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# New Weakly Generalized locally Closed Sets

**Research Article**

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**Abstract:** The aim of this paper is to introduce and study the classes of $g^{\#}$-locally closed sets, $g^{\#}$-lc$^{\#}$ sets and $g^{\#}$-lc$^{\#\#}$ sets which are weaker forms of the class of locally closed sets. Furthermore the relations with other notions connected with the forms of locally closed sets are investigated.

**MSC:** 54C10, 54C08, 54C05.

**Keywords:** Topological space, g-closed set, $g^{\#}$-closed set, $g^{\#}$-lc$^{\#}$ set, $g^{\#}$-lc$^{\#\#}$ set.



## 1. Introduction

The first step of locally closedness was done by Bourbaki [3]. He defined a set A to be locally closed if it is the intersection of an open set and a closed set. In literature many general topologists introduced the studies of locally closed sets. Extensive research on locally closedness and generalizing locally closedness were done in recent years. Stone [16] used the term FG for a locally closed set. Ganster and Reilly used locally closed sets in [6] to define LC-continuity and LC-irresoluteness. Balachandran et al [1] introduced the concept of generalized locally closed sets.

In this paper, we introduce three forms of locally closed sets called $g^{\#}$-locally closed sets, $g^{\#}$-lc$^{\#}$ sets and $g^{\#}$-lc$^{\#\#}$ sets. Properties of these new concepts are studied as well as their relations to the other classes of locally closed sets are investigated.

## 2. Preliminaries

Throughout this paper (X, $\tau$) (or X) represents topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, $\tau$), cl(A), int(A) and A$^c$ denote the closure of A, the interior of A and the complement of A, respectively.

We recall the following Definitions, Remarks, Corollary and Theorem which are useful in the sequel.

**Definition 2.1.** *A subset A of a space (X, $\tau$) is called*

1. *semi-open set [9] if $A \subseteq cl(int(A))$;*

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2. *$\alpha$-open set [12] if $A \subseteq int(cl(int(A)))$;*

3. *regular open set [17] if $A=int(cl(A))$.*

*The complements of the above mentioned open sets are called their respective closed sets.*

*The semi-closure [4] of a subset A of X, denoted by scl(A), is defined to be the intersection of all semi-closed sets of $(X, \tau)$ containing A. It is known that scl(A) is a semi-closed set.*

*The $\alpha$-closure [11] of a subset A of X, denoted by $\alpha cl(A)$, is defined to be the intersection of all $\alpha$-closed sets of $(X, \tau)$ containing A. It is known that $\alpha cl(A)$ is a $\alpha$-closed set.*

**Definition 2.2.** *A subset A of a space $(X, \tau)$ is called*

1. *a generalized closed (briefly g-closed) set [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in $(X, \tau)$. The complement of g-closed set is called g-open set.*

2. *a regular generalized closed (briefly rg-closed) set [13] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in $(X, \tau)$. The complement of rg-closed set is called rg-open set.*

3. *a semi-generalized closed (briefly sg-closed) set [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in $(X, \tau)$. The complement of sg-closed set is called sg-open set.*

4. *an $\alpha g$-closed [10] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in $(X, \tau)$. The complement of $\alpha g$-closed set is called $\alpha g$-open.*

5. *$g^{\#}$-closed [19] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\alpha g$-open in $(X, \tau)$.*

6. *$g^{\#}$-open [19] if $A^c$ is $g^{\#}$-closed.*

**Remark 2.3.** *The collection of all $g^{\#}$-closed (resp. g-closed) sets in X is denoted by $G^{\#}C(X)$ (resp. GC(X)). The collection of all $g^{\#}$-open (resp. g-open) sets in X is denoted by $G^{\#}O(X)$(resp. GO(X)). We denote the power set of X by P(X).*

**Definition 2.4.** *A subset S of a space $(X, \tau)$ is called*

1. *locally closed (briefly lc) [6] if $S=U \cap F$, where U is open and F is closed in $(X, \tau)$.*

2. *generalized locally closed (briefly glc) [1] if $S=U \cap F$, where U is g-open and F is g-closed in $(X, \tau)$.*

3. *semi-generalized locally closed (briefly sglc) [14] if $S=U \cap F$, where U is sg-open and F is sg-closed in $(X, \tau)$.*

4. *generalized locally semi-closed (briefly glsc) [7] if $S=U \cap F$, where U is g-open and F is semi-closed in $(X, \tau)$.*

5. *locally semi-closed (briefly lsc) [7] if $S=U \cap F$, where U is open and F is semi-closed in $(X, \tau)$.*

6. *$\alpha$-locally closed (briefly $\alpha$-lc) [7] if $S=U \cap F$, where U is $\alpha$-open and F is $\alpha$-closed in $(X, \tau)$.*

   *The class of all locally closed (resp. generalized locally closed, generalized locally semi-closed, locally semi-closed) sets in X is denoted by LC(X) (resp. GLC(X), GLSC(X), LSC(X)).*

**Definition 2.5** ([15])**.** *For any $A \subseteq X$, the $g^{\#}$-interior of a subset A of X, $g^{\#}$-int(A), is defined as the union of all $g^{\#}$-open sets contained in A. i.e., $g^{\#}\text{-}int(A)=\cup\{G : G \subseteq A$ and G is $g^{\#}$-open$\}$.*

**Definition 2.6** ([15])**.** *For every set $A \subseteq X$, we define the $g^{\#}$-closure of A to be the intersection of all $g^{\#}$-closed sets containing A. i.e., $g^{\#}\text{-}cl(A)=\cap\{F : A \subseteq F \in G^{\#}C(X)\}$.*

**Definition 2.7** ([15]). *A space (X, $\tau$) is called a $T^{\#}{}_{\frac{1}{2}}$-space if every $g^{\#}$-closed set in it is closed.*

Recall that a subset A of a space (X, $\tau$) is called dense if cl(A)=X.

**Definition 2.8.** *A topological space (X, $\tau$) is called*

1. *submaximal [5, 18] if every dense subset is open.*
2. *g-submaximal [1] if every dense subset is g-open.*
3. *rg-submaximal [13] if every dense subset is rg-open.*

**Remark 2.9.** *For a topological space (X, $\tau$), the following statements hold:*

1. *Every closed set is $g^{\#}$-closed but not conversely [19].*
2. *Every $g^{\#}$-closed set is g-closed but not conversely [19].*
3. *A subset A of X is $g^{\#}$-closed if and only if $g^{\#}$-cl(A)=A [15].*
4. *A subset A of X is $g^{\#}$-open if and only if $g^{\#}$-int(A)=A [15].*

**Corollary 2.10** ([15]). *If A is a $g^{\#}$-closed set and F is a closed set, then A∩F is a $g^{\#}$-closed set.*

**Theorem 2.11** ([18]). *Let (X, $\tau$) be a topological space.*

1. *If X is g-submaximal, then X is rg-submaximal.*
2. *The converse of the above need not be true in general.*

# 3. $g^{\#}$-Locally Closed Sets

We introduce the following definition.

**Definition 3.1.** *A subset A of (X, $\tau$) is called $g^{\#}$-locally closed (briefly $g^{\#}$-lc) if A=S∩G, where S is $g^{\#}$-open and G is $g^{\#}$-closed in (X, $\tau$).*
*The class of all $g^{\#}$-locally closed sets in X is denoted by $G^{\#}LC(X)$.*

**Proposition 3.2.** *Every $g^{\#}$-closed (resp. $g^{\#}$-open) set is $g^{\#}$-lc set but not conversely.*

*Proof.* It follows from Definition 3.1. □

**Example 3.3.** *Let X={a, b, c} with $\tau$={∅, {a}, X}. Then the set {a} is $g^{\#}$-lc set but it is not $g^{\#}$-closed and the set {b, c} is $g^{\#}$-lc set but it is not $g^{\#}$-open in (X, $\tau$).*

**Proposition 3.4.** *Every lc set is $g^{\#}$-lc set but not conversely.*

*Proof.* It follows from Remark 2.9 (1). □

**Example 3.5.** *Let X={a, b, c} with $\tau$={∅, {b, c}, X}. Then the set {b} is $g^{\#}$-lc set but it is not lc set in (X, $\tau$).*

**Proposition 3.6.** *Every $g^{\#}$-lc set is a (1) glc set and (2) sglc set. However the separate converses are not true.*

*Proof.* It follows from Remark 2.9 (2) and the fact that every $g^{\#}$-closed set is *sg*-closed. □

**Example 3.7.** *Let X={a, b, c} with $\tau$={∅, {a}, {a, b}, X}. Then the set {a, c} is glc set but it is not $g^{\#}$-lc set in (X, $\tau$). Moreover, the set {a, c} is sglc set but it is not $g^{\#}$-lc set in (X, $\tau$).*

**Remark 3.8.** *The concepts of $\alpha$-lc set and $g^{\#}$-lc set are independent of each other.*

**Example 3.9.** *The set {a, c} in Example 3.7 is $\alpha$-lc set but it is not a $g^{\#}$-lc set in (X, $\tau$).*

**Example 3.10.** *Let X={a, b, c} with $\tau$={∅, {a, b}, X}. Then set {a} is $g^{\#}$-lc set but it is not an $\alpha$-lc set in (X, $\tau$).*

**Remark 3.11.** *The concepts of lsc set and $g^{\#}$-lc set are independent of each other.*

**Example 3.12.** *The set {b} in Example 3.3 is lsc set but it is not a $g^{\#}$-lc set in (X, $\tau$) and the set {a} in Example 3.10 is $g^{\#}$-lc set but it is not a lsc set in (X, $\tau$).*

**Remark 3.13.** *The concepts of $g^{\#}$-lc set and glsc set are independent of each other.*

**Example 3.14.** *The set {a, c} in Example 3.3 is glsc set but it is not a $g^{\#}$-lc set in (X, $\tau$) and the set {a, c} in Example 3.10 is $g^{\#}$-lc set but it is not a glsc set in (X, $\tau$).*

**Theorem 3.15.** *For a $T^{\#}{}_{\frac{1}{2}}$-space (X, $\tau$), the following properties hold:*

1. *$G^{\#}$LC(X)=LC(X).*
2. *$G^{\#}$LC(X)⊆GLC(X).*
3. *$G^{\#}$LC(X)⊆GLSC(X).*

*Proof.* 1. Since every $g^{\#}$-open set is open and every $g^{\#}$-closed set is closed in (X, $\tau$), $G^{\#}$LC(X)⊆LC(X) and hence $G^{\#}$LC(X)=LC(X).

(2), (3) since for any space (X, $\tau$), LC(X)⊆GLC(X), LC(X)⊆GLSC(X).

□

**Corollary 3.16.** *If GO(X)=$\tau$, then $G^{\#}$LC(X)⊆GLSC(X)⊆LSC(X).*

*Proof.* GO(X)=$\tau$ implies that (X, $\tau$) is a $T^{\#}{}_{\frac{1}{2}}$-space and hence by Theorem 3.15, $G^{\#}$LC(X)⊆GLSC(X). Let A∈GLSC(X). Then A=U∩F, where U is g-open and F is semi-closed. By hypothesis, U is open and hence A is a lsc set and so A∈LSC(X). □

**Definition 3.17.** *A subset A of a space (X, $\tau$) is called*

1. *$g^{\#}$-$lc^{\#}$ set if A=S∩G, where S is $g^{\#}$-open in (X, $\tau$) and G is closed in (X, $\tau$).*
2. *$g^{\#}$-$lc^{\#\#}$ set if A=S∩G, where S is open in (X, $\tau$) and G is $g^{\#}$-closed in (X, $\tau$).*

*The class of all $g^{\#}$-$lc^{\#}$ (resp. $g^{\#}$-$lc^{\#\#}$) sets in a topological space (X, $\tau$) is denoted by $G^{\#}LC^{\#}$(X) (resp. $G^{\#}LC^{\#\#}$(X)).*

**Proposition 3.18.** *Every lc set is $g^{\#}$-$lc^{\#}$ set but not conversely.*

*Proof.* It follows from Definitions 2.4 (1) and 3.17 (1). □

**Example 3.19.** *The set {b} in Example 3.10 is $g^{\#}$-$lc^{\#}$ set but it is not a lc set in (X, $\tau$).*

**Proposition 3.20.** *Every lc set is $g^{\#}$-$lc^{\#\#}$ set but not conversely.*

*Proof.* It follows from Definitions 2.4 (1) and 3.17 (2). □

**Example 3.21.** *The set $\{a, c\}$ in Example 3.10 is $g^{\#}$-lc$^{\#\#}$ set but it is not a lc set in $(X, \tau)$.*

**Proposition 3.22.** *Every $g^{\#}$-lc$^{\#}$ set is $g^{\#}$-lc set but not conversely.*

*Proof.* It follows from Definitions 3.1 and 3.17 (1). □

**Example 3.23.** *The set $\{a, c\}$ in Example 3.10 is $g^{\#}$-lc set but it is not a $g^{\#}$-lc$^{\#}$ set in $(X, \tau)$.*

**Proposition 3.24.** *Every $g^{\#}$-lc$^{\#\#}$ set is $g^{\#}$-lc set.*

*Proof.* It follows from Definitions 3.1 and 3.17 (2).
The converse of Proposition 3.24 is not true as shown by the following Example.
Let $X = \mathbb{R}$, the set of all real numbers, with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$ where $\mathbb{Q}$ is the set of all rational numbers. Then $\mathbb{N}$, the set of natural numbers, is $g^{\#}$-lc set but not $g^{\#}$-lc$^{\#\#}$ set.
**Solution**: Since $\mathbb{N} \in \tau$, $\mathbb{N}$ is not open. Since $\alpha cl(\mathbb{N}) = \mathbb{R}$, $\mathbb{N} \subseteq \mathbb{Q}$ and $\mathbb{Q} \in \tau$, $\mathbb{N}$ is not $\alpha g$-closed and $\mathbb{R}\backslash\mathbb{N}$ is not $\alpha g$-open. It is clear that $\mathbb{R}\backslash\mathbb{N}$ is $\alpha g$-closed, being $\mathbb{R}$ is the only open set containing $\mathbb{R}\backslash\mathbb{N}$. Since $cl(\mathbb{N}) = \mathbb{R} \nsubseteq \mathbb{N}$, $\mathbb{N} \subseteq \mathbb{N}$ and $\mathbb{N}$ is $\alpha g$-open, $\mathbb{N}$ is not $g^{\#}$-closed. Since $\mathbb{R}\backslash\mathbb{N}$ is not $\alpha g$-open and $\mathbb{R}$ is the only $\alpha g$-open set containing $\mathbb{R}\backslash\mathbb{N}$, $\mathbb{R}\backslash\mathbb{N}$ is $g^{\#}$-closed and hence $\mathbb{N}$ is $g^{\#}$-open. Since $\mathbb{N}$ is $g^{\#}$-open, $\mathbb{N}$ is $g^{\#}$-lc set. Also, $\mathbb{N}$ is not $g^{\#}$-lc$^{\#\#}$ set, since $\mathbb{N}$ is neither open nor $g^{\#}$-closed. Hence the solution. □

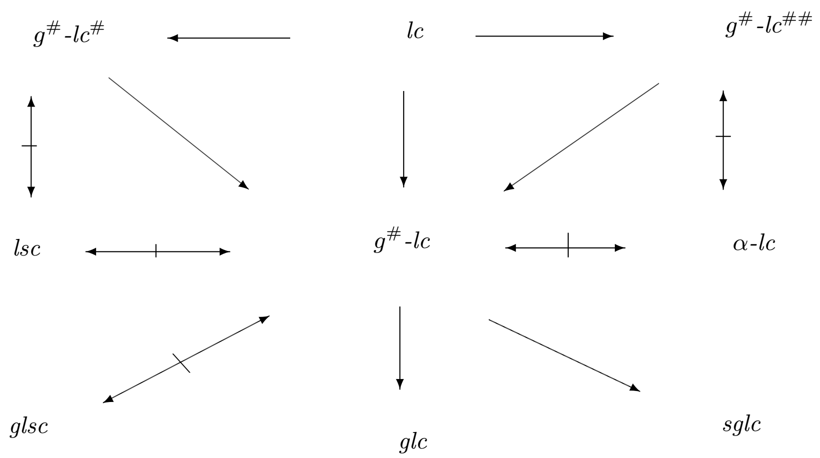
**Remark 3.25.** *The concepts of $g^{\#}$-lc$^{\#}$ set and lsc set are independent of each other.*

**Example 3.26.** *The set $\{a\}$ in Example 3.10 is $g^{\#}$-lc$^{\#}$ set but it is not a lsc set in $(X, \tau)$ and the set $\{b\}$ in Example 3.3 is lsc set but it is not a $g^{\#}$-lc$^{\#}$ set in $(X, \tau)$.*

**Remark 3.27.** *The concepts of $g^{\#}$-lc$^{\#\#}$ set and $\alpha$-lc set are independent of each other.*

**Example 3.28.** *The set $\{a, c\}$ in Example 3.10 is $g^{\#}$-lc$^{\#\#}$ set but it is not an $\alpha$-lc set in $(X, \tau)$ and the set $\{a, c\}$ in Example 3.7 is $\alpha$-lc set but it is not a $g^{\#}$-lc$^{\#\#}$ set in $(X, \tau)$.*

**Remark 3.29.** *From the above discussions we have the following implications where $A \rightarrow B$ (resp. $A \nleftrightarrow B$) represents A implies B but not conversely (resp. A and B are independent of each other).*



**Proposition 3.30.** *If GO(X)=$\tau$, then $G^{\#}LC(X)=G^{\#}LC^{\#}(X)=G^{\#}LC^{\#\#}(X)$.*

*Proof.* For any space (X, $\tau$), $\tau \subseteq G^{\#}O(X) \subseteq GO(X)$. Therefore by hypothesis, $G^{\#}O(X)=\tau$. i.e., (X, $\tau$) is a $T^{\#}_{\frac{1}{2}}$-space and hence $G^{\#}LC(X)=G^{\#}LC^{\#}(X)=G^{\#}LC^{\#\#}(X)$. □

**Remark 3.31.** *The converse of Proposition 3.30 need not be true.*
*For the topological space (X, $\tau$) in Example 3.3, $G^{\#}LC(X)=G^{\#}LC^{\#}(X)=G^{\#}LC^{\#\#}(X)$ holds. However GO(X)=$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\} \neq \tau$.*

**Proposition 3.32.** *Let (X, $\tau$) be a topological space. If GO(X)$\subseteq$LC(X), then $G^{\#}LC(X)=G^{\#}LC^{\#\#}(X)$.*

*Proof.* Let $A \in G^{\#}LC(X)$. Then A=S$\cap$G where S is $g^{\#}$-open and G is $g^{\#}$-closed. Since $G^{\#}O(X) \subseteq GO(X)$ and by hypothesis GO(X)$\subseteq$LC(X), S is locally closed. Then S=P$\cap$Q, where P is open and Q is closed. Therefore, A=P$\cap$(Q$\cap$G). By Corollary 2.10, Q$\cap$G is $g^{\#}$-closed and hence $A \in G^{\#}LC^{\#\#}(X)$. i.e., $G^{\#}LC(X) \subseteq G^{\#}LC^{\#\#}(X)$. For any topological space, $G^{\#}LC^{\#\#}(X) \subseteq G^{\#}LC(X)$ and so $G^{\#}LC(X)=G^{\#}LC^{\#\#}(X)$. □

**Remark 3.33.** *The converse of Proposition 3.32 need not be true in general. For the topological space (X, $\tau$) in Example 3.3, we have $G^{\#}LC(X)=G^{\#}LC^{\#\#}(X) = \{\emptyset, X, \{a\}, \{b, c\}\}$. But GO(X)=$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\} \nsubseteq$ LC(X)=$\{\emptyset, \{a\}, \{b, c\}, X\}$. The following results are characterizations of $g^{\#}$-lc sets, $g^{\#}$-lc$^{\#}$ sets and $g^{\#}$-lc$^{\#\#}$ sets.*

**Theorem 3.34.** *Assume that $G^{\#}C(X)$ is closed under finite intersections. For a subset A of (X, $\tau$) the following statements are equivalent:*

1. *$A \in G^{\#}LC(X)$,*
2. *A=S$\cap g^{\#}$-cl(A) for some $g^{\#}$-open set S,*
3. *$g^{\#}$-cl(A)$-$A is $g^{\#}$-closed,*
4. *A$\cup(g^{\#}$-cl(A))$^c$ is $g^{\#}$-open,*
5. *A$\subseteq g^{\#}$-int(A$\cup(g^{\#}$-cl(A))$^c$).*

*Proof.* (1)$\Rightarrow$(2). Let $A \in G^{\#}LC(X)$. Then A=S$\cap$G where S is $g^{\#}$-open and G is $g^{\#}$-closed. Since A$\subseteq$G, $g^{\#}$-cl(A)$\subseteq$G and so S$\cap g^{\#}$-cl(A)$\subseteq$A. Also A$\subseteq$S and A$\subseteq g^{\#}$-cl(A) implies A$\subseteq$S$\cap g^{\#}$-cl(A) and therefore A=S$\cap g^{\#}$-cl(A).
(2)$\Rightarrow$(3). A=S$\cap g^{\#}$-cl(A) implies $g^{\#}$-cl(A)$-$A=$g^{\#}$-cl(A)$\cap S^c$ which is $g^{\#}$-closed since $S^c$ is $g^{\#}$-closed and $g^{\#}$-cl(A) is $g^{\#}$-closed.
(3)$\Rightarrow$(4). A$\cup(g^{\#}$-cl(A))$^c$=($g^{\#}$-cl(A)$-$A)$^c$ and by assumption, ($g^{\#}$-cl(A)$-$A)$^c$ is $g^{\#}$-open and so is A$\cup(g^{\#}$-cl(A))$^c$.
(4)$\Rightarrow$(5). By assumption, A$\cup(g^{\#}$-cl(A))$^c$=$g^{\#}$-int(A$\cup(g^{\#}$-cl(A))$^c$) and hence A$\subseteq g^{\#}$-int(A$\cup(g^{\#}$-cl(A))$^c$).
(5)$\Rightarrow$(1). By assumption and since A$\subseteq g^{\#}$-cl(A), A=$g^{\#}$-int(A$\cup(g^{\#}$-cl(A))$^c$)$\cap g^{\#}$-cl(A). Therefore, $A \in G^{\#}LC(X)$. □

**Theorem 3.35.** *For a subset A of (X, $\tau$), the following statements are equivalent:*

1. *$A \in G^{\#}LC^{\#}(X)$,*
2. *A=S$\cap$cl(A) for some $g^{\#}$-open set S,*
3. *cl(A)$-$A is $g^{\#}$-closed,*
4. *A$\cup$(cl(A))$^c$ is $g^{\#}$-open.*

*Proof.* (1)⇒(2). Let A $\in G^{\#}LC^{\#}(X)$. There exist an $g^{\#}$-open set S and a closed set G such that A=S∩G. Since A⊆S and A⊆cl(A), A⊆S∩cl(A). Also since cl(A)⊆G, S∩cl(A)⊆S∩G=A. Therefore A=S∩cl(A).

(2)⇒(1). Since S is $g^{\#}$-open and cl(A) is a closed set, A=S∩cl(A)$\in G^{\#}LC^{\#}(X)$.

(2)⇒(3). Since cl(A)−A=cl(A)∩$S^c$, cl(A)−A is $g^{\#}$-closed by Corollary 2.10.

(3)⇒(2). Let S=$(cl(A)-A)^c$. Then S is $g^{\#}$-open in (X, $\tau$) and A=S∩cl(A).

(3)⇒(4). Let G=cl(A)−A. Then $G^c$=A∪$(cl(A))^c$ and A∪$(cl(A))^c$ is $g^{\#}$-open.

(4)⇒(3). Let S=A∪$(cl(A))^c$. Then $S^c$ is $g^{\#}$-closed and $S^c$=cl(A)−A and so cl(A)−A is $g^{\#}$-closed. □

**Theorem 3.36.** *Let A be a subset of (X, $\tau$). Then A$\in G^{\#}LC^{\#\#}(X)$ if and only if A=S∩$g^{\#}$-cl(A) for some open set S.*

*Proof.* Let A$\in G^{\#}LC^{\#\#}(X)$. Then A=S∩G where S is open and G is $g^{\#}$-closed. Since A⊆G, $g^{\#}$-cl(A)⊆G. We obtain A=A∩$g^{\#}$-cl(A) =S∩G∩$g^{\#}$-cl(A)=S∩$g^{\#}$-cl(A).

Converse part is trivial. □

**Corollary 3.37.** *Let A be a subset of (X, $\tau$). If A$\in G^{\#}LC^{\#\#}(X)$, then $g^{\#}$-cl(A)−A is $g^{\#}$-closed and A∪$(g^{\#}\text{-}cl(A))^c$ is $g^{\#}$-open.*

*Proof.* Let A$\in G^{\#}LC^{\#\#}(X)$. Then by Theorem 3.36, A=S∩$g^{\#}$-cl(A) for some open set S and $g^{\#}$-cl(A)−A=$g^{\#}$-cl(A)∩$S^c$ is $g^{\#}$-closed in (X, $\tau$). If G=$g^{\#}$-cl(A)−A, then $G^c$=A∪$(g^{\#}\text{-}cl(A))^c$ and $G^c$ is $g^{\#}$-open and so is A∪$(g^{\#}\text{-}cl(A))^c$. □

# 4. $g^{\#}$-dense Sets and $g^{\#}$-submaximal Spaces

We introduce the following definition.

**Definition 4.1.** *A subset A of a space (X, $\tau$) is called $g^{\#}$-dense if $g^{\#}$-cl(A)=X.*

**Example 4.2.** *Consider the topological space (X, $\tau$) in Example 3.5. Then the set A={b, c} is $g^{\#}$-dense in (X, $\tau$).*

**Proposition 4.3.** *Every $g^{\#}$-dense set is dense.*

*Proof.* Let A be an $g^{\#}$-dense set in (X, $\tau$). Then $g^{\#}$-cl(A)=X. Since $g^{\#}$-cl(A)⊆cl(A), we have cl(A)=X and so A is dense.

The converse of Proposition 4.3 need not be true as seen from the following example. □

**Example 4.4.** *The set {a, c} in Example 3.10 is a dense in (X, $\tau$) but it is not $g^{\#}$-dense in (X, $\tau$).*

**Definition 4.5.** *A topological space (X, $\tau$) is called $g^{\#}$-submaximal if every dense subset in it is $g^{\#}$-open in (X, $\tau$).*

**Proposition 4.6.** *Every submaximal space is $g^{\#}$-submaximal.*

*Proof.* Let (X, $\tau$) be a submaximal space and A be a dense subset of (X, $\tau$). Then A is open. But every open set is $g^{\#}$-open and so A is $g^{\#}$-open. Therefore (X, $\tau$) is $g^{\#}$-submaximal.

The converse of Proposition 4.6 need not be true as seen from the following example. □

**Example 4.7.** *Let X={a, b, c} with $\tau$={∅, X}. Then $G^{\#}O(X)=P(X)$. We have every dense subset is $g^{\#}$-open and hence (X, $\tau$) is $g^{\#}$-submaximal. However, the set A={a} is dense in (X, $\tau$), but it is not open in (X, $\tau$). Therefore (X, $\tau$) is not submaximal.*

**Proposition 4.8.** *Every $g^{\#}$-submaximal space is g-submaximal.*

*Proof.* Let (X, $\tau$) be a $g^{\#}$-submaximal space and A be a dense subset of (X, $\tau$). Then A is $g^{\#}$-open. But every $g^{\#}$-open set is g-open and so A is g-open. Therefore (X, $\tau$) is g-submaximal.

The converse of Proposition 4.8 need not be true as seen from the following example. □

**Example 4.9.** *Let X={a, b, c} with $\tau$={∅, {a}, X}. Then GO(X)={∅, {a}, {b}, {c}, {a, b}, {a, c}, X} and $G^{\#}O(X)$={∅, X, {a}}. Every dense subset is g-open and hence (X, $\tau$) is g-submaximal. However, the set A={a, b} is dense in (X, $\tau$), but it is not $g^{\#}$-open in (X, $\tau$). Therefore (X, $\tau$) is not $g^{\#}$-submaximal.*

**Remark 4.10.** *From Propositions 4.6, 4.8 and Theorem 2.11, we have the following diagram:*
*submaximal → $g^{\#}$-submaximal → g-submaximal → rg-submaximal*

**Theorem 4.11.** *A space (X, $\tau$) is $g^{\#}$-submaximal if and only if P(X)=$G^{\#}LC^{\#}(X)$.*

*Proof.* Necessity. Let A∈P(X) and let V=A∪$(cl(A))^c$. This implies that cl(V)=cl(A)∪ $(cl(A))^c$=X. Hence cl(V)=X. Therefore V is a dense subset of X. Since (X, $\tau$) is $g^{\#}$-submaximal, V is $g^{\#}$-open. Thus A∪$(cl(A))^c$ is $g^{\#}$-open and by Theorem 3.35, we have A∈$G^{\#}LC^{\#}$(X).
Sufficiency. Let A be a dense subset of (X, $\tau$). This implies A∪$(cl(A))^c$=A∪$X^c$=A∪∅=A. Now A∈$G^{\#}LC^{\#}$(X) implies that A=A∪$(cl(A))^c$ is $g^{\#}$-open by Theorem 3.35. Hence (X, $\tau$) is $g^{\#}$-submaximal. □

**Proposition 4.12.** *Assume that $G^{\#}O(X)$ forms a topology. For subsets A and B in (X, $\tau$), the following are true:*

1. *If A, B∈$G^{\#}LC(X)$, then A∩B∈$G^{\#}LC(X)$.*
2. *If A, B∈$G^{\#}LC^{\#}(X)$, then A∩B∈$G^{\#}LC^{\#}(X)$.*
3. *If A, B∈$G^{\#}LC^{\#\#}(X)$, then A∩B∈$G^{\#}LC^{\#\#}(X)$.*
4. *If A∈$G^{\#}LC(X)$ and B is $g^{\#}$-open (resp. $g^{\#}$-closed), then A∩B∈$G^{\#}LC(X)$.*
5. *If A∈$G^{\#}LC^{\#}(X)$ and B is $g^{\#}$-open (resp. closed), then A∩B∈$G^{\#}LC^{\#}(X)$.*
6. *If A∈$G^{\#}LC^{\#\#}(X)$ and B is $g^{\#}$-closed (resp. open), then A∩B∈$G^{\#}LC^{\#\#}(X)$.*
7. *If A∈$G^{\#}LC^{\#}(X)$ and B is $g^{\#}$-closed, then A∩B∈$G^{\#}LC(X)$.*
8. *If A∈$G^{\#}LC^{\#\#}(X)$ and B is $g^{\#}$-open, then A∩B∈$G^{\#}LC(X)$.*
9. *If A∈$G^{\#}LC^{\#\#}(X)$ and B∈$G^{\#}LC^{\#}(X)$, then A∩B∈$G^{\#}LC(X)$.*

*Proof.* By Remark 2.9 and Corollary 2.10, (1) to (8) hold.
(9). Let A=S∩G where S is open and G is $g^{\#}$-closed and B=P∩Q where P is $g^{\#}$-open and Q is closed. Then A∩B=(S∩P)∩(G∩Q) where S∩P is $g^{\#}$-open and G∩Q is $g^{\#}$-closed, by Corollary 2.10. Therefore A∩B∈$G^{\#}$LC(X). □

**Remark 4.13.** *Union of two $g^{\#}$-lc sets (resp. $g^{\#}$-lc$^{\#}$ sets, $g^{\#}$-lc$^{\#\#}$ sets) need not be an $g^{\#}$-lc set (resp. $g^{\#}$-lc$^{\#}$ set, $g^{\#}$-lc$^{\#\#}$ set) as seen from the following Examples.*

**Example 4.14.** *In Example 3.7, we have $G^{\#}LC(X)$={∅, {a}, {b}, {c}, {a, b}, {b, c}, X}. Then the sets {a} and {c} are $g^{\#}$-lc sets, but their union {a, c}∉$G^{\#}LC(X)$.*

**Example 4.15.** *In Example 3.7, we have $G^{\#}LC^{\#}(X)$=$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$. Then the sets $\{a\}$ and $\{c\}$ are $g^{\#}$-lc$^{\#}$ sets, but their union $\{a, c\} \notin G^{\#}LC^{\#}(X)$.*

**Example 4.16.** *In Example 3.7, we have $G^{\#}LC^{\#\#}(X)$= $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$. Then the sets $\{a\}$ and $\{c\}$ are $g^{\#}$-lc$^{\#\#}$ sets, but their union $\{a, c\} \notin G^{\#}LC^{\#\#}(X)$.*

We introduce the following definition.

**Definition 4.17.** *Let A and B be subsets of $(X, \tau)$. Then A and B are said to be $g^{\#}$-separated if $A \cap g^{\#}$-cl(B)=$\emptyset$ and $g^{\#}$-cl(A)$\cap$B=$\emptyset$.*

**Example 4.18.** *For the topological space $(X, \tau)$ of Example 3.10. Let A=$\{a\}$ and B=$\{b\}$. Then $g^{\#}$-cl(A)=$\{a, c\}$ and $g^{\#}$-cl(B)=$\{b, c\}$ and so the sets A and B are $g^{\#}$-separated.*

**Proposition 4.19.** *Assume that $G^{\#}O(X)$ forms a topology. For a topological space $(X, \tau)$, the following are true:*

1. *Let A, B$\in G^{\#}LC(X)$. If A and B are $g^{\#}$-separated then A$\cup$B$\in G^{\#}LC(X)$.*
2. *Let A, B$\in G^{\#}LC^{\#}(X)$. If A and B are separated (i.e., A$\cap$cl(B)=$\emptyset$ and cl(A)$\cap$B=$\emptyset$), then A$\cup$B$\in G^{\#}LC^{\#}(X)$.*
3. *Let A, B$\in G^{\#}LC^{\#\#}(X)$. If A and B are $g^{\#}$-separated then A$\cup$B$\in G^{\#}LC^{\#\#}(X)$.*

*Proof.* 1. Since A, B$\in$G$^{\#}$LC(X), by Theorem 3.34, there exist $g^{\#}$-open sets U and V of $(X, \tau)$ such that A=U$\cap g^{\#}$-cl(A) and B=V$\cap g^{\#}$-cl(B). Now G=U$\cap$(X $-g^{\#}$-cl(B)) and H=V$\cap$(X$-g^{\#}$-cl(A)) are $g^{\#}$-open subsets of $(X, \tau)$. Since A$\cap g^{\#}$-cl(B)=$\emptyset$, A$\subseteq(g^{\#}$-cl(B))$^c$. Now A=U$\cap g^{\#}$-cl(A) becomes A$\cap(g^{\#}$-cl(B))$^c$ =G$\cap g^{\#}$-cl(A). Then A=G$\cap g^{\#}$-cl(A). Similarly B=H$\cap g^{\#}$-cl(B). Moreover G$\cap$ $g^{\#}$-cl(B)=$\emptyset$ and H$\cap g^{\#}$-cl(A)=$\emptyset$. Since G and H are $g^{\#}$-open sets of $(X, \tau)$, G$\cup$H is $g^{\#}$-open. Therefore A$\cup$B=(G$\cup$H)$\cap g^{\#}$-cl(A$\cup$B) and hence A$\cup$B $\in$G$^{\#}$LC(X).

(2) and (3) are similar to (1), using Theorems 3.35 and 3.36.

□

**Remark 4.20.** *The assumption that A and B are $g^{\#}$-separated in (1) of Proposition 4.19 cannot be removed. In Example 3.7, the sets $\{a\}$ and $\{c\}$ are not $g^{\#}$-separated and their union $\{a, c\} \notin G^{\#}LC(X)$.*

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