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Rainbow Connection Number of Sunlet Graph and its Line, Middle and Total Graph

Research Article

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Abstract: A path in an edgecolored graph is said to be a rainbow path if every edge in the path has a different color. An edge colored graph is rainbow connected if there exists a rainbow path between every pair of its vertices. The rainbow connection number of a graph G, denoted by $rc(G)$, is the smallest number of colors required to color the edges of G such that G is rainbow connected. Given two arbitrary vertices u and v in G, a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. G is strongly rainbow connected if there exist a rainbow $u-v$ geodesic for any two vertices u and v in G. The strong rainbow connection number of G, denoted by $src(G)$, is the minimum number of colors required to make G strongly rainbow connected. SyafrizalSyet. al. in [2] proved that, for the sunlet graph S_n, $rc(S_n) = src(S_n) = \lfloor \frac{n}{2} \rfloor + n$ for $n \geq 2$. In this paper, we improve this result and showthat $rc(S_n) = src(S_n) = \begin{cases} n, & \text{if n is odd;} \\ \frac{3n-2}{2}, & \text{if n is even.} \end{cases}$ We also obtain the rainbow connection number and strong rainbow connection number for the line, middle and total graphs of S_n.

MSC: 05C15.

Keywords: Rainbow connection number, sunlet graph, line graph, middle graph and total graph.

1. Introduction

All graphs in this paper are finite, undirected and simple. Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \to \{1, \ldots, k\}$, $k \in N$ is defined, where adjacent edges may be colored the same. A path in G is a rainbow path if no two edges of it are colored the same. Clearly, if G is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected-just color each edge with a distinct color. Given two arbitrary vertices u and v in G, a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. G is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. The strong rainbow connection number of G, denoted by $src(G)$, is the minimum number of colors required to make G strongly rainbow connected.

Chartrand et al. in [3] introduced the concept of rainbow coloring and determined $rc(G)$ and $src(G)$ of the cycle, path, tree and wheel graphs. In [4] and [5], Li and Sun studied the rainbow connection numbers of line graphs in the light of particular properties of line graphs shown in [6] and [7]. They gave two sharp upper bounds for rainbow connection number of a line graph.

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Yuefang Sun in [1], investigated the rainbow connection number of the line graph, middle graph and total graph of a connected triangle-free graph G and obtained three (near) sharp upper bounds in terms of the number of vertex-disjoint cycles of the original graph G.

Definition 1.1. *The n-Sun let graph of 2n vertices is obtained by attaching n-pendent edges to the cycle C_n and is denoted by S_n. Figure 1 below illustrates the Sunlet graph S_n.*

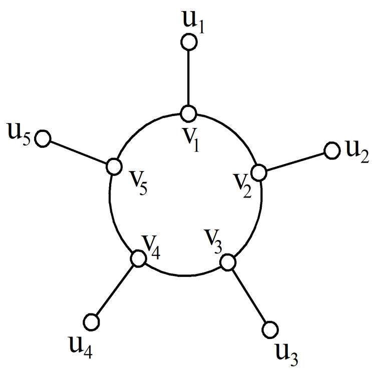

Figure 1. **Sun let graph S_5.**

Definition 1.2. *The line graph of a graph G, denoted by L(G), is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.*

Definition 1.3. *Let G be a graph with vertex set V(G) and edge set E(G). The middle graph of G, denoted by T(G), is defined as follows. The vertex set of M(G) is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of M(G) are adjacent in M(G) in case one of the following holds:*

i) x,y are in E(G) and x, y are adjacent in G.

ii) x is in V(G), y is in E(G), and x,y are incident in G.

Definition 1.4. *Let G be a graph with vertex set V(G) and edge set E(G). The Total graph of G, denoted by T(G), is defined as follows. The vertex set of T(G) is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of T(G) are adjacent in T(G) in case one of the following holds*

i) x, y are in V(G) and x is adjacent to y in G.

ii) x, y are in E(G) and x, y are adjacent in G.

iii) x is in V(G), y is in E(G), and x, y are incident in G.

2. Preliminary Result

In the following corollary SyafrizalSy et.al in [2] determined the $rc(S_n)$ and $src(S_n)$.

Corollary 2.1. *The rainbow connection number and strong rainbow connection number of a graph S_n for $n \geq 2$ are $rc(S_n) = src(S_n) = \lfloor \frac{n}{2} \rfloor + n$.*

3. Main Result

In this section, we improve the result proved by SyafrizalSy et.al in [2]. We state this result in Theorem 3.1 below.

Theorem 3.1. *If $n \geq 3$, $rc(S_n) = src(S_n) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ \frac{3n-2}{2}, & \text{if } n \text{ is even.} \end{cases}$*

Proof. Let us define the vertex set V and the edge set E of S_n as $V(S_n) = \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\}$ where v_i are the vertices of cycles taken in cyclic order and u_i are the pendent vertices such that v_iu_i is a pendent edge and $E(S_n) = \{e'_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n-1\} \cup \{e_n\}$, where e_i is the edge v_iv_{i+1} $(1 \leq i \leq n-1)$, e_n is the edge v_nv_l and e'_i is the edge v_iu_i $(1 \leq i \leq n)$.

Case 1: n is odd

Since all the paths from u_i to u_j for $1 \leq i \leq n$ and $1 \leq j \leq n$ go through the pendent edges e'_i, it is obvious that the color of the edges e'_i must be different. i.e. $c(e'_i) = i$ for $1 \leq i \leq n$. Hence

$$rc(S_n) \geq n \tag{1}$$

Now to get rainbow connectivity between any two vertices of S_n, assign the colors to the edges of cycle as $c\left(e_{(i+2)(mod\ n)}\right) = i$ for $1 \leq i \leq n$ (multiplicative modulo n). From the above assignment of colors it is clear that,

$$rc(S_n) \leq n \tag{2}$$

From (1) and (2) $rc(S_n) = src(S_n) = n$.

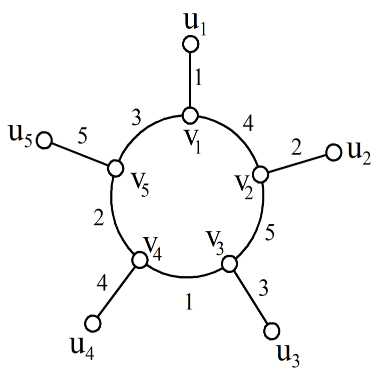

Figure 2. **Sun let Graph S_5 with $rc(S_5) = src(S_5) = 5$**

Case 2: n is even

As in case (1), let $c(e'_i) = i$ for $1 \leq i \leq n$. Assign colors to the edges of the cycle as,

$$c(e_i) = \begin{cases} \frac{i}{2} + 1, & \text{for } i = n \\ 2i, & \text{for } i = \frac{n}{2} \\ i + n, & \text{for } 1 \leq i \leq \frac{n}{2} - 1 \\ i + \frac{n}{2}, & \text{for } \frac{n}{2} + 1 \leq i \leq n - 1. \end{cases}$$

From the above assignment, it is clear that for $n = 4, 6, 8, \ldots$

$$rc(S_n) = src(S_n) = 5, 8, 11, \ldots.$$

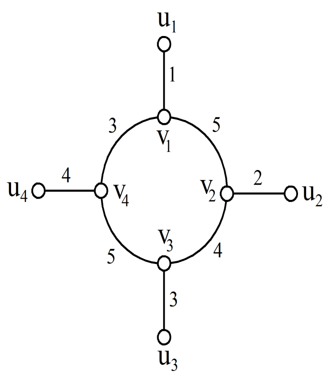

Figure 3. **Assignment of colors in S_4**

This proves $rc(S_n) = src(S_n) = \frac{3n-2}{2}$.

□

Theorem 3.2. *If $n \geq 3$ and $G = L(S_n)$, then* $rc(G) = src(G) = \begin{cases} 2, & for\ n = 3 \\ 3, & for\ n = 4 \\ 4, & for\ n = 5\ \&\ 6 \\ \lceil \frac{n}{2} \rceil + 2, & for\ n \geq 7. \end{cases}$

Proof. The vertex and edge sets of S_n are as described in Theorem 3.1. By the definition of line graph $V(G) = E(S_n) = \{u_i' : 1 \leq i \leq n\} \cup \{v_i' : 1 \leq i \leq n-1\} \cup \{v_n'\}$ where v_i' and u_i' represent the edge e_i and e_i' $(1 \leq i \leq n)$ respectively.

Case 1 : For $n = 3$, define the coloring $c : E(G) \rightarrow \{1, 2\}$ as,

$$c(v_1'v_2') = c(v_2'v_3') = c(v_3'v_1') = 1$$
$$c(v_i'u_i') = 1 \text{ for } 1 \leq i \leq 3$$
$$c(v_i'u_{i+1}') = 2 \text{ for } 1 \leq i \leq 2 \text{ and}$$
$$c(v_3'u_1') = 2, \text{ which is a rainbow coloring.}$$

In this assignment since we cannot assign more than 2 colors (which is optimum) and hence it follows that $rc(G) = src(G) = 2$.

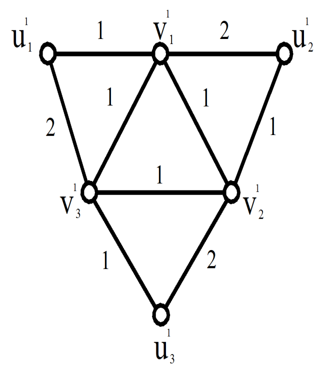

Figure 4. **Line graph of Sun let Graph** $L(S_3)$

Case 2 : For $n = 4$, define the coloring $c : E(G) \to \{1, 2, 3\}$ as,

$$c(v'_i v'_{i+1}) = \begin{cases} 1, & \text{if i is odd and } 1 \leq i \leq 4 \\ 2, & \text{if i is even and } 1 \leq i \leq 4 \end{cases}$$

$$c(v'_i u'_i) = 3 \text{ for } 1 \leq i \leq 4$$

$$c(v'_i u'_{i+1}) = 2 \text{ for } 1 \leq i \leq 3 \text{ and}$$

$$c(v'_n u'_l) = 2, \text{ which is a rainbow coloring.}$$

In this assignment since we cannot assign more than 3 colors (which is optimum) and hence it follows that $rc(G) = src(G) = 3$.

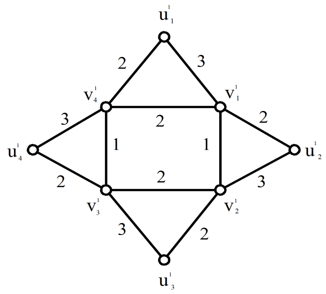

Figure 5. **Line graph of Sun let Graph** $L(S_4)$

Case 3 : For $n = 5$, define the coloring $c : E(G) \to \{1, 2, 3, 4\}$ as,

$$c(v'_i v'_{i+1}) = \begin{cases} 1, & \text{if i is odd and } 1 \leq i \leq 4 \\ 2, & \text{if i is even and } 1 \leq i \leq 5 \end{cases}$$

$$c(v'_n v'_l) = 1$$

$$c(v'_i u'_i) = 3 \text{ for } 1 \leq i \leq 5$$

$$c(v'_i u'_{i+1}) = 4 \text{ for } 1 \leq i \leq 4 \text{ and}$$

$$c(v'_5 u'_l) = 4, \text{ which is a rainbow coloring.}$$

In this assignment since we cannot assign more than 4 colors (which is optimum) and hence it follows that $rc(G) = src(G) = 4$.

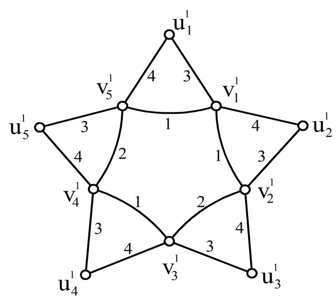

Figure 6. **Line graph of Sun let Graph** $L(S_5)$

Case 4 : For $n = 6$, define the coloring $c : E(L(S_n)) \to \{1, 2, 3, 4\}$ as,

$$c(v'_i v'_{i+1}) = \begin{cases} 1, & \text{if i is odd and } 1 \leq i \leq 6 \\ 2, & \text{if i is even and } 1 \leq i \leq 6 \end{cases}$$

$$c(v'_i u'_i) = 3 \;\text{ for }\; 1 \leq i \leq 6$$

$$c(v'_i u'_{i+1}) = 4 \;\text{ for }\; 1 \leq i \leq 5 \;\text{ and}$$

$$c(v'_6 u'_l) = 4, \quad \text{which is a rainbow coloring.}$$

In this assignment since we cannot assign more than 4 colors (which is optimum) and hence it follows that $rc(G) = src(G) = 4$.

Case 5 : If $n \geq 7$,

Let $C_n : v'_1, v'_2, \ldots, v'_n, v'_{n+1} = v'_1$ and for each i for $1 \leq i \leq n$, be the vertices of inner cycle and let the edges of C_n be $e_i = v'_i v'_{i+1}$. Define

$$c(e_i) = \begin{cases} i, & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ i - \lceil \frac{n}{2} \rceil, & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n \end{cases}$$

$$c(v'_i u'_i) = \lceil \frac{n}{2} \rceil + 1 \;\text{ for }\; 1 \leq i \leq n$$

$$c\left(u'_i, v'_{(i+1)(mod\ n)}\right) = \lceil \frac{n}{2} \rceil + 2 \;\text{ for }\; 1 \leq i \leq n \tag{3}$$

This assignment is clearly a rainbow coloring and from (3), It follows that $rc(G) = src(G) = \lceil \frac{n}{2} \rceil + 2$. □

Theorem 3.3. *If $n \geq 3$ and $G = M(S_n)$, then $rc(G) = src(G) = n + 1$.*

Proof. The vertex and edge sets of S_n are as described in Theorem 3.1. By the definition of middle graph

$$V(G) = V(S_n) \cup E(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u'_i : 1 \leq i \leq n\},$$

where v'_i and u'_i represents the edge e_i and e'_i $(1 \leq i \leq n)$ respectively. Define

$$\begin{aligned}
c(u'_i u_i) &= i && 1 \leq i \leq n \\
c(v'_i u'_i) &= n + 1 && 1 \leq i \leq n \\
c(v'_{i+1} u'_{i+1}) &= i && 1 \leq i \leq n - 1 \\
c(v_1 u'_1) &= n && \\
c(v'_i u'_{i+1}) &= (i + 2)(mod\ n) && 1 \leq i \leq n - 1 \\
c(v'_n u'_1) &= 2 && \\
c(v_i v'_i) &= i && 1 \leq i \leq n \\
c(v'_i v_{i+1}) &= i + 1 && 1 \leq i \leq n - 1 \\
c(v'_n v_1) &= 1 && \\
c(v'_i v'_{i+1}) &= i + 1 && 1 \leq i \leq n - 1 \\
c(v'_n v'_1) &= 1 &&
\end{aligned}$$

From this assignment, it follows that $rc(G) = src(G) = n + 1$.

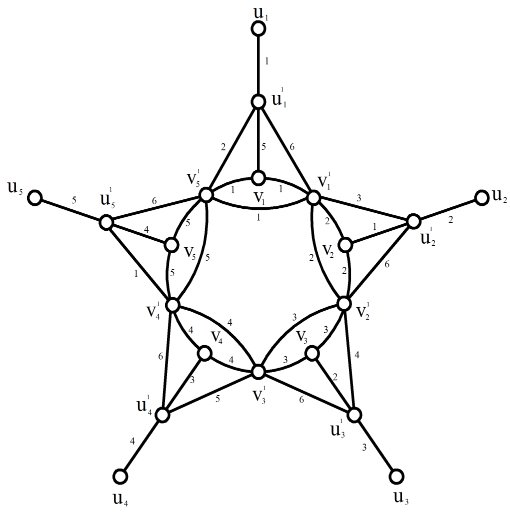

Figure 7. **The graph** $M(S_5)$

$\square$

Theorem 3.4. *If $n \geq 3$ and $G = T(S_n)$, then* $rc(G) = src(G) = \begin{cases} n, & \textit{if } n \textit{ is odd} \\ n+1, & \textit{if } n \textit{ is even} \end{cases}$

Proof. The vertex and edge sets of S_n are as described in Theorem 3.1. By the definition of total graph

$$V(G) = V(S_n) \cup E(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{v_i' : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_i' : 1 \leq i \leq n\},$$

where v_i' and u_i' represents the edge e_i and e_i' $(1 \leq i \leq n)$ respectively.

Case 1 : If n is odd

Define

$$\begin{aligned}
c(v_iu_i) &= i && 1 \leq i \leq n \\
c(v_i'u_i') &= i && 1 \leq i \leq n \\
c(v_iu_i') &= i && 1 \leq i \leq n \\
c(v_1u_1') &= n && \\
c(v_i'u_{i+1}') &= i+1 && 1 \leq i \leq n-1 \\
c(v_n'u_1') &= 1 && \\
c(v_1u_1') &= n && \\
c(u_i'u_i) &= (i+3)(mod\, n) && 1 \leq i \leq n \\
c(v_iv_i') &= i && 1 \leq i \leq n \\
c(v_i'v_{i+1}) &= i+1 && 1 \leq i \leq n-1 \\
c(v_n'v_1') &= 1 && \\
c(v_iv_{i+1}) &= (i+3)(mod\, n) && 1 \leq i \leq n-1 \\
c(v_nv_1) &= n-2 &&
\end{aligned}$$

From this assignment, it follows that $rc(G) = src(G) = n$.

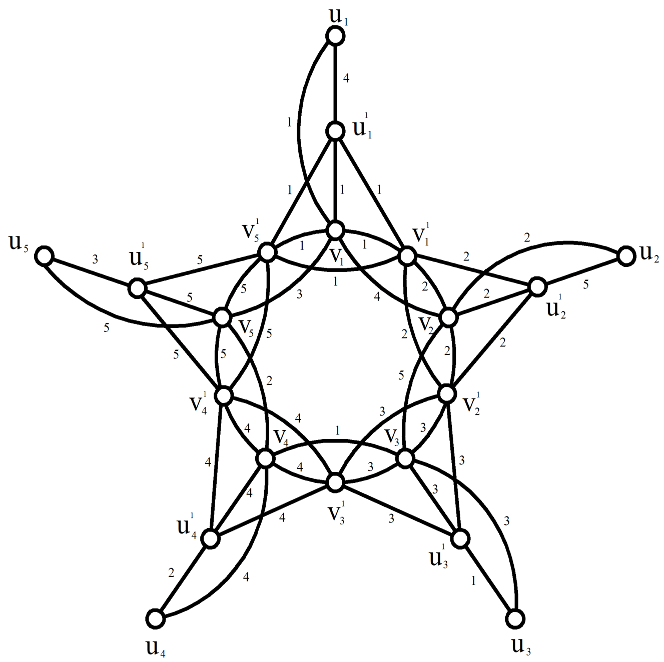

Figure 8. **The graph** $T(S_5)$

Case 2 : If n is even

Define

$$\begin{array}{ll}
c(v_iu_i) = i & 1 \le i \le n \\
c(u_i'u_i) = i & 1 \le i \le n \\
c(v_i'u_i') = n+1 & 1 \le i \le n \\
c(v_{i+1}'u_{i+1}') = i & 1 \le i \le n-1 \\
c(v_1u_1') = n & \\
c(v_i'u_{i+1}') = (i+2)(mod\ n) & 1 \le i \le n-1 \\
c(v_n'u_1') = 2 & \\
c(v_iv_i') = i & 1 \le i \le n \\
c(v_i'v_{i+1}') = i+1 & 1 \le i \le n-1 \\
c(v_n'v_1) = 1 & \\
c(v_i'v_{i+1}') = i+1 & 1 \le i \le n-1 \\
c(v_n'v_1') = 1 & \\
c(v_iv_{i+1}) = (i+3)(mod\ n) & 1 \le i \le n-1 \\
c(v_nv_1) = n-2 &
\end{array}$$

From this assignment, it follows that $rc(G) = src(G) = n+1$.

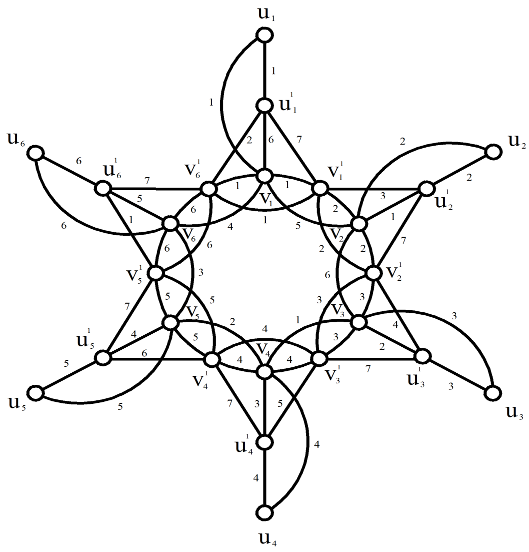

Figure 9. The graph $T(S_6)$

$\square$

4. Conclusion

SyafrizalSy et. al. in [2] proved that, for the sunlet graph S_n, $rc(S_n) = src(S_n) = \lfloor \frac{n}{2} \rfloor + n$ for $n \geq 2$. In this paper, we improve this result and show that $rc(S_n) = src(S_n) = n$ if n is odd and $\frac{3n-2}{2}$ if n is even. We also obtain the rainbow connection number and strong rainbow connection number for the line, middle and total graphs of S_n.

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References

[1] Yuefang Sun, *Rainbow connection numbers of line graphs, middle graphs and total graphs*, International Journal of Applied Mathematics and Statistics, 42(12)(2103).

[2] SyafrizalSy, GemaHistaMedika and LyraYulianti, *Rainbow Connection Numbers of fan and sun*, Applied Mathematical Sciences, 7(64)(2013), 3155-3159.

[3] G.Chartrand, G.Johns, K.McKeon and P.Zhang, *Rainbow connection in graphs*, Math. Bohem, 133(1)(2008), 8598.

[4] X.Li and Y.Sun, *Rainbow connection numbers of line graphs*, Ars Combin., 100(2011), 449463.

[5] X.Li and Y.Sun, *Upper bounds for the rainbow connection numbers of line graphs*, Graphs & Combin, to appear.

[6] S.T.Hedetniemi and P.J.Slater, *Line graphs of triangleless graphs and iterated clique graphs, in Graph Theory and Applications*, Lecture Notes in Math. 303 (ed. Y. Alavi et al.), Springer-Verlag, Berlin, Heidelberg, New York, (1972).

[7] R.Hemminger and L.Beineke, *Line graphs and line digraphs, in Selected Topics in Graph Theory*, Academic Press, London, New York, San Francisco, (1978).