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Solving Classical Nonlinear Riccati Differential Equations (RDEs) Using Differential Transformation Method (DTM)

Research Article

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Abstract: In this paper, the Differential Transformation Method (DTM) is used to solve nonlinear Riccatti differential equation of the form:

$$\left(\frac{dj}{dt}\right)^{\beta} = S(t)j + Q(t)j^2 + R(t), 0 \leq t \leq 1 \tag{1}$$

Subject to initial condition $j(0) = A$, where $S(t), Q(t), R(t), A$ are constant variables and when $\beta = 1$, the above equation (1) is called Classical Riccati differential equation. The principle of differential transformation method is briefly introduced and applied for the first derivation of the set of nonlinear Riccatti differential equations. Accuracy and efficiency of the proposed method is verified through numerical examples. The result obtained with the proposed method are in good agreement with exact solution of the problem considered. The method is simply and efficient as numerical tool for any other class of the differential equations.

Keywords: Differential Transformation Method (DTM), Nonlinear, Classical Riccatti Equation and Variable constants.

1. Introduction

The Riccati differential equation **(RDEs)** has large variety of application in applied sciences, engineering such as rheology, damping law, diffusion processes and transmission line phenomena and optimal control theory problems and to mention a few [1–6]. The **(RDEs)** is are complicated in its structure and finding exact solution for them cannot be simple. The study of **(RDEs)** attracted many researchers to work intensively both in theory and application. The most significant methods used are Adomian Decomposition Method [7], Homotopy Perturbation Method [8], Haar Wavelet Method [9] and Combination of Laplace, Adomian Decomposition and Pad Approximation Method [10].

Several numerical approaches for obtain numerical solution of the nonlinear Riccati equation are known. Balaji 2014 [11] proposed generalized Chebyshev Wavelet Operation Matrix (CWOM), Raja et-al [12] developed a stochastic technique based on particle swarm optimization and simulated annealing. A combination of finite Finite Differential Method and Pade Variation Iterative Numerical Scheme was proposed by Sweilam et-al [13].

In this work, the nonlinear Riccati equation was solve numerical by using Differential Transformation Method (DTM). The proposed method is illustrated by application and obtained results was in good agreement with exact solution of the

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problems considered.

1.1. Basic Idea of Differential Transformation Method

The concept of differential transformation method was introduced first by Zhou [14] in 1986 and it was applied to solve both linear and nonlinear initial value problems in electric circuit analysis. This method construct a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form of polynomials.

The method was used in a direct way without using linearization, perturbation or restrictive assumption [15, 16]. With this technique, it is possible to obtain highly accurate approximate solution results. Suppose the function $j(t)$ is continuously differentiable in the interval $t_0 - r \leq t \leq t_0 + r$ for $r > 0$, then we have the following definitions.

(i) The differential transform of the function $j(t)$ for the kth derivative is defined as follow

$$J(k) = \frac{1}{k!}\left[\frac{d^k j(t)}{dt^k}\right]_{t=t_0} \tag{2}$$

where $j(t)$the original function and $J(k)$ is the transformation of the function.

(ii) The inverse differential transform of $J(k)$ is defined as

$$j(t) = \sum_{k=0}^{\infty} J(k)t^k \tag{3}$$

Substituting equation (2) into equation (3) yields the following equation

$$j(t) = \sum_{k=0}^{\infty} t^k \frac{1}{k!}\left[\frac{d^k j(t)}{dt^k}\right]_{t=t_0} \tag{4}$$

Which is the Taylor's series for $j(t)$ at$t = t_0$.

From the definition (i). (ii), we can derive the following: Assume that $J(k), L(k), M(k)$, and $V_i(k)$, $i = 1, \ldots, n$ are the differential transformation of the $j(t), l(t), m(t)$ and $v_i(t)$, $\quad i = 1, \ldots, n$ respectively, then

$$\text{If } j(t) = \frac{d^n l(t)}{dt^n} \text{ then } J(k) = \frac{(k+n)!}{k} L(k+n) \tag{5}$$

$$\text{If } j(t) = l(t)m(t) \text{ then } J(k) = \sum_{r=0}^{k} L(r)M(k-r) \tag{6}$$

$$\text{If } j(t) = t^n \text{ then } J(k) = \delta(k-n), \delta \text{ is the Kronecker delta symbol} \tag{7}$$

$$\text{If } j(t) = a \text{ then } J(k) = \delta(k),\ \text{a is constant} \tag{8}$$

$$\text{If } j(t) = l(t)\int_0^t m(t)dt \text{ then } J(k) = \frac{L(k-m)}{k},\ \text{where } k \geq 1 \tag{9}$$

$$\text{If } j(t) = \ell^{\beta t} \text{ then } J(k) = \frac{\beta^t}{k!} \tag{10}$$

$$\text{If } j(t) = \beta v(t) \text{ then } J(k) = \beta V(k) \tag{11}$$

$$\text{If } j(t) = \alpha m(t) + \beta n(t) \text{ then } J(k) = \alpha M(k) \pm \beta N(k) \tag{12}$$

1.2. Error Estimate

The absolute error in each example was carried out by use of formula:

$$Error = |j_e(t) - j_N(t)| \tag{13}$$

Here $j_e(t)$ is the exact solution and $j_N(t)$ is the approximate solution at the degree of approximate N.

2. Numerical Experiment

In order to show the efficiency of the Differential Transformation Method (DTM), we applied the method to the various nonlinear Riccati equations when constant $\beta = 1$. MAPLE 18 software was used to evaluate all mathematical computation of the work.

Example 2.1. *Consider the nonlinear Riccati differential equation*

$$\left(\frac{dj}{dt}\right)^{\beta} = 2j(t) - j^2(t) + 1 \tag{14}$$

subject to initial condition

$$j(0) = 0 \;\; when \;\; \beta = 1. \tag{15}$$

The exact solution is given as

$$j_e(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \tag{16}$$

Using equations (5), (6) and (8), we obtained differential transformation of equation (14), which leads to recurrence relation given as

$$J(k+1) = \frac{2J(k) - \sum_{r=0}^{k} J(r)J(\delta + r) + \delta(k)}{(k+1)} \tag{17}$$

where J(k) is the differential transformation of j(t) and the transformation of initial condition is

$$J(0) = 0 \tag{18}$$

Substituting equation (18) and put $k = 0$, we obtained the following numerical values for J(k):

$$J(1) = 1.00000000 \tag{19}$$

$$J(2) = 1.00000000 \tag{20}$$

$$J(3) = 0.33333333 \tag{21}$$

$$J(4) = -0.33333333 \tag{22}$$

$$J(5) = -0.46666667 \tag{23}$$

$$J(6) = -0.15555555 \tag{24}$$

$$\vdots$$

$$J(15) = 0.01341069 \tag{25}$$

Hence, by substitution the above values into inverse transformation of equation (3), *we obtained the closed form approximate solution up to* $N = 15$. *Where* N *is the degree of approximant.*

$$j(t) = \sum_{k=0}^{\infty} 0t^0 + t + t^2 + 0.33333333t^3 - 0.33333333t^4 - 0.46666667t^5 - 0.5555555t^6 + \cdots + 0.01341069t^{15} \tag{26}$$

Example 2.2. *Consider the nonlinear Riccati differential equation*

$$\left(\frac{dj}{dt}\right)^{\beta} - j^2(t) = 1 \tag{27}$$

subject to initial condition

$$j(0) = 0 \;\; when \;\; \beta = 1 \tag{28}$$

The exact solution is given as

$$j(t) = \tan t \tag{29}$$

Using equations (5), (6) *and* (8), *we obtained differential transformation of equation* (**??**), *which leads to recurrence relation given as*

$$J(k+1) = \frac{\sum_{r=0}^{k} J(r)J(\delta + r) + \delta(k)}{(k+1)} \tag{30}$$

Where J(k) is the differential transformation of j(t) and the transformation of initial condition is

$$J(0) = 0 \tag{31}$$

Substituting equation (31) *and put k=0, we obtained the following numerical values for J(k):*

$$J(1) = 1.00000000 \tag{32}$$

$$J(2) = 0.00000000 \tag{33}$$

$$J(3) = 0.33333333 \tag{34}$$

$$J(4) = 0.00000000 \tag{35}$$

$$J(5) = 0.13333333 \tag{36}$$

$$J(6) = 0.00000000 \tag{37}$$

$$\vdots$$

$$J(17) = 0.00059002 \tag{38}$$

Hence, by substitution the above values into inverse transformation of equation (3), *we obtained the closed form approximate solution of to* $N = 17$.

$$j(t) = \sum_{k=0}^{\infty} t + 0.33333333t^3 - 0.13333333t^5 + \cdots + 0.0005902t^{17} \tag{39}$$

Example 2.3. *Consider the nonlinear Riccatti differential equation*

$$\left(\frac{dj}{dt}\right)^{\beta} + j^{2}(t) = \sqrt{2} \tag{40}$$

subject to initial condition

$$j(0) = 0 \ \ when \ \ \beta = 1 \tag{41}$$

The exact solution is given as

$$j(t) = \tanh(2^{1/4}t)2^{1/4} \tag{42}$$

Using equations (5),(6) and (7), we obtained differential transformation of equation (40), which leads to recurrence relation given as

$$J(k+1) = \frac{-\sum_{r=0}^{k} J(r)J(\delta+r) + \delta\left(k-\frac{1}{2}\right)}{(k+1)} \tag{43}$$

Where J(k) is the differential transformation of j(t) and the transformation of initial condition is

$$J(0) = 0 \tag{44}$$

Substituting equation (44) and put k=0, we obtained the following numerical values for J(k):

$$J(1) = 1.41421356 \tag{45}$$

$$J(2) = 0.00000000 \tag{46}$$

$$J(3) = -0.66666667 \tag{47}$$

$$J(4) = 0.00000000 \tag{48}$$

$$J(5) = 0.37712362 \tag{49}$$

$$J(6) = 0.00000000 \tag{50}$$

$$\vdots$$

$$J(15) = -0.02393350 \tag{51}$$

Hence, by substitution the above values into inverse transformation of equation (3), we obtained the closed form approximate solution of to $N = 15$.

$$j(t) = \sum_{k=0}^{\infty} 1.4142136t + 0.66666667t^{3} + 0.37712362t^{5} + \cdots + 0.0232933t^{15} \tag{52}$$

3. Numerical Result Computation

In order to obtain numerical solutions of the examples considered, we used inverse transformation equations (26), (39) and (52) respectively to obtain the following tables.

Table 1. **Numerical Result Example 1**

t	**Exact** $j_e(t)$	**DTM N=12** $j_{12}(t)$	**DTM N=15** $j_{15}(t)$	**Error** $\lvert j_e(t) - j_{12}(t)\rvert$	**Error** $\lvert j_e(t) - j_{15}(t)\rvert$
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.1102951967	0.1102951969	0.1102951969	0.0000000002	0.0000000002
0.2	0.2419767992	0.2419767997	0.2419767997	0.0000000005	0.0000000005
0.3	0.3951048481	0.3951048391	0.3951048487	0.0000000090	0.0000000006
0.4	0.5678121656	0.5678117343	0.5678121799	0.0000004313	0.0000000014
0.5	0.7560143925	0.7560060180	0.7560149094	0.0000083745	0.0000005169
0.6	0.9535662155	0.9534722943	0.9535761760	0.0000939212	0.0000099605
0.7	1.1529489660	1.1522312380	1.1530693800	0.0007177280	0.0001204140
0.8	1.3463636550	1.3422394500	1.3473929270	0.0041242050	0.0010292720
0.9	1.5269113120	1.5079071780	1.5336423670	0.0190041340	0.0067310550
1.0	1.6894983900	1.6160547260	1.6250722940	0.0734436640	0.0355739404

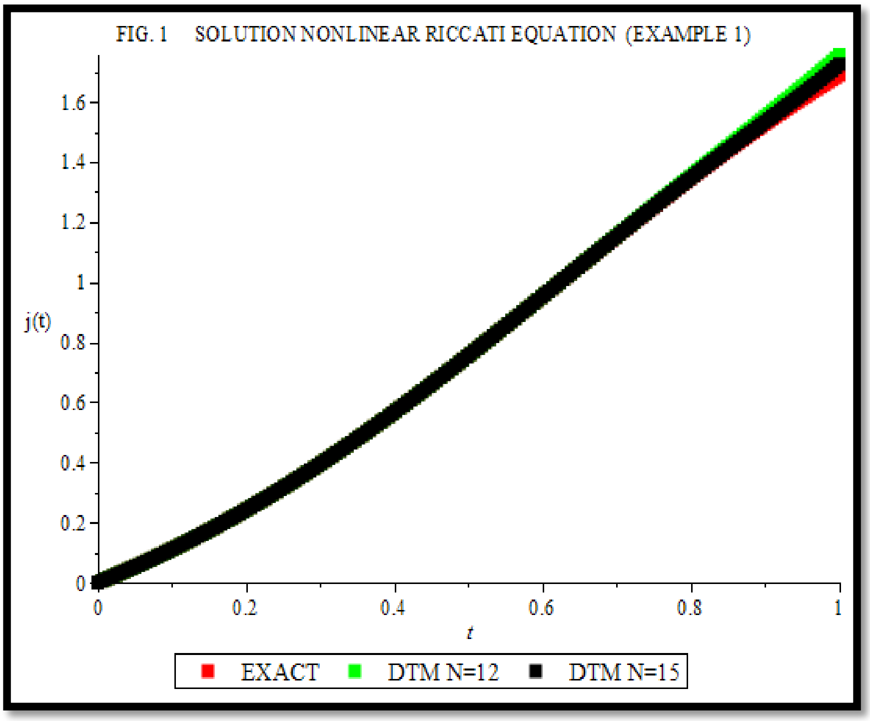


Table 2. **Numerical Result Example 2**

t	**Exact** $j_e(t)$	**DTM N=12** $j_{13}(t)$	**DTM N=17** $j_{17}(t)$	**Error** $\lvert j_e(t) - j_{13}(t)\rvert$	**Error** $\lvert j_e(t) - j_{17}(t)\rvert$
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.1003346721	0.1003346720	0.1003346720	0.0000000001	0.0000000001
0.2	0.2027100355	0.2027100356	0.2027100356	0.0000000001	0.0000000001
0.3	0.3093362496	0.3093362497	0.3093362497	0.0000000001	0.0000000001
0.4	0.4227932187	0.4227932171	0.4227932188	0.0000000016	0.0000000001
0.5	0.5463024898	0.5463024405	0.5463024894	0.0000000493	0.0000000004
0.6	0.6841368083	0.6841360070	0.6841367914	0.0000008013	0.0000000169
0.7	0.8422883805	0.8422797560	0.8422880403	0.0000008624	0.0000003402
0.8	1.0296385570	1.0295693940	1.0296339030	0.0000691630	0.0000046540
0.9	1.2601582180	1.2597119850	1.2601101280	0.0004462330	0.0000480900
1.0	1.5574077250	1.5549597730	1.5570056340	0.0024479520	0.0004020910

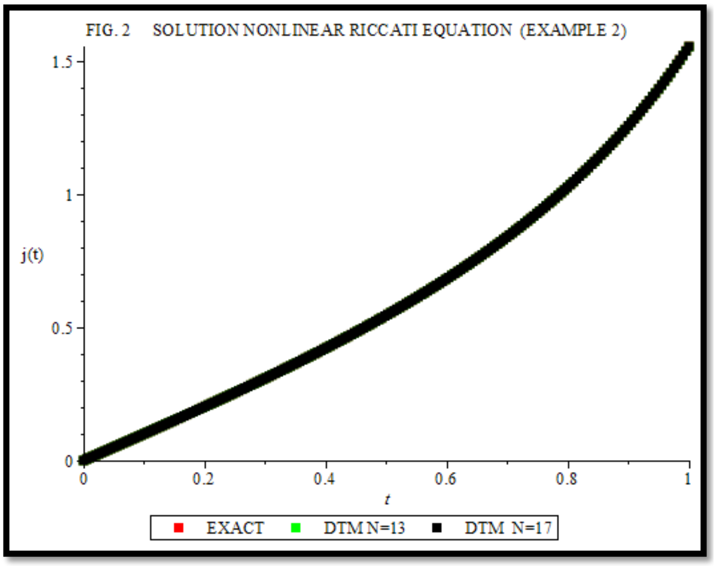
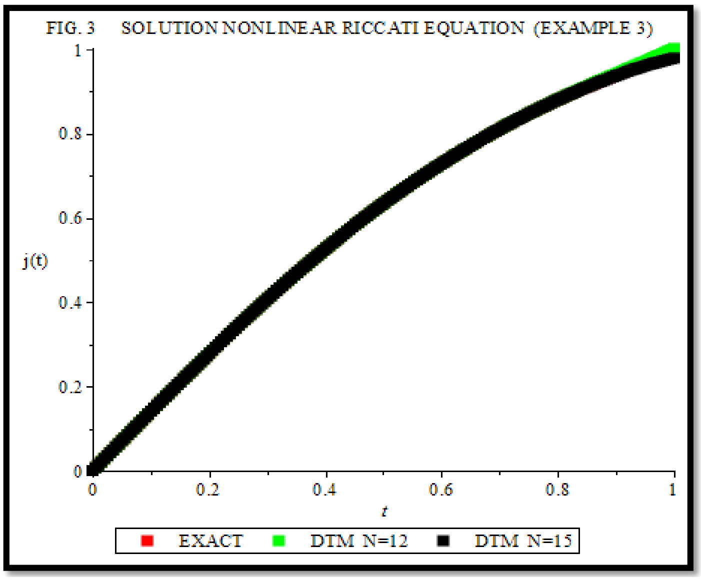


Table 3. **Numerical Result Example 3**

t	Exact $j_e(t)$	DTM N=12 $j_{12}(t)$	DTM N=15 $j_{15}(t)$	Error $\lvert j_e(t) - j_{12}(t)\rvert$	Error $\lvert j_e(t) - j_{15}(t)\rvert$
0.0	0.00000000000	0.00000000000	0.00000000000	0.00000000000	0.00000000000
0.1	0.14075843940	0.14075843930	0.14075843930	0.00000000011	0.00000000011
0.2	0.27762735750	0.27762735730	0.27762735730	0.00000000002	0.00000000002
0.3	0.40713558330	0.40713558350	0.40713558310	0.00000000002	0.00000000002
0.4	0.52655652400	0.52655654690	0.52655652190	0.00000002290	0.00000000021
0.5	0.63408339620	0.63408401760	0.63408330680	0.00000062140	0.00000008941
0.6	0.72884369970	0.72885277820	0.72884182600	0.00000907850	0.00000187371
0.7	0.81078554830	0.81087188620	0.81076129960	0.00008633790	0.00002424870
0.8	0.88048978560	0.88108939690	0.88048978350	0.00059961130	0.00021995060
0.9	0.93896119090	0.94223649590	0.93744060170	0.00327530500	0.00152058920
1.0	0.98743767620	1.00224441000	0.97895106000	0.01480673380	0.00848661620



4. Conclusion

The differential transformation method was applied to solve classical Riccati equations. Numerical solution obtained shows a good agreement with the analytical solutions of the nonlinear Riccati equation. The main advantage of this method is that it can be applied directly to differential equitation without requiring linearization, discretization or perturbation. It may be concluded that DTM is very powerful and efficient in finding analytical approximate solution for both linear and nonlinear of differential equations.

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