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Multiplication Formulae for Double Hypergeometric Functions of Exton and Kampé Defériet

Research Article

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Abstract: In this paper, we obtain two multiplication formulae for double hypergeometric functions of Exton and Kampé de Fériet, by the application of Rainville's theorem on generating function.

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1. Introduction

In 1921, Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 [11] and its seven confluent forms Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1, Ξ_2 [11] were unified and generalized by Kampé de Fériet [11]. We recall the definition of general double hypergeometric function of Kampé de Fériet in the slightly modified notation of Srivastava and Panda [12, 13]:

$$F^{A:B;D}_{E:G;H}\left[\begin{array}{lr}(a_A):(b_B);(d_D)\ ; & \\ & x,\,y\\ (e_E):(g_G);(h_H)\ ; & \end{array}\right]=\sum_{m,n=0}^{\infty}\frac{[(a_A)]_{m+n}\,[(b_B)]_m\,[(d_D)]_n\,x^m\,y^n}{[(e_E)]_{m+n}\,[(g_G)]_m\,[(h_H)]_n\,m!\,n!} \qquad (1)$$

In 1967, a unification and generalization of Lauricella's fourteen complete triple hypergeometric functions of second order F_1, F_2, F_3, $\cdots$, F_{14} [2] including Saran's ten triple hypergeometric functions F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S, F_T [11], extended triple hypergeometric function F_K of Sharma [4] and three additional triple hypergeometric functions H_A, H_B, H_C of Srivastava [7, 8], was given by Srivastava [9] in the form:

$$F^{(3)}\left[\begin{array}{lr}(a_A)::(b_B);\ (d_D);\ (e_E):(g_G);\ (h_H);\ (\ell_L); & \\ & x,\ y,\ z\\ (m_M)::(n_N);\ (p_P);\ (q_Q):(r_R);\ (s_S);\ (t_T); & \end{array}\right]$$

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$$= \sum_{i,j,k=0}^{\infty} \frac{[(a_A)]_{i+j+k}\ [(b_B)]_{i+j}\ [(d_D)]_{j+k}\ [(e_E)]_{k+i}\ [(g_G)]_i\ [(h_H)]_j\ [(\ell_L)]_k\ x^i\ y^j\ z^k}{[(m_M)]_{i+j+k}\ [(n_N)]_{i+j}\ [(p_P)]_{j+k}\ [(q_Q)]_{k+i}\ [(r_R)]_i\ [(s_S)]_j\ [(t_T)]_k\ i!\ j!\ k!} \tag{2}$$

In 1982, Exton [1] defined the following double hypergeometric series:

$$X^{A:B;D}_{E:G;H}\left[\begin{array}{ll}(a_A):(b_B);(d_D) & ; \\ & x,\ y \\ (e_E):(g_G);(h_H) & ;\end{array}\right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{2m+n}\,[(b_B)]_m\,[(d_D)]_n\,x^m\,y^n}{[(e_E)]_{2m+n}\,[(g_G)]_m\,[(h_H)]_n\,m!\,n!} \tag{3}$$

which is the generalization and unification of Horn's non confluent double hypergeometric function H_4 [11] and Horn's confluent double hypergeometric function H_7 [11]. For the sake of convenience the symbol (a_A) denotes the array of A parameters given by $a_1, a_2, a_3, \cdots, a_A$ in the contracted notation of Slater[5,p.54;6,p.41]. The symbol $\Delta(N;b)$ denotes the array of N parameters $(N \geq 1)$ given by $(b)/N,\ (b+1)/N,\ (b+2)/N,\ \cdots,\ (b+N-1)/N$. The symbol $\Delta[N;(a_A)]$ denotes the array of NA parameters given by $(a_1)/N,\ (a_1+1)/N,\ (a_1+2)/N,\ \cdots,\ (a_1+N-1)/N,\ (a_2)/N,\ (a_2+1)/N,\ (a_2+2)/N,\ \cdots,\ (a_2+N-1)/N,\ \cdots(a_A)/N,\ (a_A+1)/N,\ (a_A+2)/N,\ \cdots,\ (a_A+N-1)/N$. The asterisk in $\Delta^*(N;j+1)$ represents the fact that the (denominator) parameter N/N obtained from $\Delta(N;j+1)$ is always omitted if $0 \leq\ j\ \leq (N-1)$ so that the set $\Delta^*(N;j+1)$ obviously contains only $(N-1)$ parameters. The Pochhammer's symbol $[(a_A)]_u$ is defined by:

$$[(a_A)]_u = \prod_{m=1}^{A}\{(a_m)_u\} = \begin{cases} \prod_{m=1}^{A}\left\{\frac{\Gamma(a_m+u)}{\Gamma(a_m)}\right\} & ;\ \text{if } a_m \neq 0,-1,-2,\cdots \\ \prod_{m=1}^{A}\{(a_m)(a_m+1)(a_m+2)\cdots(a_m+u-1)\} & ;\ \text{if } u=1,2,3,\cdots \end{cases} \tag{4}$$

with similar interpretation for others and the symbol Γ stands for Gamma function.

2. Some Useful Results

Let us consider the following theorem on generating function due to Rainville [3]: If $H(u)$ have a formal power series expansion in u

$$H(u) = \sum_{m=0}^{\infty} A_m\ u^m \quad ;\ A_0 \neq 0 \tag{5}$$

and

$$(1-t)^{-c} H\left[\frac{-4xt}{(1-t)^2}\right] = \sum_{n=0}^{\infty} G_n(x)\ t^n \tag{6}$$

then

$$x^n = \frac{(c)_{2n}}{2^{2n}\ A_n} \sum_{m=0}^{n} \frac{(-1)^m (c+2m) G_m(x)}{(n-m)!\ (c)_{n+m+1}} \tag{7}$$

If we consider

$$G_n(x) = \frac{(c)_n}{n!}\ {}_{2+B}F_D\left[\begin{array}{ll}-n, c+n, (b_B) & ; \\ & x \\ (d_D) & ;\end{array}\right] \tag{8}$$

then we can prove by series rearrangement technique that:

$$\sum_{n=0}^{\infty} G_n(x)\ t^n = (1-t)^{-c} \sum_{m=0}^{\infty} \frac{(\frac{c}{2})_m(\frac{c+1}{2})_m[(b_B)]_m}{m!\ [(d_D)]_m}\left[\frac{-4xt}{(1-t)^2}\right]^m$$

$$= (1-t)^{-c} \sum_{m=0}^{\infty} A_m \left[\frac{-4xt}{(1-t)^2}\right]^m$$

$$= (1-t)^{-c} H\left[\frac{-4xt}{(1-t)^2}\right] \tag{9}$$

where

$$A_m = \frac{(\frac{c}{2})_m (\frac{c+1}{2})_m [(b_B)]_m}{m!\ [(d_D)]_m} \tag{10}$$

For $G_n(x)$ and A_m, given by (8) and (10) respectively, the relation (7) reduces in the following form:

$$\frac{x^n [(b_B)]_n}{n!\ [(d_D)]_n} = \sum_{m=0}^{n} \frac{(-1)^m\ (c+2m)\ (c)_m}{(n-m)!\ (c)_{n+m+1}\ m!}\ {}_{B+2}F_D\left[\begin{array}{lr} -m, c+m, (b_B) & ; \\ & x \\ (d_D) & ; \end{array}\right] \tag{11}$$

where ${}_{B+2}F_D$ is generalized hypergeometric polynomial of one variable [6]. In the derivations of multiplication formulae (21) and (22), we shall use (11) and following results:

$$\sum_{m=0}^{\infty}\sum_{q=0}^{\infty} \Phi(m,q) = \sum_{j=0}^{2N-1}\sum_{u=0}^{1}\sum_{m=0}^{\infty}\sum_{q=0}^{\infty} \Phi(2Nm+j, 2q+u) \tag{12}$$

which is the particular case of the following series identity due to Srivastava [10, 11]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(m,n) = \sum_{j=0}^{M-1}\sum_{k=0}^{N-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(mM+j, nN+k) \tag{13}$$

$$\sum_{m=0}^{\infty} \Psi(m) = \sum_{j=0}^{N-1}\sum_{m=0}^{\infty} \Psi(mN+j) \tag{14}$$

$$(b)_{M+N} = (b)_M\ (b+M)_N \tag{15}$$

$$[(a_A)]_{M+N} = [(a_A)]_M [(a_A)+M]_N \tag{16}$$

$$(b)_{MN} = M^{MN} \prod_{u=1}^{M} \left\{ \left(\frac{b+u-1}{M} \right)_N \right\} \tag{17}$$

$$[(a_A)]_{MN} = M^{MNA} \prod_{u=1}^{M} \left\{ \left[\frac{(a_A)+u-1}{M} \right]_N \right\} \tag{18}$$

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m} \Xi(m,\ r) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \Xi(m+r,\ r) \tag{19}$$

$${}_1F_0\left[\begin{array}{lr} a & ; \\ & z \\ — & ; \end{array}\right] = (1-z)^{-a} = \sum_{m=0}^{\infty} \frac{(a)_m\ z^m}{m!}\quad ; |z| < 1 \tag{20}$$

The identities (15), (17), (19) and (20) are given in the treatise [11]. The identities (16) and (18) can be derived in view of the definition of Pochhammer's symbol (4), (15) and (17).

3. Main Multiplication Formulae

If arguments, numerator and denominator parameters are adjusted in such a way that the each term of resulting power series is completely defined and meaningful then without any loss of convergence, we have

$$(1-xt)^{-a}\ X^{A+N:D;G}_{B:E;H}\left[\begin{array}{lr} \Delta(N;a),\ (a_A) : (d_D) ; (g_G) & ; \\ & \frac{y}{(1-xt)^{2N}},\ \frac{z}{(1-xt)^{N}} \\ (b_B) \quad : (e_E) ; (h_H) & ; \end{array}\right]$$

$$= \sum_{r=0}^{\infty}\sum_{j=0}^{2N-1}\sum_{u=0}^{1} \frac{(-1)^r\ x^{r+j}}{u!\ j!\ r!}\left(\frac{z}{N^N}\right)^u \frac{(a)_{r+Nu+j}}{(b+r)_r\ (b+2r+1)_j} \times \frac{[(a_A)]_u[(g_G)]_u[(k_K)]_{r+j}}{[(b_B)]_u[(h_H)]_u[(\ell_L)]_{r+j}}\ {}_{L+2}F_K\left[\begin{array}{ll} -r,\ b+r,\ (\ell_L)\ ; & \\ & t \\ (k_K)\ ; & \end{array}\right]\times$$

$$F^{(3)}\left[\begin{array}{l} \Delta(2N;a+r+Nu+j)::\text{---}\ ;\ \Delta[2;(a_A)+u]\ ;\ \text{---}:\ \ \Delta[2N;(k_K)+r+j]\ ;\ (d_D)\ ; \\ \text{------}::\text{---}\ ;\ \Delta[2;(b_B)+u]\ ;\ \text{---}:\ \ \Delta[2N;(\ell_L)+r+j],\Delta^*(2N;j+1),\Delta(2N;b+2r+j+1)\ ;\ (e_E)\ ; \end{array}\right.$$

$$\left.\begin{array}{ll} \Delta[2;(g_G)+u]\ ; & \\ & \frac{x^{2N}}{(2N)^{2N(1+L-K)}},\ \frac{y}{4^{(B-A-N)}},\ \frac{z^2}{4^{(B+H+1-A-N-G)}} \\ \Delta[2;(h_H)+u],\Delta^*(2;1+u)\ ; & \end{array}\right] \qquad (21)$$

$$(1-xt)^{-a}\ F^{B+N:E;H}_{D:G;L}\left[\begin{array}{ll} \Delta(N;a),\ (b_B):(e_E)\ ;\ (h_H)\ ; & \\ & \frac{y}{(1-xt)^N},\ \frac{z}{(1-xt)^N} \\ (d_D)\ \ :(g_G)\ ;\ (\ell_L)\ ; & \end{array}\right]$$

$$= \sum_{r=0}^{\infty}\sum_{j=0}^{N-1} \frac{(a)_{r+j}[(k_K)]_{r+j}(-1)^r\ x^{r+j}}{[(a_A)]_{r+j}\ (b+r)_r(b+2r+1)_j\ r!\ j!}\ {}_{A+2}F_K\left[\begin{array}{ll} -r,\ b+r,\ (a_A)\ ; & \\ & t \\ (k_K)\ ; & \end{array}\right]\times$$

$$\times F^{(3)}\left[\begin{array}{l} \Delta(N;a+r+j)::\text{---}\ ;\ (b_B)\ ;\ \text{---}: \\ \text{------}::\text{---}\ ;\ (d_D)\ ;\ \text{---}: \end{array}\right.$$

$$\left.\begin{array}{ll} \Delta[N;(k_K)+r+j]\ ;\ (e_E)\ ;\ (h_H); & \\ & \frac{x^N}{N^{N(A-K+1)}},\ y,\ z \\ \Delta[N;(a_A)+r+j],\Delta^*(N;j+1),\Delta(N;b+2r+j+1)\ ;\ (g_G)\ ;\ (\ell_L)\ ; & \end{array}\right] \qquad (22)$$

4. Derivations of (21) and (22)

Suppose left hand side of (21) is denoted by T, then its power series representation is

$$T = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(\frac{a}{N})_{2p+q}(\frac{a+1}{N})_{2p+q}\cdots(\frac{a+N-1}{N})_{2p+q}[(a_A)]_{2p+q}[(d_D)]_p[(g_G)]_q\ y^p\ z^q\ (1-xt)^{-(a+2Np+Nq)}}{[(b_B)]_{2p+q}[(e_E)]_p[(h_H)]_q\ p!\ q!}$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{N(2p+q)+m}[(a_A)]_{2p+q}[(d_D)]_p[(g_G)]_q\ y^p\ z^q\ x^m\ t^m}{[(b_B)]_{2p+q}[(e_E)]_p[(h_H)]_q\ N^{N(2p+q)}\ p!\ q!\ m!}$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{2Np+Nq+m}[(a_A)]_{2p+q}[(d_D)]_p[(g_G)]_q[(k_K)]_m\ y^p\ z^q\ x^m}{[(b_B)]_{2p+q}[(e_E)]_p[(h_H)]_q[(\ell_L)]_m\ N^{N(2p+q)}\ p!\ q!}$$

$$\times \sum_{r=0}^{m} \frac{(-1)^r\ (b+2r)(b)_r}{(b)_{m+r+1}(m-r)!\ r!}\ {}_{L+2}F_K\left[\begin{array}{ll} -r,\ b+r,\ (\ell_L)\ ; & \\ & t \\ (k_K)\ ; & \end{array}\right]$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(a)_{2Np+Nq+m+r}[(a_A)]_{2p+q}[(d_D)]_p[(g_G)]_q[(k_K)]_{m+r}\ y^p\ z^q\ x^{m+r}(-1)^r\ (b+2r)\ (b)_r}{[(b_B)]_{2p+q}[(e_E)]_p[(h_H)]_q[(\ell_L)]_{m+r}\ N^{N(2p+q)}(b)_{m+2r+1}\ p!\ q!\ r!\ m!}$$

$$\times {}_{L+2}F_K\left[\begin{array}{lr} -r,\ b+r,\ (\ell_L) & ; \\ & t \\ (k_K) & ; \end{array}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\ (-x)^r [(k_K)]_r}{(b+r)_r\ [(\ell_L)]_r\ r!}\ {}_{L+2}F_K\left[\begin{array}{lr} -r,\ b+r,\ (\ell_L) & ; \\ & t \\ (k_K) & ; \end{array}\right]$$

$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a+r)_{2Np+Nq+m}[(a_A)]_{2p+q}[(d_D)]_p[(g_G)]_q[(k_K)+r]_m\ y^p\ z^q\ x^m}{[(b_B)]_{2p+q}[(e_E)]_p[(h_H)]_q[(\ell_L)+r]_m\ N^{N(2p+q)}(b+2r+1)_m\ p!\ q!\ m!}$$

$$= \sum_{r=0}^{\infty}\sum_{j=0}^{2N-1}\sum_{u=0}^{1} \frac{(a)_r\ (-x)^r [(k_K)]_r}{r!\ (b+r)_r [(\ell_L)]_r}\ {}_{L+2}F_K\left[\begin{array}{lr} -r,\ b+r,\ (\ell_L) & ; \\ & t \\ (k_K) & ; \end{array}\right]$$

$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a+r)_{2Np+2Nq+Nu+2Nm+j}[(a_A)]_{2p+2q+u}[(d_D)]_p[(g_G)]_{2q+u}[(k_K)+r]_{2Nm+j}}{(b+2r+1)_{2Nm+j}\ N^{N(2p+2q+u)}[(b_B)]_{2p+2q+u}[(e_E)]_p[(h_H)]_{2q+u}[(\ell_L)+r]_{2Nm+j}} \times \frac{y^p\ z^{2q+u}\ x^{2Nm+j}}{p!\ (2q+u)!\ (2Nm+j)!}$$

$$= \sum_{r=0}^{\infty}\sum_{j=0}^{2N-1}\sum_{u=0}^{1} \frac{(-1)^r\ x^{r+j}\ z^u\ (a)_{r+Nu+j}}{(b+r)_r\ (b+2r+1)_j\ N^{Nu}\ r!\ u!\ j!} \frac{[(a_A)]_u[(g_G)]_u[(k_K)]_{r+j}}{[(b_B)]_u[(h_H)]_u[(\ell_L)]_{r+j}}$$

$$\times {}_{L+2}F_K\left[\begin{array}{lr} -r,\ b+r,\ (\ell_L) & ; \\ & t \\ (k_K) & ; \end{array}\right] \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\prod_{w=1}^{2N}\left\{\left(\frac{a+r+Nu+j+w-1}{2N}\right)_{m+p+q}\right\}}{\prod_{w=1}^{2}\left\{\left[\frac{(b_B)+u+w-1}{2}\right]_{p+q}\right\}}$$

$$\times \frac{\prod_{w=1}^{2}\left\{\left[\frac{(a_A)+u+w-1}{2}\right]_{p+q}\right\}\prod_{w=1}^{2N}\left\{\left[\frac{(k_K)+r+j+w-1}{2N}\right]_{m}\right\}\prod_{w=1}^{2}\left\{\left[\frac{(g_G)+u+w-1}{2}\right]_{q}\right\}}{\prod_{w=1}^{2N}\left\{\left[\frac{(\ell_L)+r+j+w-1}{2N}\right]_{m}\right\}\prod_{w=1}^{2N}\left\{\left(\frac{b+1+2r+j+w-1}{2N}\right)_{m}\right\}\prod_{w=1}^{2N}\left\{\left(\frac{1+j+w-1}{2N}\right)_{m}\right\}}$$

$$\times \frac{[(d_D)]_p\ (1)_m\ (1)_q\ (2N)^{2N(K-L-1)m}\ x^{2Nm}\ 2^{2(A-B+N)p}\ y^p\ 2^{2(A+G-1-B-H+N)q}\ z^{2q}}{\prod_{w=1}^{2}\left\{\left(\frac{1+u+w-1}{2}\right)_{q}\right\}\prod_{w=1}^{2}\left\{\left[\frac{(h_H)+u+w-1}{2}\right]_{q}\right\}[(e_E)]_p\ m!\ p!\ q!}$$

Now representing the triple power series corresponding to summation indices $m,\ p,\ q$ into its hypergeometric form (2), we get the right hand side of (21). Similarly, we can obtain (22).

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