*, then B*⊆(A*)*⊆A*. Therefore A*=B* and B⊆A*⊆B*. Hence proved.

□

Theorem 2.19. *Let (X, τ, $\mathcal{I}$) be an ideal space. If A and B are subsets of X such that A⊆B⊆A* and A is $\mathcal{I}_{g\delta}$-closed, then B is $\mathcal{I}_{g\delta}$-closed.*

Proof. Let U be any δ-open set of (X, τ, $\mathcal{I}$) such that B⊆U. Then A⊆U. Since A is $\mathcal{I}_{g\delta}$-closed, we have A*⊆U. Now B*⊆(A*)*⊆A*⊆U. Therefore B is $\mathcal{I}_{g\delta}$-closed. □

Corollary 2.20. *Let (X, τ, $\mathcal{I}$) be an ideal space. If A and B are subsets of X such that A⊆B⊆A* and A is $\mathcal{I}_{g\delta}$-closed, then A and B are $g\delta$-closed sets.*

Proof. Let A and B be subsets of X such that A⊆B⊆A* and A is $\mathcal{I}_{g\delta}$-closed. By Theorem 2.19, B is $\mathcal{I}_{g\delta}$-closed. Since A⊆B⊆A*, then A*=B* and so A and B are $\star$-dense in itself. By Theorem 2.14, A and B are $g\delta$-closed. □

The following theorem gives a characterization of $\mathcal{I}_{g\delta}$-open sets.

Theorem 2.21. *Let (X, τ, $\mathcal{I}$) be an ideal space and A⊆X. Then A is $\mathcal{I}_{g\delta}$-open if and only if F⊆int*(A) whenever F is δ-closed and F⊆A.*

Proof. Suppose A is $\mathcal{I}_{g\delta}$-open. If F is δ-closed and F⊆A, then X−A⊆X−F and so cl*(X−A)⊆X−F by Theorem 2.4. Therefore F⊆X−cl*(X−A)=int*(A). Hence F⊆int*(A).
Conversely, suppose the condition holds. Let U be a δ-open set such that X−A⊆U. Then X−U⊆A and so X−U⊆int*(A). Therefore cl*(X−A)⊆U. By Theorem 2.4, X−A is $\mathcal{I}_{g\delta}$-closed. Hence A is $\mathcal{I}_{g\delta}$-open. □

The following theorem gives a characterization of $\mathcal{I}_{g\delta}$-closed sets in terms of $\mathcal{I}_{g\delta}$-open sets.

Theorem 2.22. *Let (X, τ, $\mathcal{I}$) be an ideal space and A⊆X. Consider the following statements.*

1. *A is $\mathcal{I}_{g\delta}$-closed,*
2. *A∪(X−A*) is $\mathcal{I}_{g\delta}$-closed,*
3. *A*−A is $\mathcal{I}_{g\delta}$-open.*

Then we have (1) ⇒ (2) ⇔ (3).

Proof. (1)⇒(2) Suppose A is $\mathcal{I}_{g\delta}$-closed. If U is any δ-open set such that $A\cup(X-A^*)\subseteq U$, then $X-U\subseteq X-(A\cup(X-A^*))=X\cap(A\cup(A^*)^c)^c=A^*\cap A^c=A^*-A$. Since A is $\mathcal{I}_{g\delta}$-closed, by Theorem 2.5, it follows that $X-U=\emptyset$ and so X=U. Therefore $A\cup(X-A^*)\subseteq U$ which implies that $A\cup(X-A^*)\subseteq X$ and so $(A\cup(X-A^*))^*\subseteq X^*\subseteq X=U$. Hence $A\cup(X-A^*)$ is $\mathcal{I}_{g\delta}$-closed.

(2)⇔(3) Since $X-(A^*-A)=X\cap(A^*\cap A^c)^c=X\cap((A^*)^c\cup A)=(X\cap (A^*)^c)\cup(X\cap A)=A\cup (X-A^*)$ is $\mathcal{I}_{g\delta}$-closed. Hence A^*-A is $\mathcal{I}_{g\delta}$-open. □

Theorem 2.23. *Let (X, τ, $\mathcal{I}$) be an ideal space. Then every subset of X is $\mathcal{I}_{g\delta}$-closed if and only if every δ-open set is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}_{g\delta}$-closed. If $U\subseteq X$ is δ-open, then by hypothesis, U is $\mathcal{I}_{g\delta}$-closed and so $U^*\subseteq U$. Hence U is $\star$-closed.

Conversely, suppose that every δ-open set is $\star$-closed. Let A be a subset of X. If U is δ-open set such that $A\subseteq U$, then $A^*\subseteq U^*\subseteq U$ and so A is $\mathcal{I}_{g\delta}$-closed. □

Theorem 2.24. *The union of two $\mathcal{I}_{g\delta}$-closed sets is again $\mathcal{I}_{g\delta}$-closed.*

Proof. Suppose that $(A\cup B)\subseteq U$ and U is δ-open in (X, τ, $\mathcal{I}$), than $A\subseteq U$ and $B\subseteq U$. Since A and B are $\mathcal{I}_{g\delta}$-closed sets, $A^*\subseteq U$ and $B^*\subseteq U$. $(A\cup B)^*=A^*\cup B^*\subseteq U$. Thus, $A\cup B$ is $\mathcal{I}_{g\delta}$-closed. □

Theorem 2.25. *For each $x\in$(X, τ, $\mathcal{I}$), either $\{x\}$ is δ-closed or $\{x\}^c$ is $\mathcal{I}_{g\delta}$-closed in (X, τ, $\mathcal{I}$).*

Proof. Suppose that $\{x\}$ is not δ-closed, then $\{x\}^c$ is not δ-open and the only δ-open set containing $\{x\}^c$ is the space (X, τ, $\mathcal{I}$) itself. Therefore $cl^*(\{x\}^c)\subseteq X$ and so $\{x\}^c$ is $\mathcal{I}_{g\delta}$-closed. □

Definition 2.26. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be*

1. *a $\mathcal{X}_{\mathcal{I}}$-set if $A = U\cap V$, where U is a δ-open set and V is a $\star$-perfect set.*
2. *a $\mathcal{Y}_{\mathcal{I}}$-set if $A = U\cap V$, where U is a δ-open set and V is a $\star$-closed set.*

Theorem 2.27. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is a $\mathcal{X}_{\mathcal{I}}$-set and a $\mathcal{I}_{g\delta}$-closed set, then A is a $\star$-closed set.*

Proof. Let A be a $\mathcal{X}_{\mathcal{I}}$-set and a $\mathcal{I}_{g\delta}$-closed set. Since A is a $\mathcal{X}_{\mathcal{I}}$-set, $A = U\cap V$, where U is a δ-open set and V is a $\star$-perfect set. Now, $A = U\cap V\subseteq U$ and A is a $\mathcal{I}_{g\delta}$-closed set implies that $A^*\subseteq U$. Also, $A = U\cap V\subseteq V$ and V is $\star$-perfect set implies that $A^*\subseteq V$. Thus, $A^*\subseteq U\cap V = A$. Hence, A is a $\star$-closed set. □

Theorem 2.28. *For a subset A of an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.*

1. *A is a $\star$-closed set.*
2. *A is a $\mathcal{Y}_{\mathcal{I}}$-set and a $\mathcal{I}_{g\delta}$-closed set.*

Proof. (1) ⇒ (2): Let A be a $\star$-closed set and $A = X\cap A$, where X is δ-open set and A is a $\star$-closed set. Hence, A is a $\mathcal{Y}_{\mathcal{I}}$-set. Assume that A be a $\star$-closed set and U be a δ-open set such that $A\subseteq U$. Then $A^*\subseteq U$ and hence A is a $\mathcal{I}_{g\delta}$-closed set.

(2) ⇒ (1): Let A be a $\mathcal{Y}_{\mathcal{I}}$-set and a $\mathcal{I}_{g\delta}$-closed set. Since A is a $\mathcal{Y}_{\mathcal{I}}$-set, $A = U\cap V$, where U is a δ-open set and V is a $\star$-closed set. Now, $A\subseteq U$ and A is a $\mathcal{I}_{g\delta}$-closed set implies that $A^*\subseteq U$. Also, $A\subseteq V$ and V is a $\star$-closed set implies that $A^*\subseteq V$. Thus, $A^*\subseteq U\cap V = A$. Hence, A is a $\star$-closed set. □

Remark 2.29. *The following Examples show that the concepts of $\mathcal{Y}_{\mathcal{I}}$-sets and $\mathcal{I}_{g\delta}$-closed sets are independent.*

Example 2.30. *In Example 2.13, $\{c,\ d\}$ is $\mathcal{Y}_{\mathcal{I}}$-set but not $\mathcal{I}_{g\delta}$-closed set.*

Example 2.31. *In Example 2.13, $\{a,\ b,\ c\}$ is $\mathcal{I}_{g\delta}$-closed set but not $\mathcal{Y}_{\mathcal{I}}$-set.*

Proposition 2.32. *Every αg-closed set in $(X,\ \tau)$ is $g\delta\alpha$-closed in $(X,\ \tau)$ but not conversely.*

Example 2.33. *Let $X=\{a,\ b,\ c\}$ and $\tau=\{\emptyset,\ X,\ \{a\},\ \{b\},\ \{a,\ b\},\ \{a,\ c\}\}$. Then $\{a\}$ is $g\delta\alpha$-closed set but not αg-closed set.*

3. δ-$\star$-normal Spaces

Definition 3.1. *A space $(X,\ \tau,\ \mathcal{I})$ is said to be δ-$\star$-normal if for any two disjoint δ-closed sets A and B in $(X,\ \tau)$, there exist disjoint $\star$-open sets U, V such that $A\subseteq U$ and $B\subseteq V$.*

Theorem 3.2. *Let $(X,\ \tau,\ \mathcal{I})$ be an ideal space. Then the following are equivalent.*

1. *$(X,\ \tau,\ \mathcal{I})$ is δ-$\star$-normal.*
2. *For every pair of disjoint δ-closed sets A and B, there exist disjoint $\mathcal{I}_g$-open sets U, V such that $A\subseteq U$ and $B\subseteq V$.*
3. *For every pair of disjoint δ-closed sets A and B, there exist disjoint $\mathcal{I}_{g\delta}$-open sets U, V such that $A\subseteq U$ and $B\subseteq V$.*
4. *For each δ-closed set A and for each δ-open set V containing A, there exists an $\mathcal{I}_{g\delta}$-open set U such that $A\subseteq U\subseteq cl^*(U)\subseteq V$.*
5. *For each δ-closed set A and for each δ-open set V containing A, there exists an $\star$-open set U such that $A\subseteq U\subseteq cl^*(U)\subseteq V$.*

Proof. It is obvious that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) : Suppose that A is δ-closed and V is a δ-open set containing A. Then $A\cap V^c=\emptyset$. By assumption, there exist $\mathcal{I}_{g\delta}$-open sets U and W such that $A\subseteq U$, $V^c\subseteq W$. Since V^c is δ-closed and W is $\mathcal{I}_{g\delta}$-open, by Theorem 2.21, $V^c\subseteq \text{int}^*(W)$ and so $(\text{int}^*(W))^c\subseteq V$. Again, $U\cap W=\emptyset$ implies that that $U\cap \text{int}^*(W)=\emptyset$ and so $\text{cl}^*(U)\subseteq(\text{int}^*(W))^c\subseteq V$. Hence, U is the required $\mathcal{I}_{g\delta}$-open set such that $A\subseteq U\subseteq \text{cl}^*(U)\subseteq V$.

(4) $\Rightarrow$ (5) : Let A be a δ-closed set and V be a δ-open set such that $A\subseteq V$. By hypothesis, there exist $\mathcal{I}_{g\delta}$-open set W such that $A\subseteq W\subseteq \text{cl}^*(W)\subseteq V$. By Theorem 2.21, $A\subseteq \text{int}^*(W)$. If $U=\text{int}^*(W)$, then U is an $\star$-open set and $A\subseteq U\subseteq \text{cl}^*(U)\subseteq \text{cl}^*(W)\subseteq V$. Therefore, $A\subseteq U\subseteq \text{cl}^*(U)\subseteq V$.

(5) $\Rightarrow$ (1) : Let A and B be disjoint δ-closed sets. Then B^c is a δ-open set containing A. By assumption, there exists an $\star$-open set U such that $A\subseteq U\subseteq \text{cl}^*(U)\subseteq B^c$. If $V=(\text{cl}^*(U))^c$, then U and V are disjoint $\star$-open sets such that $A\subseteq U$ and $B\subseteq V$. □

Definition 3.3. *A function $f:(X,\ \tau,\ \mathcal{I})\rightarrow(Y,\ \sigma,\ \mathcal{J})$ is said to be $\mathcal{I}^*_{g\delta}$-continuous if $f^{-1}(A)$ is $\mathcal{I}_{g\delta}$-closed in $(X,\ \tau,\ \mathcal{I})$ for every $\star$-closed set A of $(Y,\ \sigma,\ \mathcal{J})$.*

Theorem 3.4. *Let $f:(X,\ \tau,\ \mathcal{I})\rightarrow(Y,\ \sigma,\ \mathcal{J})$ be a $\mathcal{I}^*_{g\delta}$-continuous δ-closed injection and Y is δ-$\star$-normal, then X is δ-$\star$-normal.*

Proof. Let A and B are disjoint δ-closed sets of X. Since f is δ-closed injection, f(A) and f(B) are disjoint δ-closed sets of Y. By the δ-$\star$-normality of Y, there exist disjoint $\star$-open sets U and V of Y such that f(A) $\subseteq$U and f(B)$\subseteq$V. Since f is $\mathcal{I}^*_{g\delta}$-continuous, f^{-1}(U) and f^{-1}(V) are disjoint $\mathcal{I}_{g\delta}$-open sets containing A and B respectively. It follows from Theorem 3.2 that X is δ-$\star$-normal. □

Definition 3.5. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be $\mathcal{J}^*_g$-closed if f(A) is $\mathcal{J}_g$-closed in Y for every $\star$-closed set A of X.*

Theorem 3.6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a δ-continuous (resp. continuous) $\mathcal{J}^*_g$-closed surjection and X is a δ-$\star$-normal (resp. $\star$-normal), then Y is $\star$-normal.*

Proof. Let A and B be disjoint closed sets of Y. Then f^{-1}(A) and f^{-1}(B) are disjoint δ-closed (resp. closed) sets of X. Since X is δ-$\star$-normal (resp. $\star$-normal), there exist disjoint $\star$-open sets U and V such that f^{-1}(A)$\subseteq$U and f^{-1}(B)$\subseteq$V. Now, we set K = Y − f(X − U) and L = Y − f(X − V). Then K and L are $\mathcal{J}_g$-open sets of Y such that A$\subseteq$K, B$\subseteq$L, Since A, B are disjoint closed sets and K and L are $\mathcal{J}_g$-open. We have A$\subseteq$int*(K) and B$\subseteq$int*(L) and int*(K)$\cap$int*(L) = $\emptyset$. Hence, Y is $\star$-normal. □

Theorem 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{I}$ is completely codense. Then $(X, \tau, \mathcal{I})$ is δ-normal if and only if it is δ-$\star$-normal.*

Proof. Suppose that A and B are disjoint δ-closed sets. Since X is δ-normal, there exist disjoint open sets U and V such that A$\subseteq$U and B$\subseteq$V. But every open set is $\star$-open set and Hence, X is δ-$\star$-normal.

Conversely, suppose that A and B are disjoint δ-closed sets of X. Then there exist disjoint $\star$-open sets U and V such that A$\subseteq$U and B$\subseteq$V. Since $\mathcal{I}$ is completely codense. By Lemma 1.1, $\tau^* \subseteq \tau^\alpha$ and so U, V $\in \tau^\alpha$. Hence, A$\subseteq$U $\subseteq$int(cl(int(U))) = G and B$\subseteq$V$\subseteq$int(cl(int(V))) = H. Therefore, G and H are disjoint open sets containing A and B respectively. Therefore, X is δ-normal. □

Corollary 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space, where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *$(X, \tau, \mathcal{I})$ is δ-normal.*
2. *For every pair of disjoint δ-closed sets A and B, there exist disjoint $\mathcal{I}_g$-open sets U, V such that A$\subseteq$U and B$\subseteq$V.*
3. *For every pair of disjoint δ-closed sets A and B, there exist disjoint $\mathcal{I}_{g\delta}$-open sets U, V such that A$\subseteq$U and B$\subseteq$V.*
4. *For each δ-closed set A and for each δ-open set V containing A, there exists an $\mathcal{I}_{g\delta}$-open set U such that A$\subseteq$U $\subseteq$cl*(U)$\subseteq$V.*
5. *For each δ-closed set A and for each δ-open set V containing A, there exists an $\star$-open set U such that A$\subseteq$U$\subseteq$cl*(U)$\subseteq$V.*
6. *For every pair of disjoint δ-closed sets A and B, there exist disjoint $\star$-open sets U, V such that A$\subseteq$U and B$\subseteq$V.*

If $\mathcal{I} = \mathcal{N}$, from Corollary 3.8, we get the following Corollary 3.9.

Corollary 3.9. *Let (X, τ) be a topological space. Then the following are equivalent.*

1. *X is δ-normal.*
2. *For every pair of disjoint δ-closed sets A and B, there exist disjoint αg-open sets U, V such that A$\subseteq$U and B$\subseteq$V.*

3. *For every pair of disjoint δ-closed sets A and B, there exist disjoint $g\delta\alpha$-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.*

4. *For each δ-closed set A and for each δ-open set V containing A, there exists an $g\delta\alpha$-open set U such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*

5. *For each δ-closed set A and for each δ-open set V containing A, there exists an α-open set U such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*

6. *For every pair of disjoint δ-closed sets A and B, there exist disjoint α-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.*

References

[1] A.Acikgoz, T.Noiri and S.Yuksel, *On α-I-continuous and α-I-open functions*, Acta Math. Hungar., 105(1-2)(2004), 27-37.

[2] D.Andrijevič, *Some properties of the topology of α-sets*, Mat. Vesnik, 36(1984), 1-10.

[3] C.Chattopadhyay, *On strongly pre-open sets and a decomposition of continuity*, Mat. Vesnik, 57(2005), 121-125.

[4] J.Dontchev and T.Noiri, *Quasi-normal spaces and πg-closed sets*, Acta Math. Hungar., 89(3)(2000), 211-219.

[5] J.Dontchev and M.Ganster, *On δ-generalized closed sets and $T_{3/4}$ spaces*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 17(1996), 15-31.

[6] J.Dontchev, M.Ganster and T.Noiri, *Unified operation approach of generalized closed sets via topological ideals*, Math. Japonica, 49(1999), 395-401.

[7] J.Dontchev, M.Ganster and D.Rose, *Ideal resolvability*, Topology and its Applications, 93(1999), 1-16.

[8] E.Hayashi, *Topologies defined by local properties*, Math.Ann., 156(1964), 205-215.

[9] D.Jankovic and T.R.Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, 97(4)(1990), 295-310.

[10] K.Kuratowski, *Topology*, Vol. I, Academic Press, New york, (1966).

[11] N.Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, 70(1963), 36-41.

[12] N.Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, 2(19)(1970), 89-96.

[13] H.Maki, R.Devi and K.Balachandran, *Generalized α-closed sets in topology*, Bull. Fukuoka Univ. Ed Part III, 42(1993), 13-21.

[14] A.S.Mashhour, M.E.Abd El-Monsef and S.N.El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys.Soc. Egypt, 53(1982), 47-53.

[15] A.Muthulakshmi, O.Ravi and S.Vijaya, *$g\delta$-closed sets in topological spaces*, submitted.

[16] M.Navaneethakrishnan and J. Paulraj Joseph, *g-closed sets in ideal topological spaces*, Acta Math. Hungar., 119(4)(2008), 365-371.

[17] M.Navaneethakrishnan, J.Paulraj Joseph and D.Sivaraj, *$\mathcal{I}_g$-normal and $\mathcal{I}_g$-regular spaces*, Acta Math. Hungar., 125(4)(2009), 327-340.

[18] R.L.Newcomb, *Topologies which are compact modulo an ideal*, Ph.D. Dissertation, Univ. of Cal. at Santa Barbara (1967).

[19] O.Njastad, *On some classes of nearly open sets*, Pacific J. Math., 15(1965), 961-970.

[20] T.Noiri, *A generalization of perfect functions*, J. London Math. Soc., 17(2)(1978), 540-544.

[21] T.Noiri, K.Viswanathan, M.Rajamani and S.Krishnaprakash, *On ω-closed sets in ideal topological spaces*, submitted.

[22] N.Palaniappan and K.C.Rao, *Regular generalized closed sets*, Kyungpook Math. J., 33(2)(1993), 211-219.

[23] M.Rajamani, V.Inthumathi and S.Krishnaprakash, *$\mathcal{I}_{\pi g}$-closed sets and $\mathcal{I}_{\pi g}$-continuity*, Journal of Advanced Research in Pure Mathematics, 2(4)(2010), 63-72.

[24] O.Ravi, V.Rajendran and K.Indirani, *Weakly $\mathcal{I}_{g\delta}$-closed sets*, submitted.

[25] V.Renuka Devi, D.Sivaraj and T.Tamizh Chelvam, *Codense and Completely codense ideals*, Acta Math. Hungar., 108(2005), 197-205.

[26] M.H.Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., 41(1937), 375-481.

[27] R.Vaidyanathaswamy, *Set Topology*, Chelsea Publishing Company, (1946).

[28] N.V.Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl., (2) 78(1968), 103-118.

[29] V.Zaitsev, *On certian classes of topological spaces and their bicompactifications*, Dokl. Akad. Nauk SSSR., 178(1968), 778-779.