Strongly Prime Labeling For Some Graphs

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Abstract: A graph \(G = (V, E)\) with \(n\) vertices is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding, \(n\) such that the label of each pair of adjacent vertices are relatively prime. A graph \(G\) which admits prime labeling is called a prime graph and a graph \(G\) is said to be a strongly prime graph if for any vertex, \(v\) of \(G\) there exists a prime labeling, \(f\) satisfying, \(f(v) = 1\). In this paper we prove that the graphs corona of triangular snake, corona of quadrilateral snake, corona of ladder graph and a graph obtained by attaching \(P_2\) at each vertex of outer cycle of prism \(D_n\) by \((D_n;P_2)\), helm, gearwheel are strongly prime graphs.

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1. Introduction

We begin with simple, finite, undirected and non trivial graph \(G = (V(G), E(G))\) with vertex set \(V(G)\) and edge set \(E(G)\). The set of vertices adjacent to a vertex \(u\) of \(G\) is denoted by \(N(u)\). For all other standard terminology and notations we refer to Bondy and Murthy [3]. We will give brief summary of definitions which are useful for the present investigations.

**Definition 1.1.** If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.

Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. An enormous body of literature has grown around in graph labeling in last five decades. A systematic study of various applications of graph labeling is carried out in Bloom and Golomb [2]. According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and structure of graphs. For detailed survey on graph labeling we refer to A Dynamic Survey of Graph Labeling by Gallian [6].

**Definition 1.2.** Let \(G = (V(G), E(G))\) be a graph with \(p\) vertices. A bijection \(f : V(G) \rightarrow \{1, 2, \ldots, p\}\) is called a prime labeling if for each edge \(e = uv\), \(\gcd(f(u), f(v)) = 1\). A graph which admits prime labeling is called a prime graph.

The notion of a prime labeling was originated by Entringer and was discussed in a paper by Tout et al. [9]. Many researchers have studied prime graphs. For e.g. Fu and Huang [5] have proved that \(P_n\) and \(K_{1,n}\) are prime graphs. Lee et al. [7] have proved that \(W_n\) is a prime graph if and only if \(n\) is even. Deretsky et al. [4] have proved that cycle \(C_n\) is a prime graph.

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Prime labeling of some classes of graph were discussed by S.K.Vaidya and Udayan M Prajapati in [11]. Prime labeling in the context of some graph operation was discussed by S.Meena and K.Vaithiligam [8].

Definition 1.3. A graph $G$ is said to be a strongly prime graph if for any vertex, $v$ of $G$ there exists a prime labeling, $f$ satisfying, $f(v) = 1$.

The concept of strongly prime graph was introduced by Samir K.Vaidya and Udayan M Prajapati [10] and they proved that the graphs $C_n, P_n, K_{1,n}$ and $W_n$ for every even integer $n \geq 4$ are strongly prime graphs.

Definition 1.4. Triangular snake $T_n$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$ for $1 \leq i \leq n-1$, that is every edge of path is replaced by a triangle $C_3$.

Definition 1.5. A quadrilateral snake $Q_n$ is obtained from a path $\{u_1, u_2, \ldots, u_n\}$ by joining $u_i$ and $u_{i+1}$ to two vertices $v_i$ and $w_i$, $1 \leq i \leq n-1$ respectively and then joining $v_i$ and $w_i$.

Definition 1.6. The product $P_2 \times P_n$ is called a ladder and it is denoted by $L_n$.

Definition 1.7. The corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \circ G_2$ formed by taking one copy of $G_1$ and $\{V(G_1)\}$ copies of $G_2$ where the $i$th vertex of $G_1$ is adjacent to every vertex in the $i$th copy of $G_2$.

Definition 1.8. The prism $D_n, n \geq 3$ is a trivalent graph which can be defined as the Cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on $n$ vertices. We denote a graph obtained by attaching $P_2$ at each vertex of outer cycle of $D_n$ by $(D_n; P_2)$.

Definition 1.9. The helm $H_n$ is a graph obtained from a wheel by attaching a pendant edge at each vertex of then $n$-cycle.

Definition 1.10. The gear graph $G_n$ is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle. The gear graph $G_n$ has $2n+1$ vertices and $3n$ edges.

Definition 1.11 (Bertrand’s Postulate). For every positive integer $n > 1$ there is a prime $p$ such that $n < p < 2n$.

The present work is aimed to discuss some new families of strongly prime graphs.

2. Strongly Prime Graphs

Theorem 2.1. The graph $G \circ K_1$ is a strongly prime graph where $G = T_n$ for all integer $n \geq 2$.

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ be a path of length $n$. Let $v_i, 1 \leq i \leq n-1$ be the new vertex joined to $u_i$ and $u_{i+1}$. The resulting graph is called $T_n$ and let $x_i$ be the vertex which is joined to $u_i$, $1 \leq i \leq n$, let $y_i$ be the vertex which is joined to $v_i$, $1 \leq i \leq n-1$. The resulting graph is $G_1$ (i.e.) $G \circ K_1$ where $G = T_n$ graph.

Now the vertex set of $V(G_1) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots v_{n-1}, x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_{n-1}\}$ and the edge set $E(G_1) = \{u_iu_{i+1}, u_iv_i/1 \leq i \leq n-1\} \cup \{u_ix_i/1 \leq i \leq n\} \cup \{v_iy_i/1 \leq i \leq n-1\}$. Here $|V(G_1)| = 4n-2$. Let $v$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case (i): If $v = u_j$ for some $j \in \{1, 2, \ldots n\}$ then the function $f : V(G) \rightarrow \{1, 2, \ldots 4n-2\}$ defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j - 1 & \text{if } i = 1, 2, \ldots j-1; \\ 4i - 4j + 1 & \text{if } i = j, j+1, \ldots n; \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 1 & \text{if } i = 1, 2, \ldots j-1; \\ 4i - 4j + 3 & \text{if } i = j, j+1, \ldots n-1; \end{cases}$$
is a prime labeling for $G_1$ with $f(v) = f(u_i) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j$ in $G_1$.

Case (ii): If $v = x_j$ for some $j \in \{1, 2, \ldots n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows: $f_2(u_j) = f(x_j)$, $f_2(x_j) = f(u_i)$ for $j \in \{1, 2, \ldots n\}$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling $f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = x_j$ in $G_1$.

Case (iii): If $v = v_j$ for some $j \in \{1, 2, \ldots n - 1\}$ then define a labeling $f_3$ using the labeling $f_2$ defined in case (ii) as follows: $f_3(x_j) = f_2(v_j)$, $f_3(v_j) = f_2(x_j)$ for $j \in \{1, 2, \ldots n - 1\}$ and $f_3(v) = f_3(v)$ for all the remaining vertices. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in $G_1$.

Case (iv): If $v = y_j$ for some $j \in \{1, 2, \ldots n - 1\}$ then define a labeling $f_4$ using the labeling $f_2$ defined in case (ii) as follows: $f_4(x_j) = f_2(y_j)$, $f_4(y_j) = f_2(x_j)$, $f_4(u_j) = f_2(u_j)$, $f_4(v_j) = f_2(v_j)$ for $j \in \{1, 2, \ldots n - 1\}$ and $f_4(v) = f_4(v)$ for all the remaining vertices. Then the resulting labeling $f_4$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = y_j$ in $G_1$.

In this case if $f_4(u_{j-1})$ is a multiple of 3 then interchange $f_4(u_{j-1})$ and $f_4(x_{j-1})$. Similarly $f_4(v_{j-1})$ is a multiple of 3 then interchange $f_4(v_{j-1})$ and $f_4(y_{j-1})$

Thus from all the cases described above $G_1$ is a strongly prime graph.

![Figure 1. A prime labeling of $G \odot K_1$ where $G = T_n$ having $u_4$ as label 1](image)

**Theorem 2.2.** The graph $G \odot K_1$ is a strongly prime graph where $G = Q_n$ for all integer $n \geq 2$.

**Proof.** Let $\{u_1, u_2, \ldots, u_n\}$ be a path. Let $v_i$ and $w_i$ be two vertices joined to $u_i$ and $u_{i+1}$ respectively and then join $v_i$ and $w_i$, $1 \leq i \leq n - 1$. The resulting graph is called as quadrilateral snake $Q_n$. Let $x_i$ be the new vertex joined to $u_i$, $1 \leq i \leq n$. Let $y_i$ be the new vertex joined to $v_i$, $1 \leq i \leq n - 1$ and let $z_i$ be the new vertex joined to $w_i$, $1 \leq i \leq n - 1$. The resulting graph is $G_1$ (i.e.) $G \odot K_1$ where $G = Q_n$ graph.

Now the vertex set $V(G_1) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}, w_1, w_2, \ldots, w_{n-1}, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1}, z_1, z_2, \ldots, z_{n-1}\}$. The edge set $E(G_1) = \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_1x_1, u_2x_2, \ldots, u_{n-1}x_n, y_1, y_2, \ldots, y_{n-1}, z_1, z_2, \ldots, z_{n-1}\}$. The edge set $E(G_1) = \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_1x_1, u_2x_2, \ldots, u_{n-1}x_n, y_1, y_2, \ldots, y_{n-1}, z_1, z_2, \ldots, z_{n-1}\}$.
Case (i): Let \( v = u_j \) for some \( j \in \{1, 2, \ldots, n\} \) then the function \( f : V(G) \rightarrow \{1, 2, \ldots, 6n - 4\} \) defined by

\[
\begin{align*}
    f(u_i) &= \begin{cases} 
        6n + 6i - 6j - 1 & \text{if } i = 1, 2, \ldots, j - 1; \\
        6i - 6j + 1 & \text{if } i = j, j + 1, \ldots, n;
    \end{cases} \\
    f(v_j) &= \begin{cases} 
        6n + 6i - 6j - 3 & \text{if } i = 1, 2, \ldots, j - 1; \\
        6i - 6j + 3 & \text{if } i = j, j + 1, \ldots, n - 1;
    \end{cases} \\
    f(w_i) &= \begin{cases} 
        6n + 6i - 6j + 1 & \text{if } i = 1, 2, \ldots, j - 1; \\
        6i - 6j + 5 & \text{if } i = j, j + 1, \ldots, n - 1;
    \end{cases} \\
    f(x_i) &= \begin{cases} 
        6n + 6i - 6j & \text{if } i = 1, 2, \ldots, j - 1; \\
        6i - 6j + 2 & \text{if } i = j, j + 1, \ldots, n;
    \end{cases} \\
    f(y_i) &= \begin{cases} 
        6n + 6i - 6j - 2 & \text{if } i = 1, 2, \ldots, j - 1; \\
        6i - 6j + 4 & \text{if } i = j, j + 1, \ldots, n - 1;
    \end{cases} \\
    f(z_i) &= \begin{cases} 
        6n + 6i - 6j + 2 & \text{if } i = 1, 2, \ldots, j - 1; \\
        6i - 6j + 6 & \text{if } i = j, j + 1, \ldots, n - 1
    \end{cases}
\]

is a prime labeling for \( G_1 \) with \( f(v) = f(u_j) = 1 \). Thus \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( v = u_j \) in \( G_1 \).

Case (ii): Let \( v = x_j \) for some \( j \in \{1, 2, \ldots, n\} \) then define a labeling \( f_2 \) using the labeling \( f \) defined in case (i) as follows: \( f_2(u_j) = f(x_j) \), \( f_2(v_j) = f(u_j) \) for \( j \in \{1, 2, \ldots, n\} \) and \( f_2(v) = f(v) \) for all the remaining vertices. Then the resulting labeling \( f_2 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( v = x_j \) in \( G_1 \).

Case (iii): Let \( v = v_j \) for some \( j \in \{1, 2, \ldots, n - 1\} \) then define a labeling \( f_3 \) using the labeling \( f_2 \) defined in case (ii) as follows: \( f_3(x_j) = f_2(v_j) \), \( f_3(v_j) = f_2(x_j) \) for \( j \in \{1, 2, \ldots, n - 1\} \) and \( f_3(v) = f_2(v) \) for all the remaining vertices. Then the resulting labeling \( f_3 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( v = v_j \) in \( G_1 \).

Case (iv): Let \( v = w_j \) for some \( j \in \{1, 2, \ldots, n - 1\} \) then define a labeling \( f_4 \) using the labeling \( f_3 \) defined in case (iii) as follows: \( f_4(w_j) = f_3(v_j) \), \( f_4(v_j) = f_3(w_j) \) for \( j \in \{1, 2, \ldots, n - 1\} \) and \( f_4(v) = f_3(v) \) for all the remaining vertices. Then the resulting labeling \( f_4 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( v = w_j \) in \( G_1 \).

Case (v): Let \( v = z_j \) for some \( j \in \{1, 2, \ldots, n - 1\} \) then define a labeling \( f_5 \) using the labeling \( f_4 \) defined in case (iv) as follows: \( f_5(z_j) = f_4(w_j) \), \( f_5(w_j) = f_4(z_j) \) for \( j \in \{1, 2, \ldots, n - 1\} \) and \( f_5(v) = f_4(v) \) for all the remaining vertices. Then the resulting labeling \( f_5 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( v = z_j \) in \( G_1 \).

Case (vi): Let \( v = y_j \) for some \( j \in \{1, 2, \ldots, n - 1\} \) then define a labeling \( f_6 \) using the labeling \( f_2 \) defined in case (ii) as follows: \( f_6(u_j) = f_2(v_j) \), \( f_6(v_j) = f_2(u_j) \), \( f_6(x_j) = f_2(y_j) \), \( f_6(y_j) = f_2(x_j) \) for \( j \in \{1, 2, \ldots, n - 1\} \) and \( f_6(v) = f_2(v) \) for all the
remaining vertices. Then the resulting labeling $f_6$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j$ in $G_1$. Thus from all the cases described above $G_1$ is a strongly prime graph.

![Figure 2. A prime labeling of $G \odot K_1$ where $G = Q_n$ having $u_4$ as label 1](image)

**Theorem 2.3.** The graph $G \odot K_1$ is a strongly prime graph where $G = L_n$ for all integer $n \geq 2$.

**Proof.** Let $G$ be the Ladder graph with vertices $\{u_1, u_2, ... u_n, v_1, v_2, ... v_n\}$. Let $u_i$ be the new vertex joined to $u_i$, $1 \leq i \leq n$ and $v_i'$ be the new vertex joined to $v_i$, $1 \leq i \leq n$ in $G$. The resulting graph is $G_1$ (i.e.) $G \odot K_1$ where $G = L_n$ graph. Now the vertex set $V(G_1) = \{u_1, u_2, ... u_n, v_1, v_2, ... v_n, u_1', u_2', ... u_n', v_1', v_2', ... v_n'\}$.

The edge set $E(G_1) = \{(v_i, v_{i+1}, u_i, u_{i+1}, 1 \leq i \leq n-1) \cup \{v_i, v_i', v_i, v_i', 1 \leq i \leq n\}\}$. Here $|V(G_1)| = 4n$. Let $v$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):** If $v = u_j$ for some $j \in \{1, 2, ... n\}$ then the function $f : V(G_2) \to \{1, 2, ... 4n\}$ defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j + 1 & \text{if } i = 1, 2, ... j - 1; \\ 4i - 4j + 1 & \text{if } i = j, j + 1, j + 2, ... n; \end{cases}$$

$$f(u_i') = \begin{cases} 4n + 4i - 4j + 2 & \text{if } i = 1, 2, ... j - 1; \\ 4i - 4j + 2 & \text{if } i = j, j + 1, j + 2, ... n; \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 3 & \text{if } i = 1, 2, ... j - 2; \\ 4i - 4j + 3 & \text{if } i = j, j + 1, j + 2, ... n; \end{cases}$$

$$f(v_i') = \begin{cases} 4n + 4i - 4j + 4 & \text{if } i = 1, 2, ... j - 2; \\ 4i - 4j + 4 & \text{if } i = j, j + 1, j + 2, ... n; \end{cases}$$

$$f(v_{j-1}) = \begin{cases} 4n & \text{if } 4n - 1 \text{ is multiple of } 3; \\ 4n - 1 & \text{otherwise}; \end{cases}$$

$$f(v_{j-1}') = \begin{cases} 4n - 1 & \text{if } 4n - 1 \text{ is multiple of } 3; \\ 4n & \text{otherwise}; \end{cases}$$

is a prime labeling for $G_1$ with $f(v) = f(u_j) = 1$. Thus $f$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j$ in $G_1$ graph.

**Case (ii):** If $v = u_j'$ for some $j \in \{1, 2, ... n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows:

$f_2(u_j) = f(u_j'), f_2(v_j') = f(u_j)$ for $j \in \{1, 2, ... n\}$ and $f_2(v) = f(v)$ for all the remaining vertices. Then the resulting labeling
$f_2$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j$ in $G_1$.

**Case (iii):** If $v = v_j$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_3$ using the labeling $f$ defined in case (i) as follows:

$f_3(u_i) = f(v_i)$, $f_3(v_i) = f(u_i)$, $f_3(u_j') = f(v_j')$, $f_3(v_j') = f(u_j')$ for $1 \leq i \leq n$. Then the resulting labeling $f_3$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in $G_1$ graph.

**Case (iv):** If $v = v_j'$ for some $j \in \{1, 2, ..., n\}$ then define a labeling $f_4$ using the labeling $f_3$ defined in case (iii) as follows:

$f_4(v_j) = f_3(v_j)$, $f_4(v_j') = f_3(v_j')$ for $j \in \{1, 2, ..., n\}$ and $f_4(v_i) = f_3(v_i)$ for all the remaining vertices. Then the resulting labeling $f_4$ is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = v_j'$ in $G_1$. Thus from all the cases described above gives $G_1$ graph is a strongly prime graph.

![Figure 3](image.png)

**Figure 3.** A prime labeling of $G \oplus K_1$ where $G = L_n$ having $u_4$ as label 1

\[\Box\]

**Theorem 2.4.** The graph obtained by attaching $P_2$ at each vertex of outer cycle of prism $D_n$ by $(D_n; P_2)$ for all integer $n \geq 3$, is a strongly prime graph.

**Proof.** Let $u_i$ and $v_i$ be the vertices of the inner and outer cycle of $(D_n; P_2)$ respectively in which $u_i$ and $v_i$ are adjacent, $1 \leq i \leq n$. Let $w_i$ be the pendant vertex which is joined with $v_i$, $1 \leq i \leq n$. The vertex set $V(D_n; P_2) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_{n-1}, w_1, w_2, ..., w_n\}$. The edge set $E(D_n; P_2) = \{u_i u_{i+1}, v_i v_{i+1}/1 \leq i \leq n - 1\} \cup \{u_n u_1, v_n v_1\} \cup \{u_i v_i, v_i w_i/1 \leq i \leq n\}$.

Here $|V(D_n; P_2)| = 3n$. Let $v$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case (i):** If $v$ is the vertex of the inner cycle. Let $v = u_j$ for some $j \in \{1, 2, ..., n\}$ then the function $f : V(D_n; P_2) \to \{1, 2, ...3n\}$ defined by

\[
f(u_i) = \begin{cases} 
3n + 3i - 3j + 1 & \text{if } i = 1, 2, ... j - 1; \\
3i - 3j + 1 & \text{if } i = j, j + 1, j + 2, ... n;
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
3n + 3i - 3j + 2 & \text{if } i = 1, 2, ... j - 2; \\
3i - 3j + 2 & \text{if } i = j, j + 1, j + 2, ... n;
\end{cases}
\]
\[ f(w_i) = \begin{cases} 
3n + 3i - 3j + 3 & \text{if } i = 1, 2, \ldots, j - 2; \\
3i - 3j + 3 & \text{if } i = j, j + 1, \ldots, n; 
\end{cases} \]

\[ f(v_{j-1}) = \begin{cases} 
3n - 1 & \text{if } 3n \text{ is even}; \\
3n & \text{otherwise}; 
\end{cases} \]

\[ f(w_{j-1}) = \begin{cases} 
3n & \text{if } 3n \text{ is even}; \\
3n - 1 & \text{otherwise}; 
\end{cases} \]

is a prime labeling for \((D_n; P_2)\) with \(f(v) = f(u_j) = 1\). Thus \(f\) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of the inner cycle \(v = u_j\) in \((D_n; P_2)\) graph.

**Case (ii):** If \(v\) is any pendent vertex. Let \(v = w_j\) for some \(j \in \{1, 2, \ldots, n\}\), then define a labeling \(f_2\) using the labeling \(f\) defined in case (i) as follows: \(f_2(u_j) = f(w_j), f_2(w_j) = f(u_j)\) for \(j \in \{1, 2, \ldots, n\}\) and \(f_2(v) = f(v)\) for all the remaining vertices. Then the resulting labeling \(f_2\) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of the pendent vertex \(v = w_j\) in \((D_n; P_2)\) graph.

**Case (iii):** If \(v\) is the vertex of the outer cycle. Let \(v = v_j\) for some \(j \in \{1, 2, \ldots, n\}\), then define a labeling \(f_3\) using the labeling \(f_2\) defined in case (ii) as follows: \(f_2(v_j) = f(w_j), f_2(w_j) = f(u_j)\) for \(j \in \{1, 2, \ldots, n\}\) and \(f_3(v) = f_2(v)\) and \(f_3(v) = f_2(v)\) for all other remaining vertices. Then the resulting labeling \(f_3\) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of outer cycle \(v = v_j\) in \((D_n; P_2)\) graph. Thus from all the cases described above gives \((D_n; P_2)\) graph is a strongly prime graph.

**Figure 4.** A prime labeling of \((D_n; P_2)\) having \(u_5\) as label 1

**Theorem 2.5.** The Helm \(H_n\) is a strongly prime graph.

**Proof.** Let \(v_0\) be the apex vertex \(v_1, v_2, \ldots, v_n\) be the consecutive rim vertices of \(H_n\) and \(v'_1, v'_2, \ldots, v'_n\) be the pendent vertices of \(H_n\). Let \(v\) be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:
Case (i): If $v$ is the apex vertex $v = v_0$ then the function $f : V(H_n) \to \{1, 2, 2n + 1\}$ defined as

$$f(v_0) = 1,$$
$$f(v_1) = 2,$$
$$f(v'_1) = 3,$$

if $v_i = 2i + 1$ if $2 \leq i \leq n$, $f(v'_i) = 2i$ if $2 \leq i \leq n$, then clearly $f$ is an injection. For an arbitrary edge $e = ab$ of $H_n$ we claim that $(f(a), f(b)) = 1$.

Subcase (i): If $e = v_0v_i$ for some $i \in \{2, 3, ..., n\}$ then $gcd(f(v_0), f(v_i)) = gcd(1, f(v_i)) = 1$.

Subcase (ii): If $e = v_iv_{i+1}$ for some $i \in \{1, 2, ..., n - 1\}$ then $gcd(f(v_i), f(v_{i+1})) = gcd(2i + 1, 2i + 3) = 1$ as $2i + 1$, $2i + 3$ are consecutive odd positive integers. If $e = v_1v_2$ then $gcd(f(v_1), f(v_2)) = gcd(2, 5) = 1$ and if $e = v_nv_1$ then $gcd(f(v_n), f(v_1)) = gcd(2n + 2, 1) = 1$ as $2n + 1$ is an odd integer.

Subcase (iii): If $e = v_iv'_i$ for some $i \in \{2, 3, ..., n\}$ then $gcd(f(v_i), f(v'_i)) = gcd(2i + 1, 2i) = 1$ as $2i + 1$, $2i$ are consecutive positive integers and if $e = v_1v'_1$ then $gcd(f(v_1), f(v'_1)) = gcd(2, 3) = 1$ as $2$ and $3$ are consecutive positive integers.

Case (ii): If $v = v_j$ for some $j \in \{1, 2, ..., n\}$, $v$ is one of the rim vertices then we may assume that $v = v_1$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as follows: $f_2(v_0) = f(v_1)$, $f_2(v_1) = f(v_0)$ and $f_2(v) = f(v)$ for all other remaining vertices. Clearly $f$ is an injection. For an arbitrary edge $e = ab$ of $G$ we claim that $gcd(f(a), f(b)) = 1$. To prove our claim the following cases are to be considered.

Subcase (i): If $e = v_0v_i$ for some $i \in \{2, 3, ..., n\}$ then $gcd(f(v_0), f(v_i)) = gcd(2, 2i + 1) = 1$ as $2i + 1$ is an odd positive integer and it is not divisible by 2. If $e = v_0v_1$ then $gcd(f(v_0), f(v_1)) = gcd(2, 1) = 1$. When $e = v_1v_{i+1}$ for some $i \in \{2, 3, ..., n - 1\}$ are as same as subcase (ii) in case (i).

Subcase (ii): When $e = v_iv'_i$ for some $i \in \{2, 3, ..., n\}$ are as same as subcase (iii) in case (i). If $e = v_1v'_1$ then $gcd(f(v_1), f(v'_1)) = gcd(1, 3) = 1$.

Case (iii): If $v$ is one of the pendent vertices then we assume that $v = v_i$ for $i = \frac{p-1}{2}$ (or) $\frac{p+3}{2}$, where $p$ is the largest prime less than or equal to $2n + 1$. According to Bertrand’s postulate such a prime $p$ exist with $\frac{2n+1}{2} < p < 2n + 1$.

Subcase (i): If $n \neq 3k + 1$ where $k$. Define a function $f : V(H_n) \to \{1, 2, ...2n + 1\}$

$$f(v_i) = \begin{cases} p & \text{if } i = 0; \\ 2i + 1 & \text{if } i \in \{1, 2, 3, ..., n\} - \{\frac{p-1}{2}\}; \\ p - 1 & \text{if } i = \frac{p+3}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

$$f(v'_i) = \begin{cases} 2i & \text{if } i \in \{1, 2, 3, ..., n\} - \{\frac{p-1}{2}\}; \\ 1 & \text{if } i = \frac{p+3}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

Subcase (ii): If $f\left(V_{\frac{p+1}{2}}\right)$ and $f\left(V_{\frac{p-1}{2}}\right)$ are multiple of 3 then the above case $gcd \left(\left(\frac{p-1}{2}\right), f\left(\frac{p+1}{2}\right)\right) \neq 1$ then define a function $f : V(H_n) \to \{1, 2, ...2n + 1\}$ as

$$f(v_i) = \begin{cases} p & \text{if } i = 0; \\ 2i + 1 & \text{if } i \in \{1, 2, 3, ..., n\} - \{\frac{p+1}{2}, \frac{p-3}{2}\}; \\ p - 2 & \text{if } i = \frac{p-1}{2}; \\ p - 3 & \text{if } i = \frac{p+3}{2}; \\ 0 & \text{otherwise.} \end{cases}$$
Case (iv): When \( n = 3k + 1 \) then define the labeling \( f_2 \) using labeling \( f \) defined in subcase (i) of case (iii) as follows: 

\[
f_2(v_i) = \begin{cases} 
2i & \text{if } \{1, 2, 3, \ldots, n\} - \{\frac{p-3}{2}\} \\
1 & \text{if } i = \frac{p-3}{2};
\end{cases}
\]

Subcase (i): If \( e = v_0v_i \) for some \( i \in \{1, 2, 3, \ldots, n\} \) then \( \gcd(f(v_0), f(v_i)) = \gcd(p, f(v_i)) = 1 \) as \( p \) is co-prime to every integer from \{1, 2, 3, \ldots, n+1\} – \{p\}.

Subcase (ii): If \( e = v_i v_{i+1} \) for some \( i \in \{1, 2, 3, \ldots, n-1\} \) then \( \gcd(f(v_i), f(v_{i+1})) = \gcd(2i + 1, 2i + 3) = 1 \) as \( 2i + 1, 2i + 3 \) are consecutive odd positive integers. If \( e = V_{\frac{p-1}{2}}, V_{\frac{p-3}{2}} \) then \( \gcd(f(V_{\frac{p-1}{2}}), f(V_{\frac{p-3}{2}})) = \gcd(p - 1, p - 2) = 1 \) as \( p - 1 \) and \( p - 2 \) are consecutive positive integers. If \( e = V_{\frac{p+1}{2}}, V_{\frac{p+3}{2}} \) then \( \gcd(f(V_{\frac{p+1}{2}}), f(V_{\frac{p+3}{2}})) = \gcd(p - 1, p + 2) = 1 \) as \( p - 1 \) is even and it is differ by 3. Similarly we prove for any arbitrary edge \( e = ab \) of \( H_n \) have \( \gcd(f(a), f(b)) = 1 \) subcases (ii) and case (iv). Thus in all the possibilities described above \( f \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( H_n \). That is \( H_n \) is strongly prime graph.

![Figure 5. A prime labeling of Helm graph of\( H_n \) having the apex vertex \((v_0)\) as label 1](image)

**Theorem 2.6.** The Gear graph \( G_n \) is a strongly prime graph.

**Proof.** Let \( v_0 \) be the apex vertex \( v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n \) be the consecutive rim vertices. Let \( v \) be an arbitrary vertex of \( G_n \) that is \( v = v_0 \). Then the function \( f : V(G_n) \to \{1, 2, \ldots, 2n+1\} \) defined as \( f(v_i) = 2i+1 \) for \( i = 0, 1, 2, \ldots, n \), \( f(v'_i) = 2i+2 \) for \( i = 1, 2, \ldots, n-1 \),

\[
f(v'_i) = 2.
\]

Clearly \( f \) is an injection. For an arbitrary edge \( e = ab \) of \( G_n \) we claim that \( \gcd(f(a), f(b)) = 1 \). To prove our claim the following cases are to be considered.

Subcase (i): If \( e = v_0v_i \) for some \( i \in \{1, 2, 3, \ldots, n\} \) then \( \gcd(1, 2i + 1) = 1 \).

Subcase (ii): If \( e = v_i v'_i \) for some \( i \in \{1, 2, 3, \ldots, n-1\} \) then \( \gcd(f(v_i), f(v'_i)) = \gcd(2i + 1, 2i + 2) = 1 \) as \( 2i + 1, 2i + 2 \) are consecutive positive integers. If \( e = v_n v'_n \) then \( \gcd(f(v_n), f(v'_n)) = \gcd(2n + 1, 2) = 1 \) as \( 2n + 1 \) is an odd positive integer and it is not divisible by 2.

Subcase (iii): If \( e = v'_i v_{i+1} \) for some \( i \in \{1, 2, 3, \ldots, n-1\} \) then \( \gcd(f(v'_i), f(v_{i+1})) = \gcd(2i + 2, 2i + 3) = 1 \) as \( 2i + 1 \) and \( 2i + 3 \) are consecutive positive integers. If \( e = v_n v'_n \) then \( \gcd(f(v'_n), f(v_i)) = \gcd(2, 3) = 1 \) as 2 and 3 are consecutive
positive integers.

**Case (ii):** When \( v \) is of degree 2. Define a labeling \( f_2 \) using the labeling \( f \) in case (i) as follows: \( f_2(v'_n) = f(v_0) \), \( f_2(v_0) = f(v'_n) \) and \( f_2(v) = f(v) \) for all other remaining vertices. Then clearly \( f \) is an injection. For an arbitrary edge \( e = ab \) of \( G_n \) we claim that \( \gcd(f(a), f(b)) = 1 \). To prove our claim the following cases are to be considered.

**Subcase (i):** If \( e = v_0v_i \) for some \( i \in \{1, 2, 3, \ldots n\} \) then \( \gcd(f(v_0), f(v_i)) = \gcd(2, 2 + i) = 1 \) as \( 2i + 1 \) is an odd positive integer and it is not divisible by 2.

**Subcase (ii):** If \( e = v_iv'_i \) for some \( i \in \{1, 2, 3, \ldots n - 1\} \) are as same as subcase (ii) in case (i). If \( e = v_nv'_n \) for some \( i \in \{2, 3, \ldots n\} \) then \( \gcd(f(v_n), f(v'_n)) = \gcd(2n + 1, 1) = 1 \).

**Subcase (iii):** If \( e = v'_i v_{i+1} \) for some \( i \in \{1, 2, 3, \ldots n - 1\} \) are as same as subcase (iii) in case (i). If \( e = v'_nv_1 \) then \( \gcd(f(v'_n), f(v_1)) = \gcd(2n + 1, 1) = 1 \).

**Case (iii):** When \( v \) is of degree 3. We may assume that \( v = V_{\frac{p+1}{2}} \) where \( p \) is the largest prime less than or equal to \( 2n + 1 \). According to the Bertrand’s postulate such a prime \( p \) exist with \( \frac{2n + 1}{2} < p < 2n + 1 \). Now let \( f_3 \) be the labeling obtained from \( f \) in case (i) by interchanging the label \( f(v_0) \) and \( \left( v_{\frac{p+1}{2}} \right) \) and for all the remaining vertices \( f_3(v) = f(v) \). Then clearly \( f \) is an injection. For an arbitrary edge \( e = ab \) of \( G_n \) we claim that \( \gcd(f(a), f(b)) = 1 \). To prove our claim the following cases are to be considered.

**Subcase (i):** If \( e = v_0v_i \) for some \( i \in \{1, 2, 3, \ldots n\} \) then \( \gcd(f(v_0), f(v_i)) = \gcd(p, f(v_i)) = 1 \) as \( p \) is co-prime to every integer from \( \{1, 2, 3, \ldots n\} \).

**Subcase (ii):** If \( e = v_iv'_i \) for some \( i \in \{1, 2, 3, \ldots n - 1\} \) are as same as subcase (ii) in case (i). If \( e = v_{\frac{p+1}{2}}v_{\frac{p-1}{2}} \) then \( \gcd(f(v_{\frac{p+1}{2}}), f(v_{\frac{p-1}{2}})) = (1, f(v_{\frac{p-1}{2}})) = 1 \). If \( e = v_nv'_n \) then \( \gcd(f(v_n), f(v'_n)) = \gcd(2n + 1, 2) = 1 \) as \( 2n + 1 \) is an odd positive integer and it is not divisible by 2.

**Subcase (iii):** If \( e = v'_i v_{i+1} \) for some \( i \in \{1, 2, 3, \ldots n - 1\} \) are as same as subcase (iii) in case (i). If \( e = v_{\frac{p-1}{2}}v_{\frac{p+3}{2}} \) then \( \gcd(f(v_{\frac{p-1}{2}}), f(v_{\frac{p+3}{2}})) = f(v_{\frac{p-1}{2}}) = 1 \). If \( e = v_nv_1 \) then \( \gcd(f(v_n), f(v_1)) = \gcd(2, 3) = 1 \). Thus in all the possibilities described above \( f \) admits prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( G_n \).

Thus \( G_n \) is strongly prime graph for all \( n \).

![Figure 6](image_url)  
**Figure 6.** A prime labeling of Gear graph of \( G_0 \) having the apex vertex\((v_0)\) as label 1
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References


