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Counting Homomorphisms From Quasi-dihedral Group into Some Finite Groups

Research Article

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Abstract: We derive general formulae for counting the number of homomorphisms from quasi-dihedral group into each of quasi-dihedral group, quaternion group, dihedral group, and modular group by using only elementary group theory.

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Keywords: Finite groups, homomorphisms.

1. Introduction

Counting homomorphisms between two groups or rings is a basic problem in group thoery. In [2], Gallian and Buskirk enumerated the homomorphisms between two specified cyclic groups by using only elementary group theory. Also by using the elementary techniques, in [3] Gallian and Jungreis provided a method for counting homomorphisms between some specific rings. In [5], Matei *et al* present a method for computing the number of epimorphisms from a finitely presented group to a finite solvable group. But this needs advanced tools of algebra; see, also in [1]. In [4] Jeremiah Johnson, described a method of enumerating homomorphisms between two specified dihedral groups by using only elementary methods. Now we consider dihedral group, quaternion group, quasi-dihedral group and modular group. In [6], [7] and [8] authors give the enumeration of homomorphisms, monomorphisms and epimorphisms from each of dihedral group, quaternion group and modular group into each of these four groups respectively by using elementary techniques. In this paper, we consider the problem of enumerating the homomorphisms, monomorphisms and epimorphisms from a quasi-dihedral group into each of these four groups by using elementary methods.

We use the following notations: for a positive integer $n > 1$, D_n denotes the dihedral group generated by two generators x_n and y_n subject to the relations $x_n^n = e = y_n^2$ and $x_n y_n = y_n x_n^{-1}$; and for a positive integer $m > 1$, Q_m denotes the quaternion group generated by two generators a_m and b_m subject to the relations $a_m^{2m} = e = b_m^4$ and $a_m b_m = b_m a_m^{-1}$; and for a positive integer $\alpha > 3$, QD_{2^α} denotes the quasi-dihedral group generated by two generators s_α and t_α subject to the relations $s_\alpha^{2^{\alpha-1}} = e = t_\alpha^2$ and $t_\alpha s_\alpha = s_\alpha^{2^{\alpha-2}-1} t_\alpha$; and for a positive integer $\beta > 2$, M_{p^β} denotes the modular group generated by two generators r_β and f_β subject to the relations $r_\beta^{p^{\beta-1}} = e = f_\beta^p$ and $f_\beta r_\beta = r_\beta^{p^{\beta-2}+1} f_\beta$.

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2. The Number of Homomorphisms From QD_{2^α} into QD_{2^β}

Theorem 2.1. *Let $\alpha > 3$ and $\beta > 3$ be any two positive integers. Then the number of group homomorphisms from QD_{2^α} into QD_{2^β} is* $4 + 2^\beta + 2^{\beta-2}\left(\sum_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$.

Proof. Suppose ρ is a group homomorphism from QD_{2^α} into QD_{2^β}. Then $|\rho(s_\alpha)|$ divides $|s_\alpha| = 2^{\alpha-1}$ and $|\rho(t_\alpha)|$ divides $|t_\alpha| = 2$. Therefore, $\rho(s_\alpha)$ is either $s_\beta^{k_1}t_\beta$, $0 \le k_1 < 2^{\beta-1}$ or s_β^m, where $|s_\beta^m|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$; and $\rho(t_\alpha)$ is one of e or $s_\beta^{2^{\beta-2}}$ or $s_\beta^{k_2}t_\beta$, $0 \le k_2 < 2^{\beta-1}$ and k_2 is even.

Suppose $\rho(s_\alpha) = s_\beta^{k_1}t_\beta$, $0 \le k_1 < 2^{\beta-1}$ and $\rho(t_\alpha) = e$. Then ρ is well defined only when k_1 is even since $\rho(s_\alpha)^{2^{\alpha-2}} = e = \rho(s_\alpha t_\alpha)^2$. Then $\rho(s_\alpha^l t_\alpha) = (s_\beta^{k_1}t_\beta)^l$, $0 \le l < 2^{\alpha-1}$. For every k_1, $0 \le k_1 < 2^{\beta-1}$ and k_1 is even, $|s_\beta^{k_1}t_\beta| = 2$. Therefore, $|(s_\beta^{k_1}t_\beta)^l| = 1$ or 2, for every l, $0 \le l < 2^{\alpha-1}$. Then $|(s_\beta^{k_1}t_\beta)^l|$ divides $|s_\alpha^l t_\alpha|$. Thus we have $2^{\beta-2}$ homomorphisms.

Similarly suppose $\rho(s_\alpha) = s_\beta^{k_1}t_\beta$, $0 \le k_1 < 2^{\beta-1}$ and $\rho(t_\alpha) = s_\beta^{2^{\beta-2}}$, then ρ is well defined only when k_1 is even. Then $\rho(s_\alpha^l t_\alpha) = (s_\beta^{k_1}t_\beta)^l s_\beta^{2^{\beta-2}}$. If l is even, $\rho(s_\alpha^l t_\alpha) = s_\beta^{2^{\beta-2}}$ and if l is odd, $\rho(s_\alpha^l t_\alpha) = s_\beta^{k_1+2^{\beta-2}}t_\beta$. Thus in both cases $|\rho(s_\alpha^l t_\alpha)|$ divides $|s_\alpha^l t_\alpha|$. Thus we have $2^{\beta-2}$ homomorphisms.

Suppose $\rho(s_\alpha) = s_\beta^{k_1}t_\beta$, $0 \le k_1 < 2^{\beta-1}$ and $\rho(t_\alpha) = s_\beta^{k_2}t_\beta$, $0 \le k_2 < 2^{\beta-1}$ and k_2 is even. Then $\rho(s_\alpha^l t_\alpha) = (s_\beta^{k_1}t_\beta)^l s_\beta^{k_2}t_\beta$. If l is even, $\rho(s_\alpha^l t_\alpha) = s_\beta^{k_2}t_\beta$ or $s_\beta^{k_1 2^{\beta-2}+k_2}t_\beta$. Since k_2 is even, $|\rho(s_\alpha^l t_\alpha)| = 2$ which divides $|s_\alpha^l t_\alpha|$. If l is odd, $\rho(s_\alpha^l t_\alpha) = s_\beta^{k_1-k_2}$ or $s_\beta^{k_1-k_2+k_1 2^{\beta-2}}$. Then ρ is a homomorphism only when $|\rho(s_\alpha^l t_\alpha)|$ divides 2 since $\rho(s_\alpha)^{2^{\alpha-2}} = e$. This is possible when $k_1 - k_2$ must be either 0 or $2^{\beta-2}$. Thus there are $2 \times 2^{\beta-2} = 2^{\beta-1}$ homomorphisms.

Suppose $\rho(s_\alpha) = s_\beta^m$, where $|s_\beta^m|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$, and $\rho(t_\alpha) = s_\beta^{k_2}t_\beta$, $0 \le k_2 < 2^{\beta-1}$ and k_2 is even. Then $\rho(s_\alpha^l t_\alpha) = s_\beta^{lm+k_2}t_\beta$. If l is even, $|s_\alpha^l t_\alpha| = 2$ and since k_2 is even, $|s_\beta^{lm+k_2}t_\beta| = 2$. If l is odd, $|s_\alpha^l t_\alpha| = 4$ and $|s_\beta^{lm+k_2}t_\beta| = 2$ or 4. Thus in both cases $|\rho(s_\alpha^l t_\alpha)|$ divides $|s_\alpha^l t_\alpha|$. Since $\rho(s_\alpha)$ has $\left(\sum_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$ choices and $\rho(t_\alpha)$ has $2^{\beta-2}$ choices, in this case we have $2^{\beta-2}\left(\sum_{k|\gcd(2^{\alpha-1},2^{\beta-1})} \phi(k)\right)$ homomorphisms.

Suppose $\rho(s_\alpha) = s_\beta^m$, where $|s_\beta^m|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$, and $\rho(t_\alpha) = e$. Then $\rho(s_\alpha^l t_\alpha) = s_\beta^{lm}$. Suppose l is even, ρ is a homomorphism when $|s_\beta^{lm}|$ divides $|s_\alpha^l t_\alpha| = 2$. Therefore, m is one of 0, $2^{\beta-2}$, $2^{\beta-3}$ or 3 $2^{\beta-3}$. Suppose l is odd and $\rho(s_\alpha)$ is one of e, $s_\beta^{2^{\beta-2}}$, $s_\beta^{2^{\beta-3}}$ or $s_\beta^{3\ 2^{\beta-3}}$and $\rho(t_\alpha) = e$, then $|\rho(s_\alpha^l t_\alpha)|$ must divide 2, since $\rho(s_\alpha)^{2^{\alpha-2}} = e$. Thus we have 2 choices for m that are 0 and $2^{\beta-2}$. Thus we have 2 homomorphisms.

Similarly if $\rho(s_\alpha) = s_\beta^m$, where $|s_\beta^m|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$, and $\rho(t_\alpha) = s_\beta^{2^{\beta-2}}$. Then ρ is a homomorphism only when m is either 0 or $2^{\beta-2}$. Thus we have 2 homomorphisms. Hence we get the result. □

Corollary 2.1. *Let $\alpha, \beta > 3$. Then the number of monomorphisms from QD_{2^α} into QD_{2^β} is $2^{2\alpha-4}$, if $\alpha = \beta$; 0, otherwise. Also the number of automorphisms on QD_{2^α} is $2^{2\alpha-4}$.*

Proof. Suppose $\alpha > \beta$, then there is no monomorphism from QD_{2^α} into QD_{2^β} since there is no element in QD_{2^α} has order 2^β. So, assume that $\alpha \le \beta$. If ρ is a group monomorphism from QD_{2^α} into QD_{2^β}. Then $\rho(s_\alpha) = s_\beta^m$, where $|s_\beta^m| = 2^{\alpha-1}$ and $\rho(t_\alpha) = s_\beta^{k_2}t_\beta$, $0 \le k_2 < 2^{\beta-1}$ and k_2 is even. Then $\rho(s_\alpha^l t_\alpha) = s_\beta^{lm+k_2}t_\beta$. If l is even, $|s_\alpha^l t_\alpha| = 2$ and since k_2 is even, $|s_\beta^{lm+k_2}t_\beta| = 2$. If l is odd, $|s_\alpha^l t_\alpha| = 4$ and $|s_\beta^{lm+k_2}t_\beta| = 4$ only when m is odd. Thus if $\alpha = \beta$, we have $2^{\alpha-2}\phi(2^{\alpha-1}) = 2^{2\alpha-4}$ monomorphisms from QD_{2^α} into QD_{2^β}; and if $\alpha \ne \beta$, there is no monomorphism from QD_{2^α} into QD_{2^β}. □

Corollary 2.2. *Let $\alpha, \beta > 3$. Then the number of epimorphisms from QD_{2^α} onto QD_{2^β} is $2^{2\beta-4}$, if $\alpha \ge \beta$; 0, otherwise.*

Proof. Suppose $\alpha < \beta$, then clearly there is no epimorphism epimorphisms from QD_{2^α} onto QD_{2^β}. So, assume that $\alpha \ge \beta$. If $\rho(s_\alpha) = s_\beta^m$, where $|s_\beta^m| = 2^{\beta-1}$ and $\rho(t_\alpha) = s_\beta^{k_2}t_\beta$, $0 \le k_2 < 2^{\beta-1}$ and k_2 is even. Then $\rho(s_\alpha)$ and $\rho(t_\alpha)$ generate the group QD_{2^β}. Then ρ is a epimorphism. Thus we have $2^{2\beta-4}$ epimorphisms, if $\alpha \ge \beta$; 0, otherwise. □

3. The Number of Homomorphisms From QD_{2^α} into D_n

Theorem 3.1. *Let n be a positive odd integer and $\alpha > 3$, then the number of group homomorphisms from QD_{2^α} into D_n is $3n+1$.*

Proof. Let $\rho : QD_{2^\alpha} \to D_n$ be a group homomorphism. Then $|\rho(s_\alpha)|$ divides $|s_\alpha| = 2^{\alpha-1}$, and since n is odd, $\rho(s_\alpha)$ must be either e or $x_n^{k_1}y_n, 0 \le k_1 < n$. Also since $|\rho(t_\alpha)|$ divides $|t_\alpha| = 2$, $\rho(t_\alpha) = e$ or $\rho(t_\alpha) = x_n^{k_2}y_n, 0 \le k_2 < n$.
Suppose $\rho(s_\alpha) = e$ and $\rho(t_\alpha) = x_n^{k_2}y_n, 0 \le k_2 < n$, then $\rho(s_\alpha^m t_\alpha) = x_n^{k_2}y_n$ and $|x_n^{k_2}y_n| = 2$ divides $|s_\alpha^m t_\alpha|$ for every $0 \le m < 2^{\alpha-1}$. Thus we have n such homomorphisms. Suppose $\rho(s_\alpha) = x_n^{k_1}y_n, 0 \le k_1 < n$ and $\rho(t_\alpha) = e$, then $\rho(s_\alpha^m t_\alpha) = (x_n^{k_2}y_n)^m$. If m is even, then $|\rho(s_\alpha^m t_\alpha)| = 1$ and $|s_\alpha^m t_\alpha| = 2$; and if m is odd, then $|\rho(s_\alpha^m t_\alpha)| = 2$ and $|s_\alpha^m t_\alpha| = 4$. Therefore, in both cases $|\rho(s_\alpha^m t_\alpha)|$ divides $|s_\alpha^m t_\alpha|$. Thus we have n homomorphisms in this case.
Suppose $\rho(s_\alpha) = x_n^{k_1}y_n, 0 \le k_1 < n$ and $\rho(t_\alpha) = x_n^{k_2}y_n, 0 \le k_2 < n$, then $\rho(s_\alpha^m t_\alpha) = (x_n^{k_1}y_m)^m x_n^{k_2}y_n$. If m is even, $\rho(s_\alpha^m t_\alpha) = x_n^{k_2}y_n$, and if m is odd, $\rho(s_\alpha^m t_\alpha) = x_n^{k_1-k_2}$. Therefore, ρ is a homomorphism if $|x_n^{k_1-k_2}|$ divides $|s_\alpha^m t_\alpha| = 4$. Since n is odd, this is possible only when $k_1 = k_2$. Thus there are n such homomorphisms. Thus in addition to the trivial homomorphism, totally there are $3n+1$ homomorphisms. □

Theorem 3.2. *Let n be a positive even integer and $\alpha > 3$. Then the number of group homomorphisms from QD_{2^α} into D_n is $4 + 4n + n\left(\sum_{k\mid \gcd(n,2^{\alpha-2})} \phi(k)\right)$.*

Proof. Let ρ be a group homomorphism from QD_{2^α} into D_n. Since n is even, $\rho(s_\alpha)$ can be of the form x_n^β, where $|x_n^\beta|$ divides both $2^{\alpha-1}$ and n, or $\rho(s_\alpha) = x_n^{k_1}y_n, 0 \le k_1 < n$; and $\rho(t_\alpha)$ is one of e, $x_n^{\frac{n}{2}}$, or $x_n^{k_2}y_n, 0 \le k_2 < n$.
Suppose $\rho(t_\alpha) = e$ and $\rho(s_\alpha) = x_n^\beta$, where $|x_n^\beta|$ divides both $2^{\alpha-1}$ and n. Then $\rho(s_\alpha^m t_\alpha) = x_n^{m\beta (mod\ n)}$ and $|x_n^{m\beta(mod\ n)}|$ divides $|s_\alpha^m t_\alpha|$. Suppose $n \equiv 2(mod\ 4)$, this is possible when $\beta = 0$ or $\frac{n}{2}$; and if $n \equiv 0(mod\ 4)$, then the possible values of β are 0, $\frac{n}{4}$, $\frac{n}{2}$, $\frac{3n}{4}$. But if $\beta = \frac{n}{4}$ or $\frac{3n}{4}$, ρ is not well defined since $\rho(s_\alpha)^{2^{\alpha-2}} = e$ but $\rho(s_\alpha t_\alpha)^2 \ne e$. As in the proof of Theorem 3.1, $\rho(t_\alpha) = e$ and $\rho(s_\alpha) = x_n^{k_1}y_n, 0 \le k_1 < n$, is a homomorphism. So, there are $n+2$ homomorphisms send t_α to e.
Similarly, there are $n+2$ homomorphisms send t_α to $x_n^{\frac{n}{2}}$. Suppose $\rho(s_\alpha) = x_n^{k_1}y_n, 0 \le k_1 < n$, and $\rho(t_\alpha) = x_n^{k_2}y_n, 0 \le k_2 < n$, then $\rho(s_\alpha^m t_\alpha) = (x_n^{k_1}y_m)^m x_n^{k_2}y_n$. If m is even, $\rho(s_\alpha^m t_\alpha) = x_n^{k_2}y_n$, and if m is odd, $\rho(s_\alpha^m t_\alpha) = x_n^{k_1-k_2}$. Therefore, ρ is a homomorphism if $|x_n^{k_1-k_2}|$ divides 2 since $\rho(s_\alpha^{2^{\alpha-2}}) = e$. Then this is possible when $k_1 = k_2$ or $k_1 - k_2 = \frac{n}{2}$. Thus there are $2n$ such homomorphisms.
Suppose $\rho(s_\alpha) = x_n^\beta$, where $|x_n^\beta|$ divides both $2^{\alpha-1}$ and n, and $\rho(t_\alpha) = x_n^{k_2}y_n, 0 \le k_2 < n$, then $\rho(s_\alpha^m t_\alpha) = x_n^{m\beta + k_2 (mod\ n)}y_n$. Then $\rho(s_\alpha^m t_\alpha)^2 = e = \rho(s_\alpha^{2^{\alpha-2}})$, $|\rho(s_\alpha)|$ must divide both $2^{\alpha-2}$ and n. Thus there are $n\left(\sum_{k\mid \gcd(n,2^{\alpha-2})} \phi(k)\right)$ homomorphisms. Hence we obtain the result. □

Corollary 3.1. *Let n be a positive integer and $\alpha > 3$. Then there is no monomorphism from QD_{2^α} into D_n; and the number of epimorphism from QD_{2^α} onto D_n is $n\ \phi(n)$, if n divides $2^{\alpha-2}$; 0, otherwise.*

Proof. The group QD_{2^α} contains $2 + 2^{\alpha-2}$ elements having order 4 while the group D_n contains at most 2 elements having order 4. Thus there is no monomorphism from QD_{2^α} into D_n.
Suppose n does not divide $2^{\alpha-1}$, then there is no epimorphism from QD_{2^α} onto D_n. So, assume that n divides $2^{\alpha-1}$. Then by the Theorem 3.2, $\rho(s_\alpha) = x_n^\beta$, where $|x_n^\beta| = n \ne 2^{\alpha-1}$ and $\rho(t_\alpha) = x_n^{k_2}y_n, 0 \le k_2 < n$ is a homomorphism. Since $\rho(s_\alpha)$ and $\rho(t_\alpha)$ generate the group D_n, these homomorphisms are epimorphisms. Thus we have $n\ \phi(n)$ epimorphism from QD_{2^α} onto D_n, if n divides $2^{\alpha-2}$; 0, otherwise. □

4. The Number of Homomorphisms From QD_{2^α} into Q_n

Theorem 4.1. *Let $\alpha > 3$ be a positive integer and n be positive even integer. Then the number of group homomorphisms from QD_{2^α} into Q_n is 8.*

Proof. Suppose that $\rho : QD_{2^\alpha} \to Q_n$ is a group homomorphism, where $\alpha > 3$ is a positive integer and n is positive even integer. Since $|\rho(s_\alpha)|$ divides $|s_\alpha|$, it must be the case that $\rho(s_\alpha) = a_n^x b_n, 0 \leq x < 2n$ or $\rho(s_\alpha) = a_n^y$, where a_n^y is an element of Q_n whose order divides both $2^{\alpha-1}$ and $2n$, and since $|\rho(t_\alpha)|$ divides $|t_\alpha|$, either $\rho(t_\alpha) = a_n^n$ or e. But not all of these choices for $\rho(s_\alpha)$ yield homomorphisms, as can be seen when we consider where ρ sends the remaining elements in QD_{2^α} of the form $s_\alpha^l t_\alpha$, where $0 \leq l < 2^{\alpha-1}$.

If $\rho(t_\alpha) = e$ and $\rho(s_\alpha) = a_n^y$, where $|a_n^y|$ divides both $2^{\alpha-1}$ and $2n$, then $\rho(s_\alpha t_\alpha) = a_n^y$ and $|a_n^y|$ divides $|s_\alpha t_\alpha| = 4$, then $\rho(s_\alpha)$ must be one of $e, a_n^n, a_n^{\frac{n}{2}}$ or $a_n^{\frac{3n}{2}}$, there are 4 homomorphisms exist such that $\rho(t_\alpha) = e$ and $\rho(s_\alpha) = e, a_n^n, a_n^{\frac{n}{2}}$ or $a_n^{\frac{3n}{2}}$.

Suppose $\rho(t_\alpha) = e$ and $\rho(s_\alpha) = a_n^x b_n, 0 \leq x < 2n$, then $\rho(s_\alpha^{2^{\alpha-2}}) = e$. Since $s_\alpha^{2^{\alpha-2}} = (s_\alpha t_\alpha)^2$, $|s_\alpha t_\alpha|$ must divide 2. But in this case $\rho(s_\alpha t_\alpha) = a_n^x b_n$. Therefore, this ρ is not a homomorphism. Similarly, if $\rho(t_\alpha) = a_n^n$ and $\rho(s_\alpha) = a_n^x b_n, 0 \leq x < 2n$, then ρ is not a homomorphism.

Suppose $\rho(t_\alpha) = a_n^n$ and $\rho(s_\alpha) = a_n^y$, where $|a_n^y|$ divides both $2^{\alpha-1}$ and $2n$, then $\rho(s_\alpha t_\alpha) = a_n^{y+n}$ and $|a_n^{y+n}|$ divides $|s_\alpha t_\alpha|$, then $\rho(s_\alpha)$ must be one of $e, a_n^n, a_n^{\frac{n}{2}}$ or $a_n^{\frac{3n}{2}}$, there are 4 homomorphisms exist such that $\rho(t_\alpha) = a_n^n$ and $\rho(s_\alpha) = e, a_n^n, a_n^{\frac{n}{2}}$ or $a_n^{\frac{3n}{2}}$. Hence we get the result. □

Theorem 4.2. *Let $\alpha > 3$ be a positive integer and n be positive odd integer. Then the number of group homomorphisms from QD_{2^α} into Q_n is 4.*

Proof. Let as assume that $\rho : QD_{2^\alpha} \to Q_n$ is a group homomorphism, where α is positive integer and n is positive odd integer. As in the proof Theorem 4.1, when n is odd, the possible choices for $\rho(s_\alpha)$ are e, a_n^n or $a_n^x b_n, 0 \leq x < 2n$ and the possible choices for $\rho(t_\alpha)$ are either e or a_n^n.As in the proof of the Theorem 4.1, if $\rho(t_\alpha) = e$ or a_n^n and $\rho(s_\alpha) = a_n^x b_n, 0 \leq x < 2n$, then ρ is not a homomorphism. Thus we have 4 homomorphisms. □

Corollary 4.1. *Let $\alpha > 3$, n be any two positive integers. Then there is no monomorphism and epimorphism from QD_{2^α} into Q_n.*

Proof. The group QD_{2^α} contains $1 + 2^{\alpha-2}$ elements having order 2, while Q_n contains only one element having order 2. Thus there is no monomorphism from QD_{2^α} into Q_n.

By the Theorem 4.1, 4.2, we have at most 8 homomorphisms from QD_{2^α} into Q_n. We can verify that none of these homomorphisms are onto. □

5. The Number of Homomorphisms From QD_{2^α} into M_{p^β}

Theorem 5.1. *Let $p \neq 2$ be a prime number, $\alpha > 3$ and $\beta > 2$. Then there is only the trivial homomorphism from QD_{2^α} into M_{p^β}.*

Proof. Suppose $\rho : QD_{2^\alpha} \to M_{p^\beta}$ is a group homomorphism. Then $|\rho(s_\alpha)|$ divides $|s_\alpha| = 2^{\alpha-1}$ and $|\rho(t_\alpha)|$ divides $|t_\alpha| = 2$. That is the trivial homomorphism is the only homomorphism exist from QD_{2^α} into M_{p^β}, $p \neq 2$. □

Theorem 5.2. *Suppose $\alpha > 3$ and $\beta > 2$ are two positive integers. Then the number of homomorphisms from QD_{2^α} into M_{2^β} is 16.*

Proof. Suppose ρ is a group homomorphism from QD_{2^α} into M_{2^β}. Then $\rho(s_\alpha) = r_\beta^k$, where $|r_\beta^k|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$ or $\rho(s_\alpha) = r_\beta^k f_\beta$, where $|r_\beta^k|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$, and $\rho(t_\alpha) = r_\beta^{m_1 2^{\beta-2}} f_\beta^{m_2}, m_1, m_2 = 0, 1$.

Suppose $\rho(s_\alpha) = r_\beta^k$, where $|r_\beta^k|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$, and $\rho(t_\alpha) = r_\beta^{m_1 2^{\beta-2}} f_\beta^{m_2}$, where $m_1, m_2 = 0, 1$. Then $\rho(s_\alpha^l t_\alpha) = r_\beta^{lk+m_1 2^{\beta-2}} f_\beta^{m_2}$. Since ρ is a homomorphism, $|r_\beta^{lk+m_1 2^{\beta-2}} f_\beta^{m_2}| = |r_\beta^{lk+m_1 2^{\beta-2}}|$ must divide $|s_\alpha^l t_\alpha|$. This is possible only when $|r_\beta^k|$ divides 4. Then $\rho(s_\alpha)^{2^{\alpha-2}} = e$ and so $|\rho(s_\alpha^l t_\alpha)|$ must divide 2. Thus we have 2 choices for $\rho(s_\alpha)$ and 4 choices for $\rho(t_\alpha)$. Hence we get 8 homomorphisms in this case.

Suppose $\rho(s_\alpha) = r_\beta^k f_\beta$, where $|r_\beta^k|$ divides both $2^{\alpha-1}$ and $2^{\beta-1}$, and $\rho(t_\alpha) = r_\beta^{m_1 2^{\beta-2}} f_\beta^{m_2}$, where $m_1, m_2 = 0, 1$. Then $\rho(s_\alpha^l t_\alpha) = (r_\beta^k f_\beta)^l (r_\beta^{m_1 2^{\beta-2}} f_\beta^{m_2}) = r_\beta^{lk+lk2^{\beta-2}} f_\beta^l r_\beta^{m_1 2^{\beta-2}} f_\beta^{m_2}$. If l is even, $|s_\alpha^l t_\alpha| = 2$, and $\rho(s_\alpha^l t_\alpha) = r_\beta^{lk+lk2^{\beta-2}+m_1 2^{\beta-2}} f_\beta^{m_2}$. If l is odd $|s_\alpha^l t_\alpha| = 4$, and $\rho(s_\alpha^l t_\alpha) = r_\beta^{lk+lk2^{\beta-2}} (r_\beta^{m_1 2^{\alpha-2}})^{2^{\alpha-2}+1} f_\beta^{1+m_2}$. Then $|\rho(s_\alpha^l t_\alpha)|$ divides $|s_\alpha^l t_\alpha| = 4$, only when $|r_\beta^k|$ divides 4. Then $\rho(s_\alpha)^{2^{\alpha-2}} = e$ and so $|\rho(s_\alpha^l t_\alpha)|$ must divide 2. Thus we have 2 choices for $\rho(s_\alpha)$ and 4 choices for $\rho(t_\alpha)$. Hence we get 8 homomorphisms in this case. Hence we get the result. □

Corollary 5.1. *Suppose $\alpha > 3$ and $\beta > 2$ are two positive integers. Then there is no monomorphism and epimorphism from QD_{2^α} into M_{2^β}.*

Proof. The group QD_{2^α} contains $1 + 2^{\alpha-2}$ elements having order 2. But M_{2^α} has only two elements of order 2. Therefore there is no monomorphism from QD_{2^α} into M_{2^α}. Also we can verify that none of the homomorphisms obtained in the Theorem 5.2 are epimorphism. □

References

[1] M.Bate, *The number of homomorphisms from finite groups to classical groups*, J. Algebra, 308(2007), 612-628.

[2] J.A.Gallian and J.Van Buskirk, *The number of homomorphisms from $\mathbb{Z}_m$ into $\mathbb{Z}_n$*, Amer. Math. Monthly, 91(1984), 196-197.

[3] J.A.Gallian and D.S.Jungreis, *Homomorphisms from $\mathbb{Z}_m[i]$ into $\mathbb{Z}_n[i]$ and $\mathbb{Z}_m[\rho]$ into $\mathbb{Z}_n[\rho]$, where $i^2 + 1 = 0$ and $\rho^2 + \rho + 1 = 0$*, Amer. Math. Monthly, 95(1988), 247-249.

[4] Jeremiah Johnson, *The number of group homomorphisms from D_m into D_n*, The College Mathematics Journal, 44(2013), 190-192.

[5] D.Matei and A.Suciu, *Counting homomorphisms onto finite solvable groups*, J. Algebra, 286(2005), 161-186.

[6] R.Rajkumar, M.Gayathri, T.Anitha, *The number of homomorphisms from dihedral group in to some finite groups*, Mathematical Sciences International Research Journal, 4 (2015), 161-165.

[7] R.Rajkumar, M.Gayathri, T.Anitha, *The number of homomorphisms from quaternion group in to some finite groups*, International Journal of Mathematics and its Applications, 3(3–A)(2015), 23-30.

[8] R.Rajkumar, M.Gayathri, T.Anitha, *Enumeration of homomorphisms from modular group in to some finite groups*, International Journal of Mathematics and its Applications, 3(3–B)(2015), 15-19.