International Journal of Mathematics And its Applications
Volume 3, Issue 4–A (2015), 1–7.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Integral Solutions of the Octic Equation With Five Unknowns $(x-y)(x^3+y^3)=4(w^2-p^2)T^6$

Research Article

S.Vidhyalakshmi[1], A.Kavitha[1] and M.A.Gopalan[1]*

1 Department of Mathematics, Shrimati Indira Gandhi College, Trichy, Tamilnadu, India.

Abstract: The non-homogeneous octic equation with five unknowns represented by the Diophantine equation $(x-y)(x^3+y^3)=4(w^2-p^2)T^6$ is analyzed for its patterns of non-zero distinct integral solutions and seven different patterns of integral solutions are illustrated. Various interesting relations between the solutions and special numbers, namely, Pyramidal numbers, Pronic numbers, Stella octangular numbers, Gnomonic numbers, polygonal numbers, four dimensional figurate numbers are exhibited.

MSC: 11D41

Keywords: Octic non-homogeneous equation, Pyramidal numbers, Pronic numbers, Fourth, fifth and sixth dimensional figurate numbers.

1. Introduction and Notations

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1–4]. In [5–12] heptic equations with three, four and five unknowns are analyzed. This communication analyses a non homogeneous octic equation with five unknowns given by $(x-y)(x^3+y^3)=4(w^2-p^2)T^6$ for determining its infinitely many non-zero integer quintuples (x,y,w,p,T) satisfying the above equation are obtained. Various interesting properties among the values of x, y, p, w and T are presented. In this paper we use the following notations.

$t_{m,n}$: Polygonal number of rank n with size m

SO_n : Stella octangular number of rank n

Pr_n : Pronic number of rank n

G_n : Gnomonic number of rank n

CP_n^m : Centered Pyramidal number of rank n with size m.

J_n : Jacobsthal number of rank n

j_n : Jacobsthal-Lucas number of rank n

ky_n : keynea number of rank n.

$F_{4,n,6}$: Four dimensional hexagonal figurate number of rank n

* *E-mail: mayilgopalan@gmail.com*

2. Method of Analysis

The non-homogeneous octic equation with five unknowns to be solved for its distinct non-zero integral solutions is

$$(x-y)(x^3+y^3)=4(w^2-p^2)T^6 \tag{1}$$

Introduction of the linear transformations,

$$\left.\begin{aligned} x=u+v,\ \ w=uv+1 \\ y=u-v,\ \ p=uv-1 \end{aligned}\right\} \tag{2}$$

in (1) leads to

$$u^2+3v^2=4T^6 \tag{3}$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

Pattern: 1

Let

$$T=(a^2+3b^2) \tag{4}$$

Write 4 as

$$4=(1+i\sqrt{3})(1-i\sqrt{3}). \tag{5}$$

Using (4), (5) in (3) and applying the method of factorization define

$$u+i\sqrt{3}v=(1+i\sqrt{3})(\alpha+i\sqrt{3}\beta) \tag{6}$$

where $\alpha+i\sqrt{3}\beta=(a+i\sqrt{3}b)^6$ from which we have

$$\left.\begin{aligned} \alpha&=a^6-45a^4b^2+135a^2b^4-27b^6 \\ \beta&=6a^5b-60a^3b^3+54ab^5 \end{aligned}\right\} \tag{7}$$

Equating real and imaginary parts in (6), we have

$$\left.\begin{aligned} u=\alpha-3\beta \\ v=\alpha+\beta \end{aligned}\right\} \tag{8}$$

Using (8) and (2) the value of x, y, w, T and p are given by

$$x(a,b)=\alpha-2\beta$$
$$y(a,b)=-4\beta$$
$$p(a,b)=\alpha^2-2\alpha\beta-3\beta^2-1$$
$$w(a,b)=\alpha^2-2\alpha\beta-3\beta^2+1$$
$$T(a,b)=a^2+3b^2$$

Properties:

(1) $2x(1,n)-y(1,n)=2[1+5t_{4,n}(27t_{4,n}-9)-(3t_{4,n})^3]$

(2) $y(n,1) = 24[CP_n^6(t_{4,n} - 10) - CP_{19,n} + 9t_{4,n} - 1]$

(3) $T(2^n, 2^n)$ is a square number

Pattern: 2

Instead of (5) we write

$$4 = \frac{(2+8i\sqrt{3})(2-8i\sqrt{3})}{49} \tag{9}$$

Substituting (4) and (9) in (3) and employing the factorization method, define $u + i\sqrt{3}v = \frac{1}{7}[(2\alpha - 24\beta) + i\sqrt{3}(8\alpha + 2\beta)]$. Equating real and imaginary pats, we have

$$\left.\begin{aligned} u &= \tfrac{1}{7}(2\alpha - 24\beta) \\ v &= \tfrac{1}{7}(8\alpha + 2\beta) \end{aligned}\right\} \tag{10}$$

As our interest is on finding integer solutions, we choose a and b suitably so that u and v are integers. Replace a by 7a and b by 7b in (7) substituting the corresponding values of α and β in (10) and employing (2) non-zero integral solutions to (1) are found to be

$$x(a,b) = 7^5(10\alpha - 22\beta)$$
$$y(a,b) = 7^5(-6\alpha - 26\beta)$$
$$w(a,b) = 7^{10}(16\alpha^2 - 188\alpha\beta - 48\beta^2) + 1$$
$$p(a,b) = 7^{10}(16\alpha^2 - 188\alpha\beta - 48\beta^2) - 1$$
$$T(a,b) = T^2(a^2 + 3b^2)$$

Properties:

(1) $3x(n,1) + 5y(n,1) = -14^2 7^5.6[t_{4,n}(2CP_n^3 + 2t_{15,n} - 13t_{4,n}) + CP_{18,n} - 9t_{4,n} - 1]$

(2) $13x(n,1) - 11y(n,1) = 7^5 14^2\{t_{4,n}[6F_{4,n,6} - 2CP_n^9 - Pr_n - 46t_{4,n} + 135] - 27\}$

(3) $T(2^n, 2^{n-1}) = 7^2[Ky_n - j_{2n+1} + 9J_{2n-2} + 3]$

Pattern: 3

(3) can be written as

$$u^2 + 3v^2 = 4T^6 * 1 \tag{11}$$

Write 1 as

$$1 = \frac{(11 + i5\sqrt{3})(11 - i5\sqrt{3})}{14^2} \tag{12}$$

Substituting (4), (5) and (12) in (11) and employing the factorization method, define

$$(u + i\sqrt{3}v) = \frac{1}{14}[-4(\alpha + 12\beta) + i\sqrt{3}(4\alpha - \beta)] \tag{13}$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} u &= -\tfrac{1}{14}.4(\alpha + 12\beta) \\ v &= \tfrac{1}{14}.4(4\alpha - \beta) \end{aligned}\right\} \tag{14}$$

Using (14) in (2) and proceeding as in Pattern: 2, the corresponding integer solutions to (1) are as follows

$$x(a,b)=4.14^5(3\alpha-13\beta)$$
$$y(a,b)=-4.14^5(5\alpha+11\beta)$$
$$w(a,b)=-14^{10}4^2(\alpha+12\beta)(4\alpha-\beta)+1$$
$$p(a,b)=-14^{10}4^2(\alpha+12\beta)(4\alpha-\beta)-1$$
$$T=14^2(a^2+3b^2)$$

Properties:

(1) $5x(n,1)+3y(n,1)=2^3 7^2 14^5.6\{CP_n^6(t_{4,n}-10)-54\}$

(2) $11x(n,1)-13y(n,1)=2^3 7^2 14^5 t_{4,n}[24F_{4,n,3}-6CP_n^5-2CP_n^3-CP_{8,n}-52t_{4,n}+136]-27$

(3) $T(2^n,2^{n-1})=14^2[9J_{2n-2}+j_{2n}+2]$

Pattern: 4

Let

$$\left.\begin{aligned} u&=2U\\ v&=2V \end{aligned}\right\} \tag{15}$$

Substituting (15) in (3), we get

$$U^2+3V^2=T^6 \tag{16}$$

Using (4) and (16) and employing the method of factorization, define

$$U+i\sqrt{3}V=(\alpha+i\sqrt{3}\beta) \tag{17}$$

Equating real and imaginary parts, in (17) we get

$$U=\alpha \text{ and } V=\beta \tag{18}$$

Substituting (18) in (15) and using (2), the non-zero distinct integral solutions to (1) are

$$x(a,b)=2(\alpha+\beta)$$
$$y(a,b)=2(\alpha-\beta)$$
$$w(a,b)=4\alpha\beta+1$$
$$p(a,b)=4\alpha\beta-1$$
$$T(a,b)=a^2+3b^2$$

Pattern: 5

(16) can be written as,

$$U^2+3V^2=T^6*1 \tag{19}$$

Using (4) and (12) and following the procedure as presented in Pattern: 3, the corresponding non-zero integral solutions to (1) are given by

$$x(a,b) = 7^5(16\alpha - 4\beta)$$

$$y(a,b) = 7^5(6\alpha - 26\beta)$$

$$w(a,b) = 7^{10}(11\alpha - 15\beta)(5\alpha + 11\beta) + 1$$

$$p(a,b) = 7^{10}(11\alpha - 15\beta)(5\alpha + 11\beta) - 1$$

$$T(a,b) = 7^2(a^2 + 3b^2)$$

Pattern: 6

Instead of (2), we can write

$$\left.\begin{aligned} x = 2u + v, y = 2u - v \\ w = u + v, p = u - v \end{aligned}\right\} \tag{20}$$

in (1) which leads to

$$v^2 + 2u^2 = T^6 \tag{21}$$

Let

$$T(a,b) = a^2 + 2b^2 \tag{22}$$

Using (22) in (21) and applying the method of factorization, we define

$$v + i\sqrt{2}u = \alpha_1 + i\sqrt{2}\beta_1 \tag{23}$$

where $\alpha_1 + i\sqrt{2}\beta_1 = (a + \sqrt{2}b)^6$ from which we have

$$\left.\begin{aligned} \alpha_1 = a^6 - 30a^4b^2 + 60a^2b^4 - 8b^6 \\ \beta_1 = 6a^5b - 40a^3b^3 + 24ab^5 \end{aligned}\right\} \tag{24}$$

Equating real and imaginary parts, we find

$$v = \alpha_1 \text{ and } u = \beta_1 \tag{25}$$

Using (25) and (2), the values of x, y, w, p and T are given by

$$x(a,b) = \alpha_1 + 2\beta_1$$

$$y(a,b) = -\alpha_1 + 2\beta_1$$

$$w(a,b) = \alpha_1 + \beta_1$$

$$p(a,b) = -\alpha_1 + \beta_1$$

$$T(a,b) = a^2 + 2b^2$$

Pattern: 7

Consider (21) as

$$v^2 + 2u^2 = T^6 * 1 \tag{26}$$

Write 1 as

$$1=\frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{3^2} \tag{27}$$

Substituting (22) and (27) in (26) and employing the factorization method, define

$$v+i\sqrt{2}u=\frac{(1+i2\sqrt{2})}{3}(a+i\sqrt{2}b)^6$$
$$v+i\sqrt{2}u=\frac{1}{3}[(\alpha_1-4\beta_1)+i\sqrt{2}(2\alpha_1+\beta_1)]$$

Equating real and imaginary parts, we have

$$u=\frac{(2\alpha_1+\beta_1)}{3} \quad v=\frac{(\alpha_1-4\beta_1)}{3} \tag{28}$$

Using (28) in (20) and following as in Pattern: 2, the corresponding integer solutions to (1) are

$$x(a,b)=3^5(5\alpha_1-2\beta_1)$$
$$y(a,b)=3^5(3\alpha_1+6\beta_1)$$
$$w(a,b)=3^5(3\alpha_1-3\beta_1)$$
$$p(a,b)=3^5(\alpha_1+5\beta_1) \quad \text{along with (22)}$$

3. Conclusion

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous octic equation with five unknowns. As the octic equations are rich in variety, one may search for other forms of octic equation with variables greater than or equal to five and obtain their corresponding properties.

Acknowledgement

The finical support from the UGC, New Delhi (F.MRP-5123/14(SERO/UGC) dated march 2014) for a part of this work is gratefully acknowledged.

References

[1] L.E.Dickson, *History of Theory of Numbers*, Vol.11, Chelsea Publishing company, New York (1952).

[2] L.J.Mordell, *Diophantine eequations*, Academic Press, London, (1969).

[3] S.G.Telang, *Number theory*, Tata Mc Graw Hill publishing company, New Delhi, (1996).

[4] R.D.Carmichael, *The theory of numbers and Diophantine Analysis*, Dover Publications, New York, (1959).

[5] Manjusomnath, G.Sangeetha and M.A.Gopalan, *On the non-homogeneous heptic equations with three unknowns* $x^3+(2^p-1)y^5=z^7$, Diophantus J Maths., 1(2)(2012), 117-121.

[6] M.A.Gopalan and G.Sangeetha, *Parametric integral solutions of the heptic equation with five unknown* $x^4-y^4+2(x^3+y^3)(x-y)=2(X^2-Y^2)z^5$, Bessel J Math., 1(1)(2011), 17-22.

[7] M.A.Gopalan and G.Sangeetha, *On the heptic diophantine equations with five unknowns* $x^4-y^4=(X^2-Y^2)z^5$, Antarctica J Math., 9(5)(2012), 371-375.

[8] M.A.Gopalan, G.Sumathi and S.Vidhyalakshmi, *On the non-homogeneous heptic equation with four unknowns* $xy(x+y)+2zw^6$, International Journal of Engineering Sciences and Research Technology, 2(5)(2013), 1313-1317.

[9] M.A.Gopalan, G.Sumathi and S.Vidhyalakshmi, *On the non-homogeneous heptic equation with four unknowns*$(x^2+y^2)(x+y)^4=z^4w^3$, International Journal of Engineering Research Online, 1(2)(2013), 252-255.

[10] M.A.Gopalan, G.Sumathi and S.Vidhyalakshmi, *On the non-homogeneous heptic equations with four unknowns* $xy(x+y)+2zw^6=0$, International Journal of Engineering Sciences and Research Technology, 2(5)(2013), 1113-1117.

[11] M.A.Gopalan, G.Sumathi and S.Vidhyalakshmi, *On the non-homogeneous heptic equations with five unknowns* $(x^2+y^2)(x+y)^4=z^4w^3$, International Journal of Engineering Research Online, 1(2)(2013), 252-255.

[12] M.A.Gopalan, G.Sumathi and S.Vidhyalakshmi, *On the non-homogeneous octic equations with five unknowns* $(x^4-y^4)=T^6(z^2-w^2)$, Scholar Journal of Physics,mathematics and Statistics, 1(2)(2014), 84-87.