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Left-invertible Linear Transformations

Research Article

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Abstract: In this paper we consider a linear transformation T of an n -dimensional vector space V into an m-dimensional vectorspace W and we provide necessary and sufficient conditions for T to have an inverse from the left side. Moreover, we characterize the class of all left inverses of T.

MSC: 15C02.

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1. Introduction

A linear transformation $T : V \to W$ of vector spaces is said to be an invertible if there is another linear transformation denoted by $T^{-1} : W \to V$ such that $T^{-1} \circ T = I_V = T \circ T^{-1}$, where I_V is an identity operator of V. In this case this T^{-1} is unique and is called an inverse of T. It is generally known that a linear transformation of vector spaces is an invertible if and only if it is a bijection. Otherwise, it is non invertible [3]. But there are some linear transformations which have an inverse from one side only; from the left side or from the right side. In this paper we provide necessary and sufficient conditions to those linear transformations having an inverse from the left side only and we characterize the class of all left inverses of these transformations in the case of finite dimensional vector spaces.

2. Preliminaries

Definition 2.1. *Let V and W be vector spaces over a field F. A mapping $T : V \to W$ is called a linear transformation of V into W if it satisfies the following properties;*

$$T(x + y) = Tx + Ty \text{ for all } x, y \in V \text{ and}$$
$$T(\alpha x) = \alpha Tx \text{ for all } x \in V \text{ and all scalars } \alpha \in F.$$

These two properties are called the linearity properties [1–3].

Definition 2.2. *Let $T : V \to W$ be a linear transformation. Define kernel of T (or the null space of T) and Image of T respectively by: $\ker T = \{x \in V : Tx = 0\}$ and $Img\ T = \{Tx : x \in V\}$. Then both $\ker T$ and $Img\ T$ are subspaces of V and W respectively [1].*

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Theorem 2.3 ([1–3])**.** *For any linear transformation* $T : V \to W$*. If* V *and* W *are finite dimensional say* $\dim V = n$ *and* $\dim W = m$*, then* $\dim V = \dim(\ker\ T) + \dim(Img\ T)$*.*

3. Left-Invertible Linear Transformations

Definition 3.1. *Let* V *and* W *be vector spaces over a field* F*. A linear transformation* $T : V \to W$ *is called an injection if:* $Tx = Ty \Rightarrow x = y$ *for all* $x, y \in V$*.*

Theorem 3.2 ([2])**.** *A linear transformation* $T : V \to W$ *is an injection if and only if* $\ker T = \{0\}$*.*

Proof. Suppose that T is an injection. Then it is clear that $T(0) = 0 \Longrightarrow 0 \in \ker T \Longrightarrow \{0\} \subseteq \ker T$. On the other hand:

$$\begin{aligned} x \in \ker T &\Longrightarrow Tx = 0 = T0 \\ &\Longrightarrow Tx = T0 \\ &\Longrightarrow x = 0 \quad (\because \text{T is an injection}) \\ &\Longrightarrow \ker T \subseteq \{0\} \subseteq \ker T \end{aligned}$$

Therefore, $\ker T = \{0\}$.

Conversely suppose that $\ker T = \{0\}$. For any $x,\ y \in V$,

$$Tx = Ty \Longrightarrow Tx - Ty = 0 \Longrightarrow T(x - y) = 0 \Longrightarrow x - y \in \ker T = \{0\} \Longrightarrow x - y = 0 \Longrightarrow x = y$$

Therefore T is an injection. □

Theorem 3.3. *Let* V *and* W *be finite dimensional vector spaces over the given field* F*. If a linear transformation* $T : V \to W$ *is an injection, then* $\dim V \le \dim W$ *.*

Proof. It is clear that $Img\ T$ is a subspace of W and hence $\dim(Img\ T) \le \dim W$. Also, from theorem 1 we have

$$\begin{aligned} \dim V &= \dim(\ker T) + \dim(Img\ T) \\ \Longrightarrow \dim V &= 0 + \dim(Img\ T)\ (\because \ker T = \{0\}) \\ \Longrightarrow \dim V &= \dim(Img\ T) \le \dim W . \end{aligned}$$

□

Definition 3.4. *A linear transformation* $T : V \to W$ *is is said to be left invertible if there exists a linear transformation* $T_l : W \to V$ *such that* $T_l \circ T = I_V$ *where* I_V *is an identity operator of* V*.*

Definition 3.5. *A linear transformation* $T : V \to W$ *is is said to be left cancellable if for any vector space* U *over the same field* F *and any linear transformations* $T_1,\ T_2 : U \to V$*;* $T \circ T_1 = T \circ T_2 \Longrightarrow T_1 = T_2$*.*

The next theorem gives us two equivalent conditions (necessary and sufficient conditions) to a given linear transformation to have a left inverse and it is included here for the completeness of the paper.

Theorem 3.6. *Let* V *and* W *be finite dimensional vector spaces over a field* F *such that* $\dim V = n$ *and* $\dim W = m$*. Then the following are equivalent for any linear transformation* $T : V \to W$*.*

(i) T is an injection

(ii) T is Left Invertible

(iii) T is Left cancellable

Proof. (i) $\Longrightarrow$ (ii) Suppose that T is an injection and let $\{x_1, x_2, \ldots, x_n\}$ be an arbitrary basis for V. Then $\{Tx_1, Tx_2, \ldots, Tx_n\}$ forms a basis for $Img\ T$ in W. For;

$$y \in Img\ T \Longrightarrow y = Tx \text{ for some } x \in V.$$
$$\Longrightarrow y = T(\alpha_1 x + \alpha_2 x_2 + \cdots + \alpha_n x_n) \text{ for some } \alpha_1,\ \alpha_2, \ldots, \alpha_n \in F \ (\because \{x_1, x_2, \ldots, x_n\} \text{ is a basis for V})$$
$$\Longrightarrow y = \alpha_1 Tx_1 + \alpha_2 Tx_2 + \cdots + \alpha_n Tx_n \ (\because \text{T is linear})$$

Therefore, $\{Tx_1, Tx_2, \ldots, Tx_n\}$ generates $Img\ T$. Also let $\alpha_1 Tx_1 + \alpha_2 Tx_2 + \cdots + \alpha_n Tx_n = 0$ for some scalars $\alpha_1,\ \alpha_2, \ldots, \alpha_n \in F$

$$\Longrightarrow T(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) = 0$$
$$\Longrightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \in \ker T = \{0\} \ (\because \text{T is an injection})$$
$$\Longrightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$$
$$\Longrightarrow \alpha_1 = 0,\ \alpha_2 = 0,\ \ldots, \alpha_n = 0 \ (\because \{x_1, x_2, \ldots, x_n\} \text{ is a basis for V})$$

Therefore $\{Tx_1, Tx_2, \ldots, Tx_n\}$ is linearly independent and hence a basis for Im T. Put $y_1 = Tx_1$, $y_2 = Tx_2, \ldots, y_n = Tx_n$. Then we have a linearly independent set $\{y_1, y_2, \ldots, y_n\}$ in W. If $n = m$, then $\{y_1, y_2, \ldots, y_n\}$ forms a basis for W and hence Img $T = W$. Thus T is a surjection also and hence invertible. If $n < m$ such that $m - n = r$, then Img $T \subset W$ so that we can choose an element y_{n+1} in $W - Img\ T$. Since $y_{n+1} \notin Img\ T$ then it is not a scalar combination of these y_i's and hence the set $\{y_1, y_2, \ldots, y_n,\ y_{n+1}\}$ becomes linearly independent in W. let U_1 be the subspace of W generated by $\{y_1, y_2, \ldots, y_n,\ y_{n+1}\}$. Therefore since $\{y_1, y_2, \ldots, y_n,\ y_{n+1}\}$ is linearly independent we get that $\dim U_1 = n + 1$. If $n+1 = m$ then $U_1 = W$ and if $n+1 < m$, then we can choose another element y_{n+2} in $W - U$ and hence $y_{n+2} \notin U_1$, so that this y_{n+2} is not a linear combination of vectors $y_1, y_2, \ldots, y_n,\ y_{n+1}$. thus, the set $\{y_1, y_2, \ldots, y_n,\ y_{n+1},\ y_{n+2}\}$ is linearly independent in W. Similarly doing this process r times, we get $r = m - n$ vectors $y_{n+1}, \ldots, y_{n+r}$ in $W - Img\ T$ such that the set $\{y_1, y_2, \ldots, y_n,\ y_{n+1}, \ldots, y_{n+r} = y_m\}$ is linearly independent in W and hence forms a basis for W. Therefore for any $y \in W$ there exists some scalars $\alpha_1,\ \alpha_2, \ldots, \alpha_n, \ldots, \alpha_m$ in F such that $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m$. Now for any $y \in W$, define a mapping $f : W \to V$ by: $f(y) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ if $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m$. Since any element y in W can be uniquely expressed as a linear combination of elements of a given basis, it follows that f is well defined. Now we prove that this f is a linear transformation of W into V.

$y,\ z \in W \Longrightarrow\ y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m$ and $z = \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_n y_n + \cdots + \beta_m y_m$ for some scalars α_i's and β_i's in F, $1 \le i \le m$. Therefore;

$$\begin{aligned} f(y+z) &= f((\alpha_1+\beta_1) y_1 + (\alpha_2+\beta_2) y_2 + \cdots + (\alpha_n+\beta_n) y_n + \cdots + (\alpha_m+\beta_m) y_m) \\ &= (\alpha_1+\beta_1) x_1 + (\alpha_2+\beta_2) x_2 + \cdots + (\alpha_n+\beta_n) x_n \\ &= \alpha_1 x_1 + \beta_1 x_1 + \alpha_2 x_2 + \beta_2 x_2 + \cdots + \alpha_n x_n + \beta_n x_n \\ &= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \\ &= f(y) + f(z) \end{aligned}$$

Also, for any scalar α, consider:

$$\begin{aligned} f(\alpha y) &= f(\alpha(\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m)) \\ &= f(\alpha\alpha_1 y_1 + \alpha\alpha_2 y_2 + \cdots + \alpha\alpha_n y_n + \cdots + \alpha\alpha_m y_m) \\ &= \alpha\alpha_1 x_1 + \alpha\alpha_2 x_2 + \cdots + \alpha\alpha_n x_n \\ &= \alpha(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \\ &= \alpha f(y) \end{aligned}$$

This shows that f is a linear transformation. Moreover, $f(y_i)= x_i$ for all $1 \leq i \leq n$. Since $\{x_1, x_2, \ \ldots, x_n\}$ is a basis for V, any vector x in V is of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. Now for any $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ in V, consider:

$$\begin{aligned} f \circ T(x) &= f(T(x)) \\ &= f(T(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n)) \\ &= f(\alpha_1 T x_1 + \alpha_2 T x_2 + \cdots + \alpha_n T x_n) \quad (\because \ \text{T is linear}) \\ &= f(\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n) \quad (\because T x_i = y_i \text{ for all } 1 \leq i \leq n) \\ &= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \\ &= x \end{aligned}$$

Thus $f \circ T = I_V$, where I_V is an identity operator on V and therefore this f is a left inverse of T and hence T is left invertible.

(ii) $\Longrightarrow$ (iii) Suppose that T is left invertible and let f be the left inverse of T. For any vector space U over the same field F, let T_1 and T_2 be linear transformations of U into V such that;; $T \circ T_1 = T \circ T_2 \Longrightarrow f \circ \ T \circ T_1 = f \circ T \circ T_2 \Longrightarrow T_1 = T_2$ and hence T is left cancellable.

(iii) $\Longrightarrow$ (i) Suppose that T is left cancellable. Let x_1 and x_2 be any arbitrary vectors in V such that $T x_1 = T x_2$. Consider a subspace U of V generated by $\{x_1, \ \ x_2\}$ and define T_1 and $T_2 : U \to V$ by: $T_1(\alpha_1 x_1 + \alpha_2 x_2) = (\alpha_1 + \alpha_2) x_1$ and $T_2(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 x_1 + \alpha_2 x_2$ for all scalars α_1 and α_1 in F. If $x = \alpha_1 x_1 + \alpha_2 x_2$ and $y = \beta_1 x_1 + \beta_2 x_2$ are any vectors in U and α is any scalar in F, then consider;

$$\begin{aligned} T_1(x+y) &= T_1(\alpha_1 x_1 + \alpha_2 x_2 + \beta_1 x_1 + \beta_2 x_2) \\ &= T_1((\alpha_1 + \beta_1) x_1 + (\alpha_2 + \beta_2) x_2) \\ &= ((\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)) x_1 \\ &= ((\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)) x_1 \\ &= (\alpha_1 + \alpha_2) x_1 + (\beta_1 + \beta_2) x_1 \\ &= T_1(\alpha_1 x_1 + \alpha_2 x_2) + T_1(\beta_1 x_1 + \beta_2 x_2) \\ &= T_1(x) + T_1(y) \end{aligned}$$

Also,

$$\begin{aligned} T_1(\alpha x) &= T_1(\alpha\alpha_1 x_1 + \alpha\alpha_2 x_2) \\ &= (\alpha\alpha_1 + \alpha\alpha_2)x_1 \\ &= \alpha(\alpha_1 + \alpha_2)x_1 \\ &= \alpha T_1(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \alpha T_1(x) \end{aligned}$$

Thus T_1 is a linear transformation. Similarly:

$$\begin{aligned} T_2(x+y) &= T_2(\alpha_1 x_1 + \alpha_2 x_2 + \beta_1 x_1 + \beta_2 x_2) \\ &= T_2((\alpha_1 + \beta_1)x_1 + (\alpha_2 + \beta_2)x_2) \\ &= (\alpha_1 + \beta_1)x_1 + (\alpha_2 + \beta_2)x_2 \\ &= \alpha_1 x_1 + \beta_1 x_1 + \alpha_2 x_2 + \beta_2 x_2 \\ &= \alpha_1 x_1 + \alpha_2 x_2 + \beta_1 x_1 + \beta_2 x_2 \\ &= T_2(\alpha_1 x_1 + \alpha_2 x_2) + T_2(\beta_1 x_1 + \beta_2 x_2) \\ &= T_2(x) + T_2(y) \end{aligned}$$

Also,

$$\begin{aligned} T_2(\alpha x) &= T_2(\alpha\alpha_1 x_1 + \alpha\alpha_2 x_2) \\ &= \alpha\alpha_1 x_1 + \alpha\alpha_2 x_2 \\ &= \alpha(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \alpha T_2(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \alpha T_2(x) \end{aligned}$$

Therefore T_2 is again a linear transformation. Now for any $x = \alpha_1 x_1 + \alpha_2 x_2$ in U, consider;

$$\begin{aligned} T \circ T_1(x) &= T(T_1(\alpha_1 x_1 + \alpha_2 x_2)) \\ &= T((\alpha_1 + \alpha_2)x_1) \\ &= (\alpha_1 + \alpha_2)Tx_1 \ \ (\because \text{ T is linear}) \end{aligned}$$

On the other hand, consider;

$$\begin{aligned} T \circ T_2(x) &= T(T_2(\alpha_1 x_1 + \alpha_2 x_2)) \\ &= T(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \alpha_1 Tx_1 + \alpha_2 Tx_2 \ \ (\because \text{T is linear}) \\ &= \alpha_1 Tx_1 + \alpha_2 Tx_1 \ \ (\because Tx_1 = Tx_2) \\ &= (\alpha_1 + \alpha_2)Tx_1 \end{aligned}$$

Therefore we have that, $T \circ T_1(x) = T \circ T_2(x)$ for all $x \in U$. Thus $T \circ T_1 = T \circ T_2$ and since T is left cancellable it follows that, $T_1 = T_2$;; that is $T_1(x) = T_2(x)$ for all $x \in U$. In particular, $T_1(x_2) = T_2(x_2) \implies x_1 = x_2$ and hence T is an injection. □

Remark 3.7. *We observe from the above theorem that any injective linear transformation has at least one left inverse and in fact it is not necessarily unique. So, the question in this case is that how many left inverses can be there for a given injective linear transformation? In the next theorem we characterize the set of all left inverses of a given injective linear transformation.*

Theorem 3.8. *Let V and W be finite dimensional vector spaces over the field F such that $\dim V = n$ and $\dim W = m$ and $T : V \to W$ be an injection. For any arbitrary basis $\{x_1, x_2, \ldots, x_n\}$ for V, if we let $\mathfrak{L}(T) =$ be the class of all left inverses of T and, $\mathfrak{B}(T) = \{B : B$ is a basis for W, containing vectors $Tx_1, Tx_2, \ldots, Tx_n$ and span $[B_1 - \{Tx_1, Tx_2, \ldots, Tx_n\}] \cap$ span $[B_2 - \{Tx_1, Tx_2, \ldots, Tx_n\}] = \{0\}$ for any $B_1 \neq B_2 \in \mathfrak{B}(T)\}$ Then there is a one to one correspondence between $\mathfrak{L}(T)$ and $\mathfrak{B}(T)\}$.*

Proof. If $B \in \mathfrak{B}(T)$, then B is a basis for W containing $Tx_1, Tx_2, \ldots, Tx_n$. If $B = \{y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m\}$ then by simple rearrangement of elements of B we can assume that $y_i = Tx_i$ for all $1 \leq i \leq n$. Therefore any y in W can be expressed as $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m$ for some scalar α_i's. Now for any $B \in \mathfrak{B}(T)$, define $f_B : W \to V$ by:

$$f_B(\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

Then as it is observed from the above theorem we get that f_B is a a left inverse of T, so that $f_B \in \mathfrak{L}(T)$. Now define h: $\mathfrak{B}(T) \to \mathfrak{L}(T)$ by: $h(B) = f_B$ for all $B \in \mathfrak{B}(T)$. It is clear that this h is well defined. Now we prove that h is a one-to-one correspondence.

Let $B_1 = \{y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m\}$ and $B_2 = \{z_1, z_2, \ldots, z_n, z_{n+1}, \ldots, z_m\} \in \mathfrak{B}(T)$ such that $B_1 \neq B_2$. Therefore $y_i = z_i = Tx_i$ for all $1 \leq i \leq n$ and *span* $[B_1 - \{y_1, y_2, \ldots, y_n\}] \cap$ *span* $[B_2 - \{y_1, y_2, \ldots, y_n\}] = \{0\}$. Now choose exactly one element $y \in W - Img\ T$, then $y \notin Img\ T$ and hence y is not a linear combination of $y_1, y_2, \ldots, y_n$.

Considering a basis B_1, y can be expressed as $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m$ for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$. On the other hand, considering a basis B_2 y can also be expressed as $y = \beta_1 z_1 + \cdots + \beta_n z_n + \cdots + \beta_m z_m = \beta_1 y_1 + \cdots + \beta_n y_n + \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$ for some scalars $\beta_1, \beta_2, \ldots \beta_m$. Now our claim is to see that $\alpha_i \neq \beta_i$ for some $1 \leq i \leq n$ and we use proof by contradiction. Suppose if possible that $\alpha_i = \beta_i$ for all $1 \leq i \leq n$.

$$\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m = \beta_1 y_1 + \cdots + \beta_n y_n + \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$$
$$\implies \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m = \alpha_1 y_1 + \cdots + \alpha_n y_n + \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$$
$$\implies \alpha_{n+1} y_{n+1} + \alpha_{n+2} y_{n+2} + \cdots + \alpha_m y_m = \beta_{n+1} z_{n+1} + \beta_{n+2} z_{n+2} + \cdots + \beta_m z_m$$

If $u = \alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m = \beta_{n+1} z_{n+1} + \cdots + \beta_m z_m$. Then $u \in$ *span* $[B_1 - \{y_1, \ldots, y_n\}] \cap$ *span* $[B_2 - \{y_1, \ldots, y_n\}] = \{0\}$. So that $u = 0$. Thus we have that: $\alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m = 0$ and $\beta_{n+1} z_{n+1} + \cdots + \beta_m z_m = 0$. Since each y_i's and z_i's are linearly independent to each other for $n < i \leq m$ it follows that $\alpha_i = 0 = \beta_i$ for all $n < i \leq m$. Therefore, $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n$ and hence $y \in Img\ T$ which is a contradiction to our choice of y.

Thus $\alpha_i \neq \beta_i$ for some i, $1 \leq i \leq n$. Therefore $f_{B_1}(y) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \neq \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = f_{B_2}(y)$;; that is, we get an element $y \in W - Img\ T$ such that $f_{B_1}(y) \neq f_{B_2}(y)$ and hence $f_{B_1} \neq f_{B_2}$ which implies that $h(B_1) \neq h(B_2)$.

Therefore h is a one to one map. Furthermore, we prove that h is an onto; for, let $f \in \mathfrak{L}(T)$, then f is a left inverse of T; that is, f is a linear transformation of W into V such that $f \circ T = I_V$ (an identity operator on V). So that $f(Tx) = x$ for all $x \in V$. In particular, $f(Tx_i) = x_i$ and hence $f(y_i) = x_i$ for all $1 \le i \le n$. $x \in V \Longrightarrow x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$

$$\Longrightarrow x = \alpha_1 f(Tx_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n)$$

$$\Longrightarrow x = f(\alpha_1 Tx_1 + \alpha_2 Tx_2 + \cdots + \alpha_n Tx_n)$$

$$\Longrightarrow x \in Img\ f$$

$$\Longrightarrow V \subseteq Img\ f \subseteq V$$

$$\Longrightarrow Img\ f = V \text{ and hence } \dim(Img\ f) = \dim V = n.$$

Since f is a linear transformation of W into V, we have that:

$$\dim(\ker f) + \dim(Img\ f) = \dim W \quad \Longrightarrow \dim(\ker f) + n = m$$

In this case if $m \neq n$, then $\dim(\ker f) = m - n > 0$, and hence $\ker f$ is a nontrivial subspace of W with dimension $m - n$. So that we can choose $m - n$ linearly independent vectors $y_{n+1}, y_{n+2}, \ldots, y_m$ in $\ker f$. Thus $f(y_{n+i}) = 0$ for all $1 \le i \le m - n$.

Claim 1: The set $B = \{y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m\}$ is linearly independent in W.

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$;

$$\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m = 0 \tag{1}$$

$$\Longrightarrow f(\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m) = 0$$

$$\Longrightarrow \alpha_1 f(y_1) + \alpha_2 f(y_2) + \cdots + \alpha_n f(y_n) + \cdots + \alpha_m f(y_m) = 0$$

$$\Longrightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \alpha_{n+1} f(y_{n+1}) + \cdots + \alpha_m f(y_m) = 0 \ \ (\because f(y_i) = x_i \text{ for all } 1 \le i \le n)$$

$$\Longrightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \ \ (\because f(y_{n+i}) = 0 \text{ for all } 1 \le i \le m - n)$$

$$\Longrightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0 \ \ (\because \{x_1, \ldots, x_n\} \text{ is linearly independent in V})$$

Substituting the value of each α_i's in equation (1) we have that: $\alpha_{n+1} y_{n+1} + \cdots + \alpha_m y_m = 0$ and since each y_{n+i}'s are linearly independent to each other, we get that $\alpha_{n+i} = 0$ for all $1 \le i \le m - n$. This says that, $\alpha_i = 0$ for all $1 \le i \le m$. Therefore he set $B = \{y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_m\}$ is linearly independent in W and since B has exactly m elements, then it becomes a basis for W containing $y_1, y_2, \ldots, y_n$ so that $B \in \mathfrak{L}(T)$.

Claim 2: $f = f_B = h(B)$

Since B forms a basis for W, any $y \in W$ can be expressed as $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m$, then

$$\begin{aligned} f(y) &= f(\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m) \\ &= \alpha_1 f(y_1) + \alpha_2 f(y_2) + \cdots + \alpha_n f(y_n) + \cdots + \alpha_m f(y_m) \ \ (\because \text{f is linear}) \\ &= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \alpha_{n+1} f(y_{n+1}) + \cdots + \alpha_m f(y_m) \ \ (\because \ f(y_i) = x_i \text{ for all } 1 \le i \le n) \\ &= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \ \ (\because f(y_{n+i}) = 0 \text{ for all } 1 \le i \le m - n) \\ &= f_B(\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n + \cdots + \alpha_m y_m) \\ &= f_B(y) \end{aligned}$$

Thus $f = f_B = h(B);$; that is, h is an onto and hence a one-to-one correspondence. Thus, $\mathfrak{B}(T)$ and $\mathfrak{L}(T)$ are equivalent to each other. $\square$

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