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Enumeration of Homomorphisms From Modular Group into Some Finite Groups

Research Article

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Abstract: We derive general formulae for counting the number of homomorphisms from modular group into each of modular group, dihedral group, quaternion group and quasi-dihedral group by using only elementary group theory.

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1. Introduction

Enumeration of homomorphisms between two groups or rings is a basic problem in abstract algebra. For instance, in [2] and [3], this problem was settled in the case of finite cyclic groups and specific type of rings respectively by using only elementary methods. But in general counting homomorphisms between groups needs advanced tools of algebra; see, for instance [1, 5]. So in [4] Jeremiah Johnson, described a method of enumerating homomorphisms between two specified dihedral groups by using only elementary methods. Now we consider dihedral group, quaternion group, quasi-dihedral group and modular group. In [6], [7] and [8] authors give the enumeration of homomorphisms, monomorphisms and epimorphisms from each of dihedral group, quaternion group and quasi-dihedral group into each of these four groups respectively by using elementary techniques. In this paper, we consider the problem of enumerating the homomorphisms, monomorphisms and epimorphisms from a modular group into each of these four groups by using elementary methods.

We use the following notations in this paper: for a positive integer $n > 1$, the dihedral group $D_n := \langle x_n, y_n \mid x_n^n = e = y_n^2, x_n y_n = y_n x_n^{-1}\rangle$; and for a positive integer $m > 1$, the quarternion group $Q_m := \langle a_m, b_m \mid a_m^{2m} = e = b_m^4, a_m b_m = b_m a_m^{-1}\rangle$; and for a positive integer $\alpha > 3$, the quasi-dihedral group $QD_{2^\alpha} := \langle s_\alpha, t_\alpha \mid s_\alpha^{2^{\alpha-1}} = e = t_\alpha^2, t_\alpha s_\alpha = s_\alpha^{2^{\alpha-2}-1} t_\alpha\rangle$; and for a positive integer $\beta > 2$, the modular group $M_{p^\beta} := \langle r_\beta, f_\beta \mid r_\beta^{p^{\beta-1}} = e = f_\beta^p, f_\beta r_\beta = r_\beta^{p^{\beta-2}+1} f_\beta\rangle$.

2. The Number of Homomorphisms From M_{p^α} into M_{q^β}

Theorem 2.1. *Let $\alpha, \beta > 3$ be any two positive integers. Then the number of homomorphisms from M_{p^α} into M_{q^β} is* $p^3\left(\sum_{k \mid \gcd(p^{\alpha-1}, p^{\beta-1})} \phi(k)\right)$, *if $p = q$; 1, if $p \neq q$.*

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Proof. Suppose $\rho : M_{p^\alpha} \to M_{q^\beta}$ is a group homomorphism. Then $|\rho(r_\alpha)|$ must divide $|r_\alpha| = p^{\alpha-1}$ and $|\rho(f_\alpha)|$ must divide $|f_\alpha| = p$.

Suppose $p \neq q$, this is possible only when $\rho(r_\alpha) = e$ and $\rho(f_\alpha) = e$ which gives the trivial homomorphism.

Suppose $p = q$, then $\rho(r_\alpha)$ must be of the form $r_\beta^{k_1} f_\beta^{k_2}$, where $|r_\beta^{k_1}|$ divides both $p^{\alpha-1}$ and $p^{\beta-1}$ and $0 \leq k_2 < p$ and $\rho(f_\alpha)$ must be of the form $r_\beta^{m_1} f_\beta^{m_2}$, where $|r_\beta^{m_1}|$ divides p and $0 \leq m_2 < p$. Then by simple calculation, we can verify that $|\rho(r_\alpha^l f_\alpha)|$ divides $|r_\alpha^l f_\alpha|$. Since $\rho(r_\alpha)$ has $p \left(\sum_{k|\gcd(p^{\alpha-1}, p^{\beta-1})} \phi(k) \right)$ choices and $\rho(f_\alpha)$ has p^2 choices , we get the result. □

Corollary 2.1. *Let $\alpha > 2$ be any positive integer. Then the number of monomorphism from M_{p^α} into M_{q^β} is $(p^3 - p)p^{\alpha-2}$, if $p = q$ and $\alpha \leq \beta$; 0, otherwise. The number of epimorphisms from M_{p^α} onto M_{q^β} is $(p^3 - p)p^{\beta-2}$, if $p = q$ and $\alpha \geq \beta$; 0, otherwise.*

Proof. By the Theorem 2.1, the trivial homomorphism is the only homomorphism from M_{p^α} to M_{q^β},which is neither $1-1$ nor onto. So, assume that $p = q$. If $\alpha > \beta$, then there is no element in M_{p^β} having order $p^{\alpha-1}$. Thus there is no monomorphisms in this case. Now, assume that $\alpha \leq \beta$. Then the homomorphisms $\rho(r_\alpha) = r_\beta^{k_1} f_\beta^{l_1}$, where $|r_\beta^{k_1}| = p^{\alpha-1}$ and $0 \leq l_1 < p$, and $\rho(f_\alpha) = r_\beta^{k_2 p^{\beta-2}} f_\beta^{l_2}, 0 \leq k_2 < p$ and $0 \leq l_2 < p$ obtained in the Theorem 2.1, preserve the order of r_α and f_α. And also $|r_\alpha^k f_\alpha^l| = |\rho(r_\alpha^k f_\alpha^l)|$. Hence there are $p(p^2-1)\phi(p^{\alpha-1}) = (p^3 - p)p^{\alpha-2}$ number monomorphisms from M_{p^α} into M_{p^β}, if $\alpha \leq \beta$.

Suppose $\alpha < \beta$, then there is no epimorphism from M_{p^α} onto M_{p^β} . So, assume that $\alpha \geq \beta$. The the homomorphisms $\rho(r_\alpha) = r_\beta^{k_1} f_\beta^{l_1}$, where $|r_\beta^{k_1}| = p^{\beta-1}$ and $0 \leq l_1 < p$, and $\rho(f_\alpha) = r_\beta^{k_2 p^{\beta-2}} f_\beta^{l_2}, 0 \leq k_2 < p$ and $0 \leq l_2 < p$ obtained in the Theorem 2.1, are onto. Hence we get the result. □

3. The Number of Homomorphisms From M_{p^α} into D_n

Theorem 3.1. *Let $p \neq 2$ be a prime number and $\alpha > 2$ be any positive integer. Then the number of homomorphisms from M_{p^α} into D_n is $p \left(\sum_{k|\gcd(n, p^{\alpha-1})} \phi(k) \right)$, if n is a multiple of p; 1, if n is not a multiple of p.*

Proof. Suppose $\rho : M_{p^\alpha} \to D_n$ is a group homomorphism, where $p \neq 2$ and n is positive integer. Then $|\rho(r_\alpha)|$ must divide $|r_\alpha| = p^{\alpha-1}$ and $|\rho(f_\alpha)|$ must divide $|f_\alpha| = p$.

If n is not a multiple of p, this is possible only when $\rho(r_\alpha) = e$ and $\rho(f_\alpha) = e$ which gives the trivial homomorphism.

Next, we assume that n is a multiple of p. Then $|\rho(r_\alpha)|$ must be of the form $x_n^{k_1}$, where $|x_n^{k_1}|$ divides both n and $p^{\alpha-1}$ and $|\rho(f_\alpha)|$ must be of the form $x_n^{k_2}$, where $|x_n^{k_1}|$ divides p. Then $\rho(r_\alpha^l f_\alpha^m) = x_n^{lk_1 + mk_2}$. Then $|\rho(r_\alpha^l f_\alpha^m)|$ divides $|r_\alpha^l f_\alpha^m|$, for every $0 \leq l < p^{\alpha-1}$ and $0 \leq m < p$. For suppose $|r_\alpha^l| = p^\beta, \beta \geq 1$, then $(x_n^{lk_1+mk_2})^{p^\beta} = x_n^{lk_1 p^\beta + mk_2 p^\beta} = e$. That is $|x_n^{lk_1+mk_2}|$ divides p^β. Thus we have $p \left(\sum_{k|\gcd(n, p^{\alpha-1})} \phi(k) \right)$ homomorphisms. □

Theorem 3.2. *Let n be a positive odd integer and $\alpha > 3$ be any positive integer. Then the number of homomorphisms from M_{2^α} into D_n is $3n + 1$.*

Proof. Suppose $\rho : M_{2^\alpha} \to D_n$ is a group homomorphism. Then $|\rho(r_\alpha)|$ must divide $|r_\alpha| = 2^{\alpha-1}$ and $|\rho(f_\alpha)|$ must divide $|f_\alpha| = 2$. Since n is odd, $\rho(r_\alpha)$ must be either e or $x_n^{k_1} y_n$, $0 \leq k_1 < n$, and $\rho(f_\alpha)$ must be either e or $x_n^{k_2} y_n$, $0 \leq k_2 < n$.

Suppose $\rho(r_\alpha) = e$ and $\rho(f_\alpha) = x_n^{k_2} y_n$, $0 \leq k_2 < n$. Then $\rho(r_\alpha^l f_\alpha) = x_n^{k_2} y_n$. Since $|x_n^{k_2} y_n|$ is 2, $|\rho(r_\alpha^l f_\alpha)|$ divides $|r_\alpha^l f_\alpha|$. Thus we have n homomorphisms. Suppose $\rho(r_\alpha) = x_n^{k_1} y_n$, $0 \leq k_1 < n$ and $\rho(f_\alpha) = e$. Then $\rho(r_\alpha^l f_\alpha) = (x_n^{k_1} y_n)^l$. Since $|(x_n^{k_1} y_n)^l|$ is 1 or 2, $|\rho(r_\alpha^l f_\alpha)|$ divides $|r_\alpha^l f_\alpha|$. Thus we have another n homomorphisms.

Suppose $\rho(r_\alpha) = x_n^{k_1}y_n$, $0 \leq k_1 < n$ and $\rho(f_\alpha) = x_n^{k_2}y_n$, $0 \leq k_2 < n$. Then $\rho(r_\alpha^l f_\alpha) = x_n^{k_2}y_n$ or $x_n^{k_1-k_2}$. Then ρ is a homomorphism only when $|x_n^{k_1-k_2}|$ divides $|r_\alpha^l f_\alpha|$. Since n is odd, this is possible only when $k_1 - k_2 = 0$. Thus we have n such homomorphisms. Therefore, in addition to the trivial homomorphism, we have $3n + 1$ homomorphisms. □

Theorem 3.3. *Let n be a positive even integer and $\alpha > 3$ be any positive integer. Then the number of homomorphisms from M_{2^α} into D_n is* $4n + (n+2)\left(\sum\limits_{k|\gcd(n,2^{\alpha-1})} \phi(k)\right)$.

Proof. Suppose $\rho : M_{2^\alpha} \to D_n$ is a group homomorphism. Then $\rho(r_\alpha)$ must be either $x_n^{k_1}y_n$, $0 \leq k_1 < n$ or $x_n^{k_2}$, where $|x_n^{k_2}|$ divides both n and $2^{\alpha-1}$, and $\rho(f_\alpha)$ must be one of e, $x_n^{\frac{n}{2}}$ or $x_n^{k_3}y_n$, $0 \leq k_3 < n$.
As in the proof of the Theorem 3.2, $\rho(r_\alpha) = x_n^{k_1}y_n$, $0 \leq k_1 < n$ and $\rho(f_\alpha) = e$ is a homomorphism. Suppose $\rho(f_\alpha) = x_n^{\frac{n}{2}}$, then $\rho(r_\alpha^l f_\alpha) = x_n^{k_1-\frac{n}{2}}y_n$ or $x_n^{\frac{n}{2}}$. Then $|\rho(r_\alpha^l f_\alpha)|$ divides $|r_\alpha^l f_\alpha|$. Thus we have $2n$ homomorphisms.
Suppose $\rho(r_\alpha) = x_n^{k_1}y_n$, $0 \leq k_1 < n$ and $\rho(f_\alpha) = x_n^{k_3}y_n$, $0 \leq k_3 < n$. As in the proof of Theorem 3.2, ρ is a homomorphism only when $|x_n^{k_1-k_3}|$ divides $|r_\alpha^l f_\alpha|$. This is possible if $k_1 - k_2 = 0$ or $\frac{n}{2}$. Thus we have another $2n$ homomorphisms in this case.
Suppose $\rho(r_\alpha) = x_n^{k_2}$, where $|x_n^{k_2}|$ divides both n and $2^{\alpha-1}$, and $\rho(f_\alpha) = e$. Then $\rho(r_\alpha^l f_\alpha) = x_n^{lk_2}$. Since $|x_n^{k_2}|$ divides $|r_\alpha|$, $|x_n^{lk_2}|$ divides $|r_\alpha^l| = |r_\alpha^l f_\alpha|$. Thus we have $\left(\sum\limits_{k|\gcd(n,2^{\alpha-1})} \phi(k)\right)$ homomorphisms. Suppose $\rho(r_\alpha) = x_n^{k_2}$, where $|x_n^{k_2}|$ divides both n and $2^{\alpha-1}$, and $\rho(f_\alpha) = x_n^{\frac{n}{2}}$, then $\rho(r_\alpha^l f_\alpha) = x_n^{lk_2+\frac{n}{2}}$. This gives another $\left(\sum\limits_{k|\gcd(n,2^{\alpha-1})} \phi(k)\right)$ homomorphisms.
Suppose $\rho(r_\alpha) = x_n^{k_2}$, where $|x_n^{k_2}|$ divides both n and $2^{\alpha-1}$, and $\rho(f_\alpha) = x_n^{k_3}y_n$, $0 \leq k_3 < n$. Then $\rho(r_\alpha^l f_\alpha) = x_n^{lk_2+k_3}$. Since $|\rho(r_\alpha^l f_\alpha)|$ divides $|r_\alpha^l f_\alpha|$, we have $n\left(\sum\limits_{k|\gcd(n,2^{\alpha-1})} \phi(k)\right)$ homomorphisms. Hence we get the result. □

Corollary 3.1. *Let n be a positive even integer and $\alpha > 2$ be any positive integer. Then there is no monomorphism and epimorphism from M_{p^α} into D_n.*

Proof. Suppose $p \neq 2$, then the group M_{p^α} contains $p(p-1)$ elements having order p but D_n contains atmost $p-1$ elements having order p. And M_{2^α} contains 4 elements having order 4 but D_n contains atmost 2 elements having order 4. Thus there is no monomorphism from M_{p^α} into D_n. Also we can verify that the homomorphisms obtained in the Theorem 3.3, are not onto. □

4. The Number of Homomorphisms From M_{p^α} into Q_m

Theorem 4.1. *Let $p \neq 2$ be a prime number and $\alpha > 2$ be any positive integer. Then the number of homomorphisms from M_{p^α} into Q_m is* $p\left(\sum\limits_{k|\gcd(n,p^{\alpha-1})} \phi(k)\right)$*, if n is a multiple of p; 1, if n is not a multiple of p.*

Proof. The proof is similar to the Theorem 3.1 □

Theorem 4.2. *Let m be a positive integer and $\alpha > 2$. Then the number of homomorphisms from M_{2^α} into Q_m is* $4m + 2\left(\sum\limits_{k|\gcd(2m,2^{\alpha-1})} \phi(k)\right)$

Proof. Suppose $\rho : M_{2^\alpha} \to Q_m$ is a group homomorphism. Then $|\rho(r_\alpha)|$ divides $|r_\alpha| = 2^{\alpha-1}$ and $|\rho(f_\alpha)|$ divides $|f_\alpha| = 2$. Then $\rho(r_\alpha)$ is either $a_m^{k_1}$, where $|a_m^{k_1}|$ divides both $2m$ and $2^{\alpha-1}$, or $a_m^{k_2}b_m$, $0 \leq k_2 < 2m$ and $\rho(f_\alpha) = e$ or a_m^m.

Assume that $\rho(r_\alpha) = a_m^{k_1}$, where $|a_m^{k_1}|$ divides both $2m$ and $2^{\alpha-1}$. Suppose $\rho(f_\alpha) = e$. Then $\rho(r_\alpha^l f_\alpha) = a_m^{lk_1}$. Since $|a_m^{k_1}|$ divides $|r_\alpha|$, $|a_m^{lk_1}|$ divides $|r_\alpha^l| = |r_\alpha^l f_\alpha|$. Thus we have $\left(\sum_{k|\gcd(2m,2^{\alpha-1})} \phi(k)\right)$ homomorphisms. Suppose $\rho(f_\alpha) = a_m^m$, then $\rho(r_\alpha^{l_1} f_\alpha) = a_m^{l_1 k+m}, 0 \le l_1 < 2^{\alpha-1}$. Thus we have $\left(\sum_{k|\gcd(2m,2^{\alpha-1})} \phi(k)\right)$ homomorphisms.

Suppose $\rho(r_\alpha) = a_m^{k_2} b_m$, $0 \le k_2 < 2m$ and $\rho(f_\alpha) = e$, then $\rho(r_\alpha^l f_\alpha) = (a_m^{k_2} b_m)^l$. Since $|(a_m^{k_2} b_m)^l| = 1, 2$ or 4, then $|\rho(r_\alpha^l f_\alpha)|$ divides $|r_\alpha^l f_\alpha|$, for each $l, 0 \le l < 2^{\alpha-1}$. Suppose $\rho(f_\alpha) = a_m^m$, then $\rho(r_\alpha^l f_\alpha) = (a_m^{k_2} b_m)^l a_m^m$ is one of a_m^m, $a_m^{k_2-m} b_m$, e or $a_m^{k_2} b_m$. Then $|\rho(r_\alpha^l f_\alpha)|$ divides $|r_\alpha^l f_\alpha|$. Thus we have $4m$ homomorphisms in this case. Hence we get the result. □

Corollary 4.1. *Let m be a positive integer and $\alpha > 2$. Then there is no monomorphism and epimorphisms from M_{p^α} into Q_m.*

Proof. Ifp $\neq 2$, then the trivial homomorphism is the only homomorphism from M_{p^α} into Q_m, which is neither $1-1$ nor onto. So, assume that $p = 2$. Since M_{p^α} has 2 elements of order 2. But Q_m have only one element of order 2. Thus there is no monomorphism from M_{2^α} into Q_m.

Also, we can verify that none of the homomorphisms obtained in the proof of Theorem 4.2 generate all the elements of Q_m. Hence there is no epimorphism from M_{p^α} onto Q_m. □

5. The Number of Homomorphisms From M_{p^β} into QD_{2^α}

Theorem 5.1. *Let $p \neq 2$ be a prime number. Then there is only the trivial homomorphism from M_{p^β} into QD_{2^α}.*

Proof. Suppose $\rho : M_{p^\beta} \to QD_{2^\alpha}$ is a group homomorphism. Then $|\rho(r_\beta)|$ divides $|r_\beta| = p^{\beta-1}$ and $|\rho(f_\beta)|$ divides $|f_\beta| = p$. That is the trivial homomorphism is the only homomorphism exist from M_{p^β}, $p \neq 2$ into QD_{2^α}. □

Theorem 5.2. *Suppose $\alpha, \beta > 3$ are two positive integers. Then the number of homomorphisms from M_{2^β} into QD_{2^α} is* $2^\alpha + (2 + 2^{\alpha-1})\left(\sum_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$

Proof. Suppose $\rho : M_{2^\beta} \to QD_{2^\alpha}$ is a group homomorphism. Then $|\rho(r_\beta)|$ divides $|r_\beta| = 2^{\beta-1}$ and $|\rho(f_\beta)|$ divides $|f_\beta| = 2$. That is, $\rho(r_\beta)$ is either s_α^k, where $|s_\alpha^k|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$ or $\rho(r_\beta) = s_\alpha^{k_1} t_\alpha, 0 \le k_1 < 2^{\alpha-1}$, and $\rho(f_\beta)$ is one of e, $s_\alpha^{2^{\alpha-2}}$ or $s_\alpha^m t_\alpha, 0 \le m < 2^{\alpha-1}$ where m is even.

Suppose $\rho(r_\beta) = s_\alpha^k$, where $|s_\alpha^k|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$. First assume that $\rho(f_\beta) = e$. Then $\rho(r_\beta^l f_\beta) = s_\alpha^{lk}$. Since $|s_\alpha^k|$ divides $|\rho(r_\beta)|$, $|s_\alpha^k|$ divides $|r_\beta^l| = |r_\beta^l f_\beta|$. Thus in the case, we have $\left(\sum_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$ homomorphisms. Now, suppose $\rho(f_\beta) = s_\alpha^{2^{\alpha-2}}$. Then $\rho(r_\beta^l f_\beta) = s_\alpha^{lk+2^{\alpha-2}}$ for each k, this ρ is a homomorphism. Thus we have another $\left(\sum_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$ homomorphisms.

Suppose $\rho(r_\beta) = s_\alpha^k$, where $|s_\alpha^k|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$, and $\rho(f_\beta) = s_\alpha^m t_\alpha, 0 \le m < 2^{\alpha-1}$ and m is even. Then $\rho(r_\beta^l f_\beta) = s_\alpha^{lk+m} t_\alpha$. Since $|s_\alpha^{lk+m} t_\alpha| = 2$ or 4, $|\rho(r_\beta^l f_\beta)|$ divides $|r_\beta^l f_\beta|$. Thus we have $2^{\alpha-2}\left(\sum_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$ homomorphisms.

Suppose $\rho(r_\beta) = s_\alpha^{k_1} t_\alpha, 0 \le k_1 < 2^{\alpha-1}$ and $\rho(f_\beta) = e$, then $\rho(r_\beta^l f_\beta) = (s_\alpha^{k_1} t_\alpha)^l$. Since $|(s_\alpha^{k_1} t_\alpha)^l| = 1$ or 2 or 4 which divides $|\rho(r_\beta^l f_\beta)| = |r_\beta^l|$. Thus we have $2^{\alpha-1}$ homomorphisms.

Similarly $\rho(r_\beta) = s_\alpha^{k_1} t_\alpha, 0 \le k_1 < 2^{\alpha-1}$ and $\rho(f_\beta) = s_\alpha^{2^{\alpha-2}}$ are also homomorphisms. Thus we have another $2^{\alpha-1}$ homomorphisms.

Suppose $\rho(r_\beta) = s_\alpha^{k_1} t_\alpha, 0 \le k_1 < 2^{\alpha-1}$ and $\rho(f_\beta) = s_\alpha^{k_2} t_\alpha, 0 \le k_2 < 2^{\alpha-1}$ and k_2 is even. Then $\rho(r_\beta^l f_\beta) = (s_\alpha^{k_1} t_\alpha)^l (s_\alpha^{k_2} t_\alpha)$. If l is even, $\rho(r_\beta^l f_\beta) = s_\alpha^{k_1 2^{\alpha-2}+k_2} t_\alpha$ or $s_\alpha^{k_2} t_\alpha$. That is $|\rho(r_\beta^l f_\beta)|$ divides $|r_\beta^l f_\beta|$. If l is odd, $|r_\beta^l f_\beta| = 2^{\beta-1}$ and $\rho(r_\beta^l f_\beta) =$

$s_\alpha^{k_1+k_2(2^{\alpha-2}-1)}$ or $s_\alpha^{k_1 2^{\alpha-2}+k_1} s_\alpha^{k_2(2^{\alpha-2}-1)}$. If ρ is a homomorphism, then $|s_\alpha^{k_1-k_2}|$ divides $2^{\beta-1}$. That is, for each k_2, we have to choose k_1 such that $|s_\alpha^{k_1-k_2}|$ divides $2^{\beta-1}$. Therefore, for each k_2, we have $\left(\sum\limits_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$ choices for k_1. Thus we have $2^{\alpha-2}\left(\sum\limits_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$ homomorphisms. Hence we get the result. □

Corollary 5.1. *Let p be a prime number and $\alpha, \beta > 3$. Then there is no monomorphism and epimorphism from M_{p^β} into QD_{2^α}.*

Proof. Suppose $p \neq 2$, then by the Theorem 5.1, the trivial homomorphism is the only homomorphism from M_{p^β} into QD_{2^α}, which is not $1-1$. So, assume that $p = 2$. If $\beta > \alpha$, then there is no element in QD_{2^α} having order $2^{\beta-1}$, thus we have no monomorphism from M_{2^β} into QD_{2^α}. Suppose $\beta \leq \alpha$, $\rho(r_\beta) = s_\alpha^k$, where $|s_\alpha^k| = 2^{\beta-1}$ and $\rho(f_\beta) = s_\alpha^m t_\alpha$, $0 \leq m < 2^{\alpha-1}$ is the homomorphism which preserve the order of r_β and f_β. Then $\rho(r_\beta f_\beta) = s_\alpha^{k+m} t_\alpha$. $|r_\beta f_\beta| = 2^{\alpha-1}$ but $|s_\alpha^{k+m} t_\alpha| = 2$ or 4. Thus this ρ is not a monomorphism. Also, we can verify that the homomorphisms obtained in the Theorem 5.2 are not onto. □

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