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Some Results on Odd Mean Graphs

Research Article

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Abstract: Let $G = (V, E)$ be a graph with p vertices and q edges. A graph G is said to have an odd mean labeling if there exists a function $f : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ satisfying f is 1-1 and the induced map $f^* : E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

is a bijection. A graph that admits an odd mean labeling is called an odd mean graph. In this paper, we prove that the graphs slanting ladder SL_n for $n \geq 2$, $Q_n \odot K_1$ for $n \geq 1$, $TW(P_{2n})$ for $n \geq 2$, $H_n \odot mK_1$ for all $n \geq 1, m \geq 1$ and mQ_3 for $m \geq 1$ are odd mean graphs.

MSC: 05C78.

Keywords: Labeling, odd mean labeling, odd mean graph.

1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology we follow [3].

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. $K_{1,m}$ is called a star and it is denoted by S_m. The bistar $B_{m,n}$ is the graph obtained from K_2 by identifying the center vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of K_2 respectively. $B_{m,m}$ is often denoted by $B(m)$. The H-graph of a path P_n, denoted by H_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even. If m number of pendant vertices are attached at each vertex of G, then the resultant graph obtained from G is the graph $G \odot mK_1$. When $m = 1$, $G \odot K_1$ is the corona of G. A Twig $TW(P_n), n \geq 3$ is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices of the path.

The slanting ladder SL_n is a graph obtained from two paths $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by joining each u_i with $v_{i+1}, 1 \leq i \leq n-1$. The graph $K_2 \times K_2 \times K_2$ is called the cube, and it is denoted by Q_3. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The union of m disjoint copies of a graph G is denoted by mG.

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The graph $T_p^{(n)}$ is a tree formed from n copies of path on p vertices by joining an edge uu^0 between every pair of consecutive paths where u is a vertex in the i^{th} copy of the path and u^0 is the corresponding vertex in the $(i+1)^{th}$ copy of the path.

The graceful labelings of graphs was first introduced by Rosa in 1961[1] and R.B. Gnanajothi introduced odd graceful graphs [2]. The concept mean labeling was first introduced by S. Somasundaram and R. Ponraj [7]. Further some more results on mean graphs are discussed in [5, 6, 8, 9]. The concept of odd mean labeling was introduced and studied by K. Manickam and M. Marudai [4]. Also, odd mean property for some graphs are discussed in [10, 11].

A graph G is said to have an odd mean labeling if there exists a function $f : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ satisfying f is $l-1$ and the induced map $f^* : E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

is a bijection. A graph that admits an odd mean labeling is called an odd mean graph.
An odd mean labeling of $B_{4,4}$ is given in Figure 1.

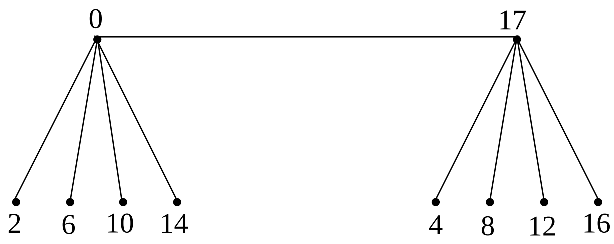

Figure 1.

In this paper, we prove that the graphs slanting ladder SL_n for $n \geq 2$, $Q_n \odot K_1$ for $n \geq 1$, $TW(P_{2n})$ for $n \geq 2$, $H_n \odot mK_1$ for $n \geq 2$, $H_n \odot mK_1$ for all $n \geq 1, m \geq 1$ and mQ_3 for $m \geq 1$ are odd mean graphs.

2. Odd Mean Graphs

Theorem 2.1. *The graph slanting ladder SL_n is an odd mean graph, $n \geq 2$.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the path of length $n-1$. The graph SL_n has $2n$ vertices and $3(n-1)$ edges.
Define $f : V(SL_n) \to \{0, 1, 2, \ldots, 2q-1 = 6n-7\}$ as follows:

$$f(u_i) = 4n + 2i - 6, \quad 1 \leq i \leq n-1$$
$$f(u_n) = 6n - 7$$
$$f(v_i) = 2i - 2, \quad 1 \leq i \leq n.$$

The induced edge labeling f^* is obtained as follows:

$$f^*(u_i u_{i+1}) = 4n + 2i - 5, \quad 1 \leq i \leq n-1$$
$$f^*(v_i v_{i+1}) = 2i - 1, \quad 1 \leq i \leq n-1$$
$$f^*(u_i v_{i+1}) = 2n + 2i - 3, \quad 1 \leq i \leq n-1.$$

Thus, f is an odd mean labeling. Hence, the graph SL_n is an odd mean graph for $n \geq 2$.

For example, an odd mean labeling of SL_9 is shown in Figure 2.

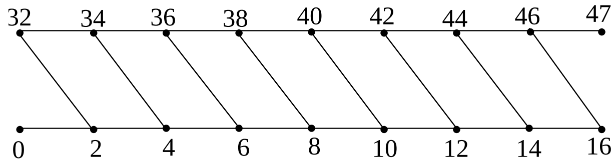

Figure 2.

□

Theorem 2.2. *$Q_n \odot K_1$ is an odd mean graph, for $n \geq 1$.*

Proof. Let Q_n be the quadrilateral snake obtained from a path $u_1, u_2, \ldots, u_{n+1}$ by joining u_i, u_{i+1} to new vertices v_i, w_i respectively and joining v_i and $w_i, 1 \leq i \leq n$.

Let $G = Q_n \odot K_1$ be the graph obtained by joining a pendant edge to each vertex of Q_n. Let $u_i' : 1 \leq i \leq n+1, v_i' : 1 \leq i \leq n$ and $w_i' : 1 \leq i \leq n$ be the new vertices made adjacent with u_i, v_i and w_i respectively. The graph G has $6n+2$ vertices and $7n+1$ edges.

$$\begin{aligned} \text{Let } V(Q_n \odot K_1) &= V(Q_n) \cup \{u_1', u_2', \ldots, u_{n+1}'\} \cup \{v_1', v_2', \ldots, v_n'\} \\ &\cup \{w_1', w_2', \ldots, w_n'\} \\ \text{and } E(Q_n \odot K_1) &= E(Q_n) \cup \{u_i u_i' : 1 \leq i \leq n+1\} \cup \{v_i v_i', w_i w_i' : 1 \leq i \leq n\}. \end{aligned}$$

Define $f : V(Q_n \odot K_1) \to \{0, 1, 2, \ldots, 2q-1 = 14n+1\}$ as follows:

$$\begin{aligned} f(u_1) &= 4 \\ f(u_i) &= 14i - 14, \quad 2 \leq i \leq n+1 \\ f(v_1) &= 2 \\ f(v_i) &= 14i - 8, \quad 2 \leq i \leq n \\ f(w_i) &= 14i - 2, \quad 1 \leq i \leq n \\ f(u_1') &= 6 \\ f(u_i') &= 14i - 13, \quad 2 \leq i \leq n+1 \\ f(v_1') &= 0 \\ f(v_i') &= 14i - 10, \quad 2 \leq i \leq n \\ f(w_i') &= 14i - 4, \quad 1 \leq i \leq n. \end{aligned}$$

The induced edge labels are given by

$$\begin{aligned} f^*(u_1 u_2) &= 9 \\ f^*(u_i u_{i+1}) &= 14i - 7, \quad 2 \leq i \leq n \\ f^*(u_1 u_1') &= 5 \\ f^*(u_i u_i') &= 14i - 13, \quad 2 \leq i \leq n+1 \end{aligned}$$

$$f^*(u_1v_1) = 3$$
$$f^*(u_iv_i) = 14i - 11, \quad 2 \le i \le n$$
$$f^*(u_{i+1}w_i) = 14i - 1, \quad 1 \le i \le n$$
$$f^*(v_1v_1') = 1$$
$$f^*(v_iv_i') = 14i - 9, \quad 2 \le i \le n$$
$$f^*(w_iw_i') = 14i - 3, \quad 1 \le i \le n$$
$$f^*(v_1w_1) = 7$$
$$f^*(v_iw_i) = 14i - 5, \quad 2 \le i \le n.$$

Thus, f is an odd mean labeling and hence $Q_n \odot K_1$ is an odd mean graph for $n \ge 1$.

For example, an odd mean labeling of $Q_7 \odot K_1$ is shown in Figure 3.

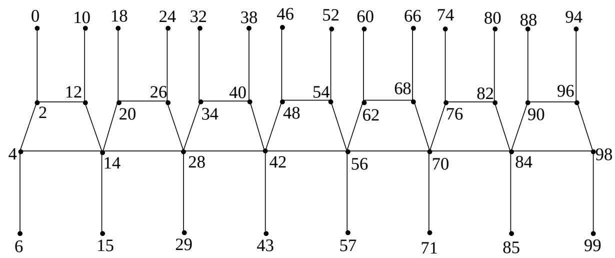

Figure 3.

□

Theorem 2.3. *$TW(P_{2n}), n \ge 2$ is an odd mean graph.*

Proof. Let $u_1, u_2, \ldots, u_{2n}$ be the vertices of the path P_{2n} and let $v_1^{(i)}, v_2^{(i)}$ be the pendant vertices at each vertex u_i for $2 \le i \le 2n - 1$.

$$\text{Let } V(TW(P_{2n})) = V(P_{2n}) \cup \left\{v_1^{(i)}, v_2^{(i)} : 2 \le i \le 2n - 1\right\}$$
$$\text{and } E(TW(P_{2n})) = E(P_{2n}) \cup \left\{u_iv_1^{(i)}, u_iv_2^{(i)} : 2 \le i \le 2n - 1\right\}.$$

Define $f : V(TW(P_{2n})) \to \{0, 1, 2, \ldots, 2q - 1 = 12n - 11\}$ as follows:

$$f(u_i) = \begin{cases} 6i - 6, & 1 \le i \le 2n \text{ and } i \text{ is odd} \\ 2, & i = 2 \\ 6i - 11, & 4 \le i \le 2n \text{ and } i \text{ is even} \end{cases}$$

$$f\left(v_1^{(i)}\right) = \begin{cases} 6i - 12, & 3 \le i \le 2n - 1 \text{ and } i \text{ is odd} \\ 6i - 8, & 2 \le i \le 2n - 1 \text{ and } i \text{ is even} \end{cases}$$

$$f\left(v_2^{(i)}\right) = \begin{cases} 6i - 8, & 3 \le i \le 2n - 1 \text{ and } i \text{ is odd} \\ 6i - 4, & 2 \le i \le 2n - 1 \text{ and } i \text{ is even} \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is obtained as follows:

$$f^*(u_iu_{i+1}) = 6i-5, \quad 1 \le i \le 2n-1$$
$$f^*\left(u_iv_1^{(i)}\right) = 6i-9, \quad 2 \le i \le 2n-1$$
$$f^*\left(u_iv_2^{(i)}\right) = 6i-7, \quad 2 \le i \le 2n-1.$$

Thus, f is an odd mean labeling of $TW(P_{2n}), n \ge 2$. Hence, $TW(P_{2n})$ is an odd mean graph for $n \ge 2$.

For example, an odd mean labeling of $TW(P_8)$ is shown in Figure 4.

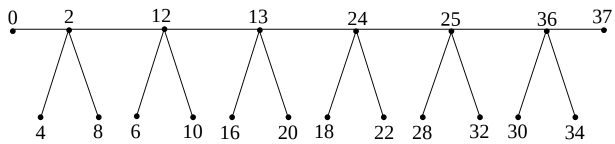

Figure 4.

□

Theorem 2.4. *The graph $H_n \odot mK_1$ is an odd mean graph for all positive integers m and n.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the path of length $n-1$. Let $x_{i,k}$ and $y_{i,k}, 1 \le k \le m$ be the pendant vertices at u_i and v_i respectively for $1 \le i \le n$. The graph $H_n \odot mK_1$ has $2n(m+1)$ vertices and $2n(m+1)-1$ edges.

Define $f : V(H_n \odot mK_1) \to \{0, 1, 2, 3, \ldots, 2q-1 = 4n(m+1)-3\}$ as follows:

For $1 \le i \le n$,

$$f(u_i) = \begin{cases} 2i + 2m(i-1), & i \text{ is odd} \\ 2i(m+1) - 4, & i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} f(u_i) + 2n(m+1) + 2m - 4, & i \text{ is odd and } n \text{ is odd} \\ f(u_i) + 2n(m+1) - 2m + 4, & i \text{ is even and } n \text{ is odd} \\ f(u_i) + 2n(m+1), & n \text{ is even.} \end{cases}$$

For $1 \le i \le n$ and $1 \le k \le m$,

$$f(x_{i,k}) = \begin{cases} 2(m+1)(i-1) + 4k - 4, & i \text{ is odd} \\ 2(m+1)(i-2) + 4k + 2, & i \text{ is even} \end{cases}$$

$$f(y_{i,k}) = \begin{cases} f(x_{i,k}) + 2n(m+1) - 2m + 4, & i \text{ is odd, } 1 \le k \le m-1 \\ & \quad \text{and } n \text{ is odd} \\ f(x_{i,k}) + 2n(m+1) + 2m - 4, & i \text{ is even and } n \text{ is odd} \\ f(x_{i,k}) + 2n(m+1), & n \text{ is even, } 1 \le k \le m-1 \end{cases}$$

$$f(y_{n,m}) = \begin{cases} f(x_{n,m}) + 2n(m+1) - 2m + 3, & n \text{ is odd} \\ f(x_{n,m}) + 2n(m+1) - 1, & n \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows:

For $1 \leq i \leq n-1$,

$$f^*(u_iu_{i+1}) = 2i(m+1) - 1$$
$$f^*(v_iv_{i+1}) = f^*(u_iu_{i+1}) + 2n(m+1).$$

For $1 \leq i \leq n$ and $1 \leq k \leq m$,

$$f^*(u_ix_{i,k}) = 2(m+1)(i-1) + 2k - 1$$
$$f^*(v_iy_{i,k}) = f^*(u_ix_{i,k}) + 2n(m+1)$$
$$f^*\left(u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}\right) = 2n(m+1) - 1, \text{ if } n \text{ is odd}$$
$$f^*\left(u_{\frac{n}{2}+1}v_{\frac{n}{2}}\right) = 2n(m+1) - 1, \text{ if } n \text{ is even.}$$

Thus, f is an odd mean labeling. Hence the graph $H_n \odot mK_1$ is an odd mean graph for all positive integers m and n.

For example, an odd mean labeling of $H_4 \odot 5K_1$ and $H_5 \odot 4K_1$ are shown in Figure 5.

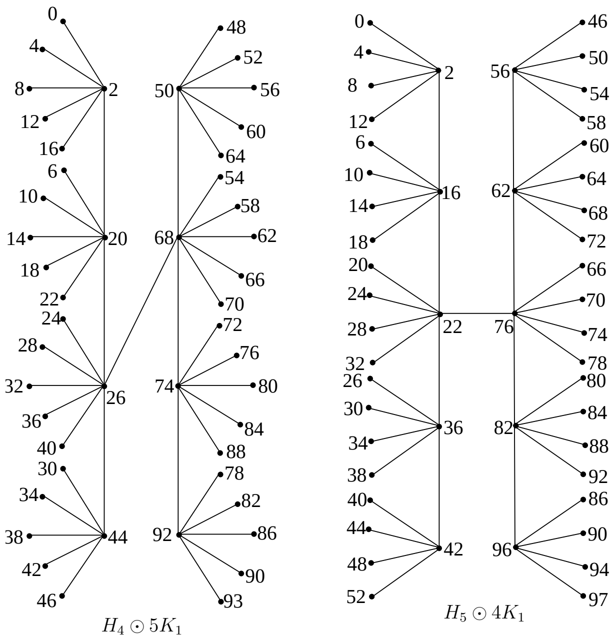

Figure 5.

□

Corollary 2.1. *For any positive integer m, the bistar graph $B(m)$ is an odd mean graph.*

Proof. By taking $n = 1$ in Theorem 2.4, the result follows. □

Theorem 2.5. *The graph mQ_3 is an odd mean graph, $m \geq 1$.*

Proof. For $1 \leq j \leq m$, let $v_1^j, v_2^j, \ldots, v_8^j$ be the vertices in the j^{th} copy of Q_3. The graph mQ_3 has $8m$ vertices and $12m$ edges.

We define $f : V(mQ_3) \to \{0, 1, 2, \ldots, 2q-1 = 24m-1\}$ as follows:

For $1 \leq j \leq m$,

$$f\left(v_i^j\right) = 24(j-1) + 2i - 2, \quad i = 1, 2, 4$$
$$f\left(v_3^j\right) = 24(j-1) + 8$$
$$f\left(v_i^j\right) = 24(j-1) + 2i + 6, \quad i = 5, 6, 8$$
$$f\left(v_7^j\right) = 24(j-1) + 23.$$

The label of the edges of the graph are $1, 3, 5, \ldots, 24m-1$. Thus, f is an odd mean labeling. Hence, the graph mQ_3 is an odd mean graph for all $m \geq 1$.

For example, an odd mean labeling of $5Q_3$ is shown in Figure 6.

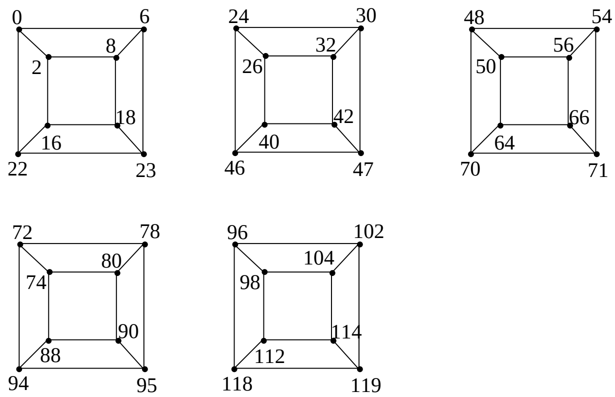

Figure 6.

□

Theorem 2.6. *For all positive integers p and n, the graph $T_p^{(n)}$ is an odd mean graph.*

Proof. Let $v_i^{(j)}$, $1 \leq i \leq p$ be the vertices of the j^{th} copy of the path on p vertices, $1 \leq j \leq n$. The graph $T_p^{(n)}$ is formed by adding an edge $v_i^{(j)} v_i^{(j+1)}$ between j^{th} and $(j+1)^{th}$ copy of the path at some i, $1 \leq i \leq p$.

Define $f : V(G) \to \{0, 1, 2, 3, \ldots, 2q-2, 2q-1 = 2np-3\}$ as follows:

For $1 \leq j \leq n-1$,

$$f\left(v_i^{(j)}\right) = \begin{cases} 2p(j-1) + 2i - 2, & 1 \leq i \leq p \text{ and } j \text{ is odd} \\ 2pj - 2i, & 1 \leq i \leq p \text{ and } j \text{ is even.} \end{cases}$$

For n is odd,

$$f\left(v_i^{(n)}\right) = \begin{cases} 2p(n-1) + 2i - 2, & 1 \leq i \leq p-1 \\ 2pn - 3, & i = p. \end{cases}$$

For n is even,

$$f\left(v_i^{(n)}\right) = \begin{cases} 2pn - 3, & i = 1 \\ 2pn - 2i, & 2 \leq i \leq p. \end{cases}$$

For the vertex labeling f, the induced edge labeling f^* is given as follows:

For $1 \leq j \leq n$ and $1 \leq i \leq p-1$,

$$f^*\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} 2p(j-1) + 2i - 1, & j \text{ is odd} \\ 2pj - 2i - 1, & j \text{ is even} \end{cases}$$
$$f^*\left(v_i^{(j)} v_i^{(j+1)}\right) = 2pj - 1.$$

Thus, f is an odd mean labeling of the graph $T_p^{(n)}$. Hence, $T_p^{(n)}$ is an odd mean graph for all positive integers p and n. For example, an odd mean labeling of $T_7^{(5)}$ and $T_6^{(4)}$ are shown in Figure 7.

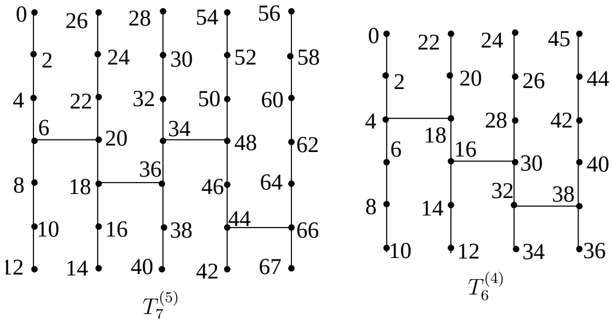

Figure 7.

□

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