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Weak Continuity via Topological Grills

Research Article

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Abstract: The aim of this paper is to introduce and characterize a new class of functions called weakly $\mathcal{G}$-precontinuous functions in ideal topological spaces by using $\mathcal{G}$-preopen sets.

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1. Introduction

The idea of grills on a topological space was first introduced by Choquet [4]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [2], [3], [13] for details). In [10], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [5] have defined new classes of sets in a grill topological space and obtained a new decomposition of continuity in terms of grills. The aim of this paper is to introduce and characterize a new class of functions called weakly $\mathcal{G}$-precontinuous functions in grill topological spaces by using $\mathcal{G}$-preopen sets.

2. Preliminaries

Let A be a subset of a topological space (X,τ). We denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of X is said to be regular open [11] if $A = \mathrm{Int}(\mathrm{Cl}(A))$. A point x of X is called a θ-cluster [12] point of A if $\mathrm{Cl}(U) \cap A \neq \varnothing$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure [12] of A and is denoted by $\mathrm{Cl}_\theta(A)$. A subset A is said to be θ-closed [12] if $\mathrm{Cl}_\theta(A) = A$. The complement of θ-closed set is called θ-open. The definition of grill on a topological space, as given by Choquet [4], goes as follows: A non-null collection $\mathcal{G}$ of subsets of a topological space (X,τ) is said to be a grill on X if

1. $\emptyset \notin \mathcal{G}$,

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2. $A \in \mathcal{G}$ and $A \subset B$ implies that $B \in \mathcal{G}$,

3. $A, B \subset X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 2.1 ([10])**.** *Let (X, τ) be a topological space and $\mathcal{G}$ a grill on X. A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G}$ for every open set U containing $x\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ.*

Definition 2.2 ([10])**.** *Let $\mathcal{G}$ be a grill on a topological space (X, τ). Then we define a map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ is a Kuratowski closure axiom. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X \backslash U) = X \backslash U\}$, where for any $A \subset X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}} \operatorname{Cl}(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subset \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X, then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.*

Definition 2.3 ([5])**.** *A subset S of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-preopen if $S \subset \operatorname{Int}(\Psi(S))$. The complement of a $\mathcal{G}$-preopen set is called a $\mathcal{G}$-preclosed set.*

Definition 2.4. *The intersection of all $\mathcal{G}$-preclosed sets containing $S \subset X$ is called the $\mathcal{G}$-preclosure of S and is denoted by $\operatorname{p}\operatorname{Cl}_{\mathcal{G}}(S)$. The family of all $\mathcal{G}$-preopen (resp. $\mathcal{G}$-preclosed) sets of $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}PO(X)$ (resp. $\mathcal{G}PC(X)$). The family of all $\mathcal{G}$-preopen (resp. $\mathcal{G}$-preclosed) sets of $(X, \tau, \mathcal{G})$ containing a point $x \in X$ is denoted by $\mathcal{G}PO(X, x)$ (resp. $\mathcal{G}PC(X, x)$).*

Definition 2.5. *A subset B_x of a topological space $(X, \tau, \mathcal{G})$ is said to be a $\mathcal{G}$-preneighbourhood of a point $x \in X$ if there exists a $\mathcal{G}$-preopen set U such that $x \in U \subset B_x$.*

Definition 2.6 ([5])**.** *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be:*

(i) $\mathcal{G}$-precontinuous at a point $x \in X$ if for each open subset V in Y containing $f(x)$, there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset V$;

(ii) $\mathcal{G}$-precontinuous if it has this property at each point of X.

Definition 2.7 ([8])**.** *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be:*

(i) almost $\mathcal{G}$-precontinuous at a point $x \in X$ if for each open subset V in Y containing $f(x)$, there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \operatorname{Int}(\operatorname{Cl}(V))$;

(ii) almost $\mathcal{G}$-precontinuous if it has this property at each point of X.

Definition 2.8 ([6])**.** *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be faintly $\mathcal{G}$-precontinuous if for each $x \in X$ and for each θ-open set V of Y containing $f(x)$, then there exist $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset V$.*

Theorem 2.1 ([6])**.** *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is faintly $\mathcal{G}$-precontinuous if and only if the inverse image of every (resp. θ-open) θ-closed subset of (Y, σ) is (resp. $\mathcal{G}$-preopen) $\mathcal{G}$-preclosed in $(X, \tau, \mathcal{G})$.*

Definition 2.9 ([7])**.** *A grill topological space $(X, \tau, \mathcal{G})$ is said to be:*

(i) $\mathcal{G}$-pre-T_1 if for each pair of distinct points x and y of X, there exists $\mathcal{G}$-preopen sets and U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

(ii) $\mathcal{G}$-pre-T_2 if for each pair of distinct points x and y of X, there exists $\mathcal{G}$-preopen sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

3. Weakly $\mathcal{G}$-precontinuous Functions

Definition 3.1. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be weakly $\mathcal{G}$-precontinuous if for each $x \in X$ and each open set V of Y containing $f(x)$ there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$.*

Theorem 3.1. *If a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is almost $\mathcal{G}$-precontinuous, then it is weakly $\mathcal{G}$-precontinuous.*

Proof. Let $x \in X$ and $V \subset Y$ be an open set with $f(x) \in V$. Then since $f(x) \in V \subset \mathrm{Cl}(V)$, $f(x) \in \mathrm{Int}(\mathrm{Cl}(V))$, which is regular open. Since f is almost $\mathcal{G}$-precontinuous, there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(V)$. Therefore, f is weakly $\mathcal{G}$-precontinuous. □

Remark 3.1. *The converse of Theorem 3.1 is not true in general as can be seen from the following example.*

Example 3.1. *Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{c\}, \{b, c\}, \{a, c\}, X\}$, $\sigma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{G} = \mathcal{P}(X)\backslash\{\varnothing, \{a\}\}$. Then the identity function $f : (X, \tau, \mathcal{G}) \to (X, \sigma)$ is weakly $\mathcal{G}$-precontinuous but not almost $\mathcal{G}$-precontinuous.*

Corollary 3.1. *Every $\mathcal{G}$-precontinuous function is weakly $\mathcal{G}$-precontinuous.*

Theorem 3.2. *If a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is weakly $\mathcal{G}$-precontinuous, then it is faintly $\mathcal{G}$-precontinuous.*

Proof. Follows from the definitions. □

Remark 3.2. *The converse of Theorem 3.2 is not true in general as can be seen from the following example.*

Example 3.2. *Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, X\}$, $\sigma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{G} = \mathcal{P}(X)\backslash\{\varnothing, \{a\}\}$. Then the identity function $f : (X, \tau, \mathcal{G}) \to (X, \sigma)$ is faintly $\mathcal{G}$-precontinuous but not weakly $\mathcal{G}$-precontinuous.*

Theorem 3.3. *For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following statements are equivalent:*

(i) f is weakly $\mathcal{G}$-precontinuous;

(ii) $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))) \subset f^{-1}(\mathrm{Cl}_\theta(A))$ for every subset A of Y;

(iii) $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(B)))) \subset f^{-1}(\mathrm{Cl}(B))$ for every open set B of Y;

(iv) $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(C))) \subset f^{-1}(C)$ for every regular closed set C of Y ;

(v) $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(D)) \subset f^{-1}(\mathrm{Cl}(D))$ for every open set D of Y;

(vi) $f^{-1}(E) \subset p\,\mathrm{Int}_{\mathcal{G}}(f^{-1}(\mathrm{Cl}(E)))$ for every open set E of Y.

Proof. (i)⇒(ii): Let A be a subset of Y and $x \in X\backslash f^{-1}(\mathrm{Cl}_\theta(A))$. Then $x \notin f^{-1}(\mathrm{Cl}_\theta(A))$, that is, $f(x) \notin \mathrm{Cl}_\theta(A)$. This means that the existence of an open set W of Y containing $f(x)$ such that $A \cap \mathrm{Cl}(W) = \varnothing$. Hence $\mathrm{Cl}_\theta(A) \cap W = \varnothing$. So, $W \subset Y\backslash \mathrm{Cl}_\theta(A)$, that is, $\mathrm{Cl}(W) \subset \mathrm{Cl}(Y\backslash \mathrm{Cl}_\theta(A))$. Since f is weakly $\mathcal{G}$-precontinuous, there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \mathrm{Cl}(W) \subset \mathrm{Cl}(Y\backslash \mathrm{Cl}_\theta(A))$. So $f(U) \cap (Y\backslash \mathrm{Cl}(Y\backslash \mathrm{Cl}_\theta(A))) = \varnothing$. Then $f(U) \cap \mathrm{Int}(\mathrm{Cl}_\theta(A)) = \varnothing$ and hence $U \cap f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(A))) = \varnothing$. This shows that $x \notin p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(A))))$. Therefore, $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))) \subset f^{-1}(\mathrm{Cl}_\theta(A))$.
(ii)⇒ (iii): This implication is follows from the fact that, $\mathrm{Cl}_\theta(A)$=$\mathrm{Cl}(A)$ for every open set B of Y.
(iii)⇒(iv): Let C be a regular closed subset of Y. Then $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(C))) = p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(C))))) \subset f^{-1}(\mathrm{Cl}(\mathrm{Int}(C)))$ $= f^{-1}(C)$.
(iv)⇒(v): Let D be an open subset of Y. Then $\mathrm{Cl}(D)$ is regular closed in Y. So, $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(D)) = p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(D))) \subset p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(D)))) \subset f^{-1}(\mathrm{Cl}(D))$, by (iv).

(v)⇒(vi): Let $x \in f^{-1}(E)$. Then $f(x) \in E$ and since $E \cap (Y \setminus \mathrm{Cl}(E)) = \varnothing$, $f(x) \notin \mathrm{Cl}(Y \setminus \mathrm{Cl}(E))$ where $x \notin f^{-1}(\mathrm{Cl}(Y \setminus \mathrm{Cl}(E)))$. Openness of $(Y \setminus \mathrm{Cl}(E))$ gives from (v) that $x \notin p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(Y \setminus \mathrm{Cl}(E)))$. This implies the existence of $U \in BO(X, x)$ such that $U \cap f^{-1}(Y \setminus \mathrm{Cl}(E)) = \varnothing$; that is, $f(U) \cap (Y \setminus \mathrm{Cl}(E)) = \varnothing$. Which assures that $f(U) \subset \mathrm{Cl}(E)$ and hence $U \subset f^{-1}(\mathrm{Cl}(E))$. Thus $x \in U \subset f^{-1}(\mathrm{Cl}(E))$ and this indicates that x is a $\mathcal{G}$-preinterior point of $f^{-1}(\mathrm{Cl}(E))$. Consequently, $f^{-1}(E) \subset p\,\mathrm{Int}_{\mathcal{G}}(f^{-1}(\mathrm{Cl}(E)))$.

(vi)⇒(i): Let $x \in X$ and V be an open subset of Y containing $f(x)$ by (vi), $x \in f^{-1}(V) \subseteq p\,\mathrm{Int}_{\mathcal{G}}(f^{-1}(\mathrm{Cl}(V)))$. Let $U = p\,\mathrm{Int}_{\mathcal{G}}(f^{-1}(\mathrm{Cl}(V)))$. Then $U \in \mathcal{G}PO(X, x)$. Now, $f(U) = f(p\,\mathrm{Int}_{\mathcal{G}}(f^{-1}(\mathrm{Cl}(V)))) \subseteq f(f^{-1}(\mathrm{Cl}(V))) \subset \mathrm{Cl}(U)$. This shows that f is weakly $\mathcal{G}$-precontinuous.

□

Theorem 3.4. *The following statements are equivalent for a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$:*

(i) f is weakly $\mathcal{G}$-precontinuous;

(ii) $f(p\,\mathrm{Cl}_{\mathcal{G}}(A)) \subset \mathrm{Cl}_{\theta}(f(A))$ for each subset A of X;

(iii) $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_{\theta}(B))$ for each subset B of Y;

(iv) $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(\mathrm{Cl}_{\theta}(B)))) \subset f^{-1}(\mathrm{Cl}_{\theta}(B))$ for every subset B of Y.

Proof. (i)⇒(ii): Let A be any subset of X and $x \in p\,\mathrm{Cl}_{\mathcal{G}}(A)$. Then $f(x) \in f(p\,\mathrm{Cl}_{\mathcal{G}}(A))$ Suppose that V be an open set of Y containing $f(x)$. Then there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Since $x \in p\,\mathrm{Cl}_{\mathcal{G}}(A)$, $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U) \cap f(A) \subset \mathrm{Cl}(V) \cap f(A)$. Therefore, we have $f(x) \in \mathrm{Cl}_{\theta}(f(A))$ and hence $f(p\,\mathrm{Cl}_{\mathcal{G}}(A)) \subset \mathrm{Cl}_{\theta}(f(A))$.

(ii)⇒(iii): Let B be any subset of Y. We have $f(p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(B))) \subset \mathrm{Cl}_{\theta}(B)$ and hence $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_{\theta}(B))$.

(iii)⇒(iv): Let B be any subset of Y. Since $\mathrm{Cl}_{\theta}(B)$ is closed in Y we have $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(\mathrm{Int}(\mathrm{Cl}_{\theta}(B)))) \subset f^{-1}(\mathrm{Cl}_{\theta}(B)) = f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_{\theta}(B))))\ f^{-1}(\mathrm{Cl}_{\theta}(B))$.

(iv)⇒(i): Let V be any open subset of Y. Then $V \subset \mathrm{Int}(\mathrm{Cl}(V)) = \mathrm{Int}(\mathrm{Cl}_{\theta}(V))$. Then $p\,\mathrm{Cl}_{\mathcal{G}}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$. It follows from Theorem 3.3 that is weakly $\mathcal{G}$-precontinuous. □

Theorem 3.5. *Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ be a function and Y be regular. Then the following statements are equivalent:*

(i) f is $\mathcal{G}$-precontinuous;

(ii) f is weakly $\mathcal{G}$-precontinuous;

(iii) f is faintly $\mathcal{G}$-precontinuous.

Definition 3.2. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be weakly continuous [9] if for each $x \in X$ and an open set V in Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset \mathrm{Cl}(V)$.*

Theorem 3.6. *If $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is $\mathcal{G}$-precontinuous and $g : (Y, \sigma) \to (Z, \eta)$ is weakly continuous, then the composition $g \circ f : (X, \tau, \mathcal{G}) \to (Z, \eta)$ is weakly $\mathcal{G}$-precontinuous.*

Proof. Let $x \in X$ and W be an open subset of Z containing $g(f(x))$. Since g is weakly continuous, then there exists an open set V of Y containing $f(x)$ such that $g(V) \subset \mathrm{Cl}(W)$. Again since f is $\mathcal{G}$-precontinuous, there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset V$. Then $g \circ f(U) \subset g(V) \subset \mathrm{Cl}(W)$. This shows that $g \circ f : (X, \tau, \mathcal{G}) \to (Z, \eta)$ is weakly $\mathcal{G}$-precontinuous. □

Theorem 3.7. *If $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is weakly $\mathcal{G}$-precontinuous and $g : (Y, \sigma) \to (Z, \eta)$ is continuous, then the composition $g \circ f : (X, \tau, \mathcal{G}) \to (Z, \eta)$ is weakly $\mathcal{G}$-precontinuous.*

Proof. Let $x \in X$ and W be an open subset of Z containing $g(f(x))$ then $g^{-1}(W)$ is an open set Y containing $f(x)$ and there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \mathrm{Cl}(g^{-1}(W))$. Since g is continuous, we obtain $(g \circ f)(U) \subset g(\mathrm{Cl}(g^{-1}(W)))$ $\subset \mathrm{Cl}(W)$. Thus, $g \circ f$ is weakly $\mathcal{G}$-precontinuous. □

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\}$ of $X \times Y$ is called the graph of f and is denoted by $G(f)$.

Theorem 3.8. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is weakly $\mathcal{G}$-precontinuous if and only if the graph function $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ define by $g(x) = (x, f(x))$ if weakly $\mathcal{G}$-precontinuous at every $x \in X$.*

Proof. Suppose f is weakly $\mathcal{G}$-precontinuous. Let $x \in X$ and W be an open subset of the product space $X \times Y$ containing $g(x)$. Then there exist $U_1 \in \tau$ and $V \in \sigma$ such that $(x, f(x)) \in U_1 \times V \subset W$. Since f is weakly $\mathcal{G}$-precontinuous, there exist $U_2 \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Let $U = U_1 \cap U_2$. Clearly $U \in \mathcal{G}PO(X, x)$ and hence $f(U) \subset f(U_2) \subset \mathrm{Cl}(V)$. Now we observe that $g(U) \subset U \times \mathrm{Cl}(V) \subset U_1 \times \mathrm{Cl}(V) \subset \mathrm{Cl}(U_1 \times V) \subset \mathrm{Cl}(W)$. This shows that g is weakly $\mathcal{G}$-precontinuous. Conversely, Suppose g is weakly $\mathcal{G}$-precontinuous. Let $x \in X$ and $f(x) \in V \in \sigma$. Then $g(x) \in X \times V \in \tau \times \sigma$ and there exists $U \in \mathcal{G}PO(X, x)$ such that $g(U) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. Hence we obtain $f(U) \subset \mathrm{Cl}(V)$, which shows that f is quasi $\mathcal{G}$-precontinuous at x. □

Recall that a topological space X is said to be rim-compact if for each point of X has a base of neighbourhoods with compact frontiers.

Theorem 3.9. *If Y is a rim-compact space and $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is weakly $\mathcal{G}$-precontinuous function with the closed graph, then f is $\mathcal{G}$-precontinuous.*

Proof. Let $x \in A$ and V be an open subset of Y containing $f(x)$. Since Y is a rim-compact, there exists an open set W such that $f(x) \in W \subset V$ and the frontier Fr(W) is compact. It is obvious that $f(x) \notin Fr(W)$. Thus, for each $y \in Fr(W)$, we have $(x, y) \notin G(f)$. Since $G(f)$ is closed, there exist open sets $U_y(x) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $f(U_y(x)) \cap V(y) = \emptyset$. The family $\{V(y); y \in Fr(W)\}$ is a cover of Fr(W) by open sets of Y. Since Fr(W) is compact, there exists a finite number of points y_1, y_2,.......y_n in Fr(W) such that $Fr(W) \subset \{V(y_i) : i = 1, 2, ...n\}$. Since f is weakly $\mathcal{G}$-precontinuous, there exists $U_0 \in \mathcal{G}PO(X, x)$ such that $f(U_0) \subset \mathrm{Cl}(W)$. Put $U = U_0 \cap (\cap\{U_{yi}(x) : i = 1, 2, ...n\})$, then by Remark 1 of [1] we have $U \in \mathcal{G}PO(X, x)$ and $f(U) \cap (Y \backslash W) = \emptyset$. This shows that $f(U) \subset V$ and hence f is weakly $\mathcal{G}$-precontinuous. □

Definition 3.3. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be weakly $\mathcal{G}$-preclosed if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \mathcal{G}PO(X, x)$ and an open set V of Y containing y such that $(U \times \mathrm{Cl}(V)) \cap G(f) = \varnothing$.*

Lemma 3.1. *The graph $G(f)$ of $f : (X, \tau) \to (Y, \sigma)$ is weakly $\mathcal{G}$-preclosed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \mathcal{G}PO(X, x)$ and an open set V of Y containing y such that $f(U) \cap \mathrm{Cl}(V) = \varnothing$.*

Proof. It follows immediately from the Definition 3.3. □

Theorem 3.10. *If $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is weakly $\mathcal{G}$-precontinuous and Y is a Urysohn space, then the graph $G(f)$ of f is weakly $\mathcal{G}$-preclosed in $X \times Y$.*

Proof. Let $(x, y) \notin G(f)$, then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V_1 and V_2 of Y containing $f(x)$ and y, respectively, such that $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. Since f is weakly $\mathcal{G}$-precontinuous, there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \mathrm{Cl}(V_1)$ and consequently $f(U) \cap \mathrm{Cl}(V_2) = \varnothing$. This shows that f is weakly $\mathcal{G}$-preclosed in $X \times Y$. □

Definition 3.4. *A grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-preconnected if it is not the union of two nonempty disjoint $\mathcal{G}$-preopen sets.*

Theorem 3.11. *If $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-preconnected space and $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is a weakly $\mathcal{G}$-precontinuous function with the weakly $\mathcal{G}$-preclosed, then f is constant.*

Proof. Sppose that f is not constant. There exist disjoint points $x, y \in X$ such that $f(x) = f(y)$. Since $(x, f(f)) \notin G(f)$, by Lemma 3.1 of there exists open sets U and V containing x and $f(x)$, respetively, such that $f(U) \cap \mathrm{Cl}(V) = \emptyset$. Since f is weakly $\mathcal{G}$-precontinuous, there exists $G \in \mathcal{G}PO(X, y)$ such that $f(G) \subset V$. Since U and V are disjoint $\mathcal{G}$-preopen sets of $(X, \tau, \mathcal{G})$. It follows that $(X, \tau, \mathcal{G})$ is not $\mathcal{G}$-preconnected. Therefore, f is constant. □

Theorem 3.12. *Let $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ be a weakly $\mathcal{G}$-precontinuous injective function. If (Y, σ) is Urysohn, then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-preT_2.*

Proof. Since f is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Since (Y, σ) is Urysohn, there exist $V_1, V_2 \in \sigma$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. This gives $f^{-1}(\mathrm{Cl}(V_1)) \cap f^{-1}(\mathrm{Cl}(V_2))) = \varnothing$. Since f is weakly $\mathcal{G}$-precontinuous $x_i \in f^{-1}(V_i) \subset p\,\mathrm{Int}_{\mathcal{G}}(f^{-1}(\mathrm{Cl}(V_i))$, $i = 1,2$. By Theorem 3.3 and this indicates that $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-preT_2. □

Definition 3.5. *If $A \subset X$, then a function $f : (X, \tau, \mathcal{G}) \to A$ is termed as weakly $\mathcal{G}$-precontinuous retraction if f is weakly $\mathcal{G}$-precontinuous and $f_{|A}$ is the identity function.*

Theorem 3.13. *Let $A \subset X$ and $f : (X, \tau, \mathcal{G}) \to A$ be a weakly $\mathcal{G}$-precontinuous retraction. If $(X, \tau, \mathcal{G})$ is T_2, then A is a $\mathcal{G}$-preclosed subset of X.*

Proof. Suppose that A is not a $\mathcal{G}$-preclosed subset of $(X, \tau, \mathcal{G})$. Then $\mathcal{G}P\,\mathrm{Cl}(A) \backslash A \neq \varnothing$. Let $x \in p\,\mathrm{Cl}_{\mathcal{G}}(A) \backslash A$. Since f is weakly $\mathcal{G}$-precontinuous retraction, $f(x) \neq x$. Since $(X, \tau, \mathcal{G})$ is T_2, then there exist disjoint open sets U and V of X such that $x \in U$, $f(x) \in V$ and this implies that $U \cap \mathrm{Cl}(V) = \varnothing$. Let $W \in \mathcal{G}PO(X, x)$. Then $U \cap W \in \mathcal{G}PO(X)$ and $x \in U \cap W$. Since $x \in p\,\mathrm{Cl}_{\mathcal{G}}(A)$, $(U \cap W) \cap A \neq \varnothing$. Let $y \in U \cap W \cap A$. Then, $y \in A$ and so $f(y) = y \in U \cap W \cap A \subset U$. Hence $f(y) \notin \mathrm{Cl}(V)$. This indicates that f is weakly $\mathcal{G}$-precontinuous and hence A is $\mathcal{G}$-preclosed in $(X, \tau, \mathcal{G})$. □

Theorem 3.14 ([9]). *Let $(X, \tau, \mathcal{G})$ and (Y, σ) be topological spaces. Then the following are equivalent:*

(i) *$f : (X, \tau) \to (Y, \sigma)$ is weakly continuous.*

(ii) *For every open set V in Y, there exists an open set G in Y such that $G \subset V$ and $f^{-1}(G) \subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$.*

Theorem 3.15. *Let $f, g : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ be functions and (Y, σ) be a Urysohn space. If f is weakly continuous and g is weakly $\mathcal{G}$-precontinuous, then the set $\{x \in X : f(x) = g(x)\}$ is $\mathcal{G}$-preclosed.*

Proof. Let $A = \{x \in X : f(x) = g(x)\}$. If $x \in X \backslash A$, then $f(x) \neq g(x)$. Since (Y, σ) is Urysohn, then there exist open sets V_1 and V_2 of Y containing $f(x)$ and $g(x)$, respectively such that $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. The weak continuity of f gives $x \in f^{-1}(V_1) \subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V_1)))$ by Theorem 3.14. Also the weakly $\mathcal{G}$-precontinuity of g gives $x \in g^{-1}(V_2) \subset p\,\mathrm{Int}_{\mathcal{G}}(g^{-1}(\mathrm{Cl}(V_2)))$ by Theorem 3.3. Let $U = \mathrm{Int}(f^{-1}(\mathrm{Cl}(V_i))) \cap p\,\mathrm{Int}_{\mathcal{G}}(g^{-1}(\mathrm{Cl}(V_2)))$. It is clear that $U \in \mathcal{G}PO(X, x)$. Again disjointness of $\mathrm{Cl}(V_i)$ for $i = 1,2$, implies that $U \cap A = \varnothing$ and hence $x \in U \subset X \backslash A$. This indicates that $X \backslash A$ is a union of $\mathcal{G}$-preopen sets. Therefore, $X \backslash A \in BO(X)$ and consequently, A is $\mathcal{G}$-preclosed in $(X, \tau, \mathcal{G})$. □

Lemma 3.2. *Let A be a subset of a space $(X, \tau, \mathcal{G})$. Then* *(i)* $A \subset B \Rightarrow p\,\mathrm{Int}_{\mathcal{G}}(A) \subset p\,\mathrm{Int}_{\mathcal{G}}(B)$;

(ii) $A \subset B \Rightarrow p\,\mathrm{Cl}_{\mathcal{G}}(A) \subset p\,\mathrm{Cl}_{\mathcal{G}}(B)$;

(iii) $p\,\mathrm{Int}_{\mathcal{G}}(X\backslash A) = X\backslash p\,\mathrm{Cl}_{\mathcal{G}}(A)$;

(iv) $p\,\mathrm{Cl}_{\mathcal{G}}(X\backslash A) = X\backslash p\,\mathrm{Int}_{\mathcal{G}}(A)$;

Theorem 3.16. *If $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is weakly $\mathcal{G}$-precontinuous and A is θ-closed in $X \times Y$, then $p_X(A \cap G(f))$ is $\mathcal{G}$-preclosed in X, where p_X represents the projection of $X \times Y$ onto X.*

Proof. Let A be a θ-closed subset of $X \times Y$ and $x \in p\,\mathrm{Cl}_{\mathcal{G}}(p_X(A \cap G(f)))$. Let $U \in \tau$ containing x and $V \in \sigma$ containing $f(x)$. Since f is weakly $\mathcal{G}$-precontinuous, by Theorem 3.3, $x \in f^{-1}(V) \subseteq p\,\mathrm{Int}_{\mathcal{G}}(f^{-1}(\mathrm{Cl}(V)))$. Then $U \cap p\,\mathrm{Int}_{\mathcal{G}}(f^{-1}(\mathrm{Cl}(V)))$ $\cap\, p_X(A \cap G(f))$ contains some point z of X. This implies that $(z, f(z)) \in A$ and $f(z) \in \mathrm{Cl}(V)$. Thus we have $\varnothing \neq (U \times \mathrm{Cl}(V))$ $\cap\, A = \mathrm{Cl}(U \times V) \cap A$ and hence $(x, f(x)) \in \mathrm{Cl}_{\theta}(A)$. Since A is θ-closed, $(x, f(x)) \in A \cap G(f)$ and $x \in p_X(A \cap G(f))$ by Lemma 3.2, $p_X(A \cap G(f))$ is $\mathcal{G}$-preclosed in $(X, \tau, \mathcal{G})$. □

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