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On Goldberg qth Order and Goldberg qth Type of an Entire Function Represented by Multiple Dirichlet Series

Research Article

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Abstract: In this paper we consider entire function represented by multiple Dirichlet series in several complex variables. Also consider product of the class of entire function, we then characterized the q^{th} order and q^{th} type of an entire function represented and express in terms of its coefficient and exponent.

Keywords: Entire function, Multiple Dirichlet series, Gol'dberg order, Gol'dberg type.

1. Introduction

In this paper we denote complex and real n-space by C^n and R^n respectively. We denote this (s$_1$, s$_{2,...}$ s$_n$) (Re s$_1$, Re s$_2$,...,Re s$_n$) with their corresponding unsuffixed symbols s, Re(s) etc. respectively. We define $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2 \ldots, y_n)$.

$$xy = (x_1y_1, x_2y_2, \ldots, x_ny_n)$$

$$||x|| = x_1 + x_2 + \cdots + x_n$$

$$x + r = (x_1 + r, x_2 + r, \ldots, x_n + r),\ r \in R \text{ and } x, y \in C.$$

For some $k \in I$, integers f^k denote the $\frac{\partial^{||k||} f}{\partial s_1^{k_1} \ldots \partial s_n^{k_n}}$. We denote the n-tuple $(\lambda_{1m_1}, \lambda_{2m_2}, \ldots, \lambda_{nm_n})$ by $\lambda_{n,m}$ and consider the multiple Dirichlet series

$$f(s_1, s_2, \ldots, s_n) = \sum_{m=1}^{\infty} a_m \exp\{s\lambda_{n,m}\} \tag{1}$$

where $a_m \in C\ \ s_j = \sigma_j + it_j \in C,\ \ j = 1, 2, 3, ..., N$ and $\lambda_{n,m}$ satisfies the conditions.

$$0 < \lambda_{p_1} < \ \lambda_{p_2} < \lambda_{p_3} < \cdots < \lambda_{pn} \to \infty, \tag{2}$$

as $k \to \infty$, where $p = 1, 2, \ldots, n$. Also

$$\lim_{m_j \to \infty} \frac{\log\ m_j}{\lambda_j m_j} = 0, \quad j = 1, 2, 3, \ldots, n \tag{3}$$

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then the domain of absolute convergence of the series (1) coincides with its domain of convergence for two entire functions f & g, their product is defined by Alam [2]

$$f \star g = \sum_{m=1}^{\infty} a_m b_m exp\{s\lambda_{n,m}\},$$

where

$$f(s) = \sum_{m=1}^{\infty} a_m exp\{s\lambda_{n,m}\}, \tag{4}$$

and

$$g(s) = \sum_{m=1}^{\infty} b_m exp\{s\lambda_{n,m}\}, \tag{5}$$

for $k \in I$, where I is the set of positive integers, and also defined by Hadamard [2]

$$f^k(s) = \sum_{m=1}^{\infty} \lambda_{n,m}^k a_m exp\{s\lambda_{n,m}\}, \tag{6}$$

$$f^k(s) \star g^k(s) = \sum_{m=1}^{\infty} \lambda_{n,m}^{2k} a_m b_m exp\{s\lambda_{n,m}\}. \tag{7}$$

Definition 1.1. *Poly half plane D as $D = \{s : s \in C^n,\ Re(s) = \sigma \leq l\}$. Then the region $D+r$ depending on the parameter $r \in R$ is defined as $D + r = \{s + r : s \in D\}$. For any $f \in F$, where F stands for the family of multiple Dirichlet series. We define the maximum modulus*

$$M_{f,D}(\sigma) = \sup\{f(s) : s \in D + r\}.$$

Also the maximum term

$$\mu_f = \mu_f(\sigma)$$

at $\sigma \in R^n$, is defined by

$$\mu_{f,D}(\sigma) = \sup_{m \in I^n} \{|a_m| exp||\sigma\lambda_{n,m}||\}. \tag{8}$$

Definition 1.2. *Let f be an entire function and D be the fundamental domain. Also let U_f be the set of all points $\alpha \in R$ such that for every $\alpha \in U_f$ then there exist real number i.e. $\sigma_o \in R$ s.t.*

$$\log\ m_{f,D}(\sigma) \leq \sigma^{\alpha}\ \ for\ \ \sigma \geq \sigma_o.$$

The infimum of the set U_f is called the Gol'dberg order $\rho(D)$ of f with respect to the region D. We say that $f(s)$ is of infinite or finite Gol'dberg order according as U_f is empty or nonempty. Next for the Gol'dberg order $\rho(D) > 0$, let $K_f(\rho)$ be the set of all $k \in R$ such that $\log\ M_{f,D}(\sigma) \leq k(\sigma)^{\rho}$ for $\sigma \geq \sigma_o$. The infimum of the set $K_f(\rho)$ is called Gol'dberg type $T(D)$ of f corresponding to $\rho(D)$. As before we say that $f(s)$ is infinite or finite Gol'dberg type according as $K_f(\rho)$ is empty or nonempty. Form the definition it follows easily that

$$\rho(D) = \lim_{\sigma \to \infty} \sup \frac{loglog M_{f,D}(\sigma)}{\log \sigma}$$

and

$$T(D) = \lim_{\sigma \to \infty} \sup \frac{log M_{f,D}(\sigma)}{\sigma^{\rho(D)}}.$$

These were defined by Gol'dberg [1]. We define q^{th} Gol'dberg order and q^{th} Gol'dberg type $T^q(D)$ of entire function f in corresponding domain D.

$$\rho^q(D) = \lim_{\sigma \to \infty} sup \frac{\log^{[q]} M_{f,D}(\sigma)}{\log \sigma} \tag{9}$$

and

$$T^q(D) = \lim_{\sigma \to \infty} sup \frac{\log[q-1] M_{f,D}(\sigma)}{\sigma^{\rho^q(D)}}. \tag{10}$$

2. Main Results

Lemma 2.1. *The function $f^k(s) \star g^k(s)$ as defined (7) is an entire function.*

Result is already proved by Alam [2].

Theorem 2.2. *Let f be an entire function defined in domain D and $k \in R$ then*

(i) $\rho^q(D+k) = \rho^q(D)$

(ii) $\rho^q(D) > 0$, *then* $T^q(D+k) = T^q(D)$.

Proof.

(i) Let $k \in R$, then from definition (9)

$$
\begin{aligned}
&= \mathop{lim\ sup}_{\sigma\to\infty} \frac{log^{[q]} M_{f\ ,D}\ (\sigma+k)}{log\ \sigma} . \frac{\log\sigma}{\log(\sigma+k)} \\
&= \rho^q\,(D+k)\,.
\end{aligned}
$$

(ii) By definition (10)

$$
\begin{aligned}
T^q(D) &= \lim_{\sigma\to\infty}\ \sup \frac{log[q-1]M_{f,D}(\sigma+k)}{(\sigma+k)^{\rho^q(D+k)}} \\
&= \lim_{\sigma\to\infty}\ \sup \frac{\log[q-1]M_{f,D}(\sigma+k)}{(\sigma)^{\rho^q(D)}} \frac{(\sigma)^{\rho^q(D)}}{(\sigma+k)^{\rho^q(D+k)}} \\
&= T^q(D+k).
\end{aligned}
$$

□

Here we may write ρ^q instead of $\rho^q(D)$. It is also know by Sarkar [3] that

$$\rho = \lim_{m\to\infty}\ \sup \frac{||\lambda_{n,m}|| \log ||\lambda_{n,m}||}{-\log|a_m|}$$

and

$$\rho_k = \lim_{m\to\infty}\ \sup \frac{||\lambda_{n,m}|| \log ||\lambda_{n,m}||}{-\log|a_m\lambda^k_{n,m}|}.$$

q^{th} order and type defined by Bajpai, Kapoor, juneja [4] as

$$\rho = \lim_{n\to\infty}\ \sup \frac{||\lambda_n|| \log^{[q-1]} ||\lambda_n||}{\log|a_n|}$$

and

$$T = \lim_{m\to\infty}\ \sup |a_n|^{\frac{\rho}{n}} \log^{[q-2]} \frac{n}{e\rho}. \tag{11}$$

We can prove in similar way for the q^{th} Gol'dberg order

$$\rho^q = \lim_{m\to\infty}\ \sup \frac{||\lambda_{n,m}|| \log^{[q-1]} ||\lambda_{n,m}||}{-\log|a_m|}$$

and

$$\rho^q_k = \lim_{m\to\infty}\ \sup \frac{||\lambda_{n,m}|| \log^{[q-1]} ||\lambda_{n,m}||}{-\log|a_m\lambda^k_{n,m}|}. \tag{12}$$

Theorem 2.3. *Let f and g be entire function, where*

$$f^k(s) = \sum_{m=1}^{\infty} \lambda_{n,m}^k a_m \exp\{s\lambda_{n,m}\}$$

and

$$g^k(s) = \sum_{m=1}^{\infty} \lambda_{n,m}^k b_m \exp\{s\lambda_{n,m}\}$$

having qth Gol'dberg order $\rho_{k_f}^q$ $(0 < \rho_{k_f}^q < \infty)$ and $\rho_{k_g}^q$ $(0 < \rho_{k_g}^q < \infty)$ respectively. Then $f^k(s) \star g^k(s)$ is an entire function with qth Gol'dberg order ρ_k^q such that $\rho_k^q \le \left(\rho_{k_f}^q \rho_{k_g}^q\right)^{\frac{1}{2}}$ provided that

$$\log \frac{1}{|\lambda_{n,m}^{2k} a_m b_m|} \sim \left\{ log \frac{1}{|\lambda_{n,m}^{2k} a_m|} \log \frac{1}{|\lambda_{n,m}^{2k} b_m|} \right\}^{\frac{1}{2}}.$$

Proof. We have $f^k(s) \star g^k(s)$ is an entire function by lemma (2.1). Now

$$\frac{1}{\rho_{k_f}^q} = \lim_{m\to\infty} inf \frac{-\log |a_m \lambda_{n,m}^k|}{||\lambda_{n,m}|| \log^{[q-1]} ||\lambda_{n,m}||}$$

and

$$\frac{1}{\rho_{k_g}^q} = \lim_{m\to\infty} \inf \frac{-\log |b_m \lambda_{n,m}^k|}{||\lambda_{n,m}|| \log^{[q-1]} ||\lambda_{n,m}||} \quad \text{for } \epsilon > 0$$

$$\frac{1}{\rho_{k_f}^q} - \frac{\epsilon}{2} \le \frac{\log \frac{1}{|a_m \lambda_{n,m}^k|}}{||\lambda_{n,m}|| \log^{[q-1]} ||\lambda_{n,m}||}$$

$$\frac{1}{\rho_{k_g}^q} - \frac{\epsilon}{2} \le \frac{\log \frac{1}{|b_m \lambda_{n,m}^k|}}{||\lambda_{n,m}|| \log^{[q-1]} ||\lambda_{n,m}||}$$

$$\frac{\log \frac{1}{|a_m \lambda_{n,m}^k|} \log \frac{1}{|b_m \lambda_{n,m}^k|}}{(||\lambda_{n,m}|| \log^{[q-1]} ||\lambda_{n,m}||)^2} \ge \left(\frac{1}{\rho_{k_f}^q} - \frac{\epsilon}{2}\right)\left(\frac{1}{\rho_{k_g}^q} - \frac{\epsilon}{2}\right)$$

$$\frac{\{\log \frac{1}{|a_m \lambda_{n,m}^k|} \log \frac{1}{|b_m \lambda_{n,m}^k|}\}^{\frac{1}{2}}}{(||\lambda_{n,m}|| log^{[q-1]} ||\lambda_{n,m}||)} \ge \left\{\left(\frac{1}{\rho_{k_f}^q} - \frac{\epsilon}{2}\right)\left(\frac{1}{\rho_{k_g}^q} - \frac{\epsilon}{2}\right)\right\}^{\frac{1}{2}}.$$

Now if

$$\log \frac{1}{||\lambda_{n,m}^{2k} a_m b_m|} \sim \left\{ \log \frac{1}{|\lambda_{n,m}^{2k} a_m|} \log \frac{1}{|\lambda_{n,m}^{2k} b_m|} \right\}^{\frac{1}{2}},$$

then

$$\frac{\log \frac{1}{|\lambda_{n,m}^{2k} a_m b_m|}}{(||\lambda_{n,m}|| log^{[q-1]} ||\lambda_{n,m}||)} \ge \left\{\left(\frac{1}{\rho_{k_f}^q} - \frac{\epsilon}{2}\right)\left(\frac{1}{\rho_{k_g}^q} - \frac{\epsilon}{2}\right)\right\}^{\frac{1}{2}}.$$

Therefore

$$\lim_{m\to\infty} \sup \frac{\log \frac{1}{|\lambda_{n,m}^{2k} a_m b_m|}}{(||\lambda_{n,m}|| \log^{[q-1]} ||\lambda_{n,m}||)} \ge \left\{ \frac{1}{\rho_{k_f}^q} \frac{1}{\rho_{k_g}^q} \right\}^{\frac{1}{2}}.$$

Then

$$\frac{1}{\rho_k^q} \ge \left\{ \frac{1}{\rho_{k_f}^q \rho_{k_g}^q} \right\}^{\frac{1}{2}}$$

$$\Rightarrow \rho_k^q \le \left(\rho_{k_f}^q \rho_{k_g}^q\right)^{\frac{1}{2}}.$$

□

Theorem 2.4. *Let f^k and g^k be an entire function with q^{th} Gol'dberg order $\rho^q_{k_f}$ $(0 < \rho^q_{k_f} < \infty)$ and $\rho^q_{k_g}$ $(0 < \rho^q_{k_g} < \infty)$ respectively and also q^{th} Gol'dberg type $T^q_{k_f}$ and T^q_{kg}. Then*

$$\rho^q_k \leq \rho^q_{k_f} + \rho^q_{k_g} \text{ and also } T^q_k \leq T^q_{k_f}(D)T^q_{k_g}(D).$$

Proof. We have from (9)

$$\rho^q_{k_f}(D) = \lim_{\sigma\to\infty} \sup \frac{log^{[q]} M_{f^k,D}(\sigma)}{\log\sigma}$$

and

$$\rho^q_{k_g}(D) = \lim_{\sigma\to\infty} \sup \frac{\log^{[q]} M_{g^k,D}(\sigma)}{\log\sigma},$$

we know that

$$(\rho^q_{k_f}\rho^q_{k_g})^{\frac{1}{2}} \leq \frac{\rho^q_{k_f} + \rho^q_{k_g}}{2} \leq \rho^q_{k_f} + \rho^q_{k_g}$$

from theorem (2.3)

$$\rho^q_k \leq \rho^q_{k_f} + \rho^q_{k_g}.$$

Again from (10)

$$\frac{\log^{[q-1]} M_{f^k,D}(\sigma)}{\sigma^{\rho^q_{k_f}(D)}} < T^q_{k_f}(D) + \epsilon$$

$$\frac{\log^{[q-1]} M_{g^k,D}(\sigma)}{\sigma^{\rho^q_{k_g}(D)}} < T^q_{k_g}(D) + \epsilon$$

$$\{\frac{\log^{[q-1]} M_{f^k,D}(\sigma)}{\sigma^{\rho^q_{k_f}(D)}}\}\{\frac{\log^{[q-1]} M_{g^k,D}(\sigma)}{\sigma^{\rho^q_{k_g}(D)}}\} < (T^q_{k_f}(D) + \epsilon)(T^q_{k_g}(D) + \epsilon)$$

$$\{\log^{[q-1]} M_{f^k,D}(\sigma)\}\{\log^{[q-1]} M_{g^k,D}(\sigma)\} \sim \log^{[q-1]} M_{f^k\star g^k,D}(\sigma)$$

then

$$\frac{\log^{[q-1]} M_{f^k\star g^k,D}(\sigma)}{\sigma^{\rho^q_{k_f}(D)+\rho^q_{k_g}(D)}} < (T^q_{k_f}(D) + \epsilon)(T^q_{k_g}(D) + \epsilon).$$

Thus if

$$\rho^q_k = \rho^q_{k_f} + \rho^q_{k_g}$$

$$\lim_{\sigma\to\infty} \sup \frac{\log^{[q-1]} M_{f^k\star g^k,D}(\sigma)}{\sigma^{\rho^q_k}} \leq T^q_{k_f}(D)T^q_{k_g}(D)$$

$$T^q_k \leq T^q_{k_f}(D)T^q_{k_g}(D).$$

□

Theorem 2.5. *If $\rho^q(D) = \lim_{\sigma\to\infty} \sup \frac{\log^{[q]} M_{f,D}(\sigma)}{\log\sigma}$ then*

$$\lim_{\sigma\to\infty} \sup \frac{\log^{[q]} \mu_{f,D}(l+\sigma)}{\log\sigma} \leq \rho$$

and

$$\lim_{\sigma\to\infty} \sup \frac{log^{[q]} \mu_{f,D}(l+\sigma+\epsilon)}{\log\sigma} \geq \frac{\rho}{k},$$

where k is positive constant.

Proof. Let f be an entire function defined in domain D and $k \in R$ then

$$\mu_f(l+\sigma) \le M_{f,D}(\sigma) \le k\mu_f(l+\sigma+\epsilon)$$

where k is positive constant. This result is proved by Sarkar [3]. By above result

$$\lim_{\sigma\to\infty}\sup\frac{\log^{[q]}\mu_{f,D}(l+\sigma)}{\log\sigma} \le \lim_{\sigma\to\infty}\sup\frac{\log^{[q]}M_{f,D}(\sigma)}{\log\sigma}$$
$$\lim_{\sigma\to\infty}\sup\frac{\log^{[q]}\mu_{f,D}(l+\sigma)}{\log\sigma} \le \rho.$$

Again

$$\lim_{\sigma\to\infty}\sup\frac{\log^{[q]}\mu_{f,D}(l+\sigma+\epsilon)}{\log\sigma} \ge \frac{1}{k}\lim_{\sigma\to\infty}\sup\frac{\log^{[q]}M_{f,D}(\sigma)}{\log\sigma}$$
$$\lim_{\sigma\to\infty}\sup\frac{\log^{[q]}\mu_{f,D}(l+\sigma+\epsilon)}{\log\sigma} \ge \frac{\rho}{k}.$$

□

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