International Journal of Mathematics And its Applications
Volume 3, Issue 4–A (2015), 15–22.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On Mild Solutions of Nonlocal Semilinear Impulsive Functional Integro-Differential Equations of Second Order

Research Article

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Abstract: This paper investigates the existence, uniqueness and continuous dependence on initial data of mild solutions of second order nonlocal semilinear functional impulsive integro-differential equations of more general type with delay in Banach spaces by using Banach contraction theorem and theory of cosine family of operators.

MSC: Primary: 34A12, Secondary: 45J05, 47H10 ,47B38.

Keywords: Existence, Uniqueness , Impulsive, Functional integro-differential equation, Fixed point, Cosine family, Second order.

1. Introduction

Impulsive differential equations in recent years have been an object of investigation with increasing interest because of the wide possibilities for their applications in various fields of science and technology such as population dynamics, medicine, economics, biology etc. For more information see, [1], [8], [9], [12], and the references cited therein. Also, as nonlocal condition is more precise to describe natural phenomena than classical initial condition, the differential problem with nonlocal condition also received much attention in recent years, for example see [2], [3], [5–7], [10].

On other hand, the problems of qualitative properties of solutions of second order functional differential equations have studied by many authors,for example see [1], [3], [4], [6], [7], [11], [13–16]. It is advantageous to treat second order abstract differential equations directly rather than to convert into first order differential system. Recently, In [4] P. M. Dhakne et. al. studied existence, uniqueness and continuous dependence on initial data of mild solutions of integro-differential equations of the type

$$x''(t) = Ax(t) + f(t, x_t, \int_0^t k(t,s)g(s,x_s)ds), \quad t \in [0,T]$$
$$x_0(t) = \phi(t), \quad -r \le t \le 0,$$
$$x'(0) = \delta$$

where A is infinitesimal generator of strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ in Banach space X, f, g, k are continuous functions, ϕ and δ are given elements of C and X respectively using Schauder fixed point theorem. Also in [6], Rupali

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Jain et. al. studied existence, uniqueness and continuous dependence on initial data of mild solutions of integro-differential equations of the type

$$
\begin{aligned}
&x''(t) = A_2x(t) + f(t, x_t), \quad t \in [0, T],\\
&x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0,\\
&x'(0) = \eta \in X
\end{aligned}
$$

using modified version of Banach contraction theorem.

In the present paper, we consider semilinear functional impulsive second order integro-differential equation with nonlocal condition of the type:

$$x''(t) = Ax(t) + f(t, x_t, \int_0^t k(t, s)h(s, x_s)ds), \quad t \in (0, T], \quad t \ne \tau_k, k = 1, 2, .., m \tag{1}$$

$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \le t \le 0, \tag{2}$$

$$x'(0) = \eta, \quad \eta \in X \tag{3}$$

$$\Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1, 2, ..., m, \tag{4}$$

$$\Delta x'(\tau_k) = \bar{I}_k x(\tau_k), \quad k = 1, 2, ..., m, \tag{5}$$

where $0 < t_1 < t_2 < ... < t_p \le T$, $p \in \mathbb{N}$, A is infinitesimal generator of strongly continuous cosine family of bounded linear operators $\{C(t)\}_{t\in\mathbb{R}}$ and I_k, $\bar{I}_k(k = 1, 2, ..., k)$ are the linear operators acting in a Banach space X. The functions f, h, g, k and ϕ are given functions satisfying some assumptions. $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0)$ and the impulsive moments τ_k are such that $0 < \tau_1 < \tau_2 < ... < \tau_k < T$, $k \in \mathbb{N}$. Equations of the form (1)-(5) or their special forms arise in some physical applications as a natural generalization of the classical initial value problems. The results for semilinear functional differential evolution nonlocal problem are extended for the case of impulsive effect. As usual, in the theory of impulsive differential equations [8], [12], we assume that at the points of discontinuity τ_i of the solution $t \to x(t)$, $x(\tau_i) \equiv x(\tau_i - 0)$. It is clear that, in general, the derivatives $x'(\tau_i)$ do not exist. On the other hand, according to (1), there exist the limits $x'(\tau_i \pm 0)$. According to the above convention, we assume $x'(\tau_i) = x'(\tau_i - 0)$.

The aim of the present paper is to study the existence, uniqueness and continuous dependence of mild solution of nonlocal IVP problem for an impulsive functional integro-differential equation. We are generalizing the results reported in papers [4],[6],[14] for the case of impulse effect. The main tool used in our analysis is based on an application of the Banach contraction theorem and theory of cosine family of operators.

The work in this paper is organized as follows. Section 2 presents the preliminaries and hypotheses. In Section 3, we prove existence and uniqueness of mild solution. In section 4 , we prove continuous dependence of solutions on initial data and finally in section 5, we give application based on our result.

2. Preliminaries and Hypotheses

Let X be a Banach space with the norm $\|\cdot\|$. $C = \mathcal{C}([-r, 0], X), 0 < r < \infty$, denote the the Banach space of all continuous functions $\psi : [-r, 0] \to X$ endowed with supremum norm $\|\psi\|_C = \sup\{\|\psi(t)\| : -r \le t \le 0\}$ and B denote the set $\{x : [-r, T] \to X | x(t)$ is continuous at $t \ne \tau_k$,left continuous at $t = \tau_k$, and the right limit $x(\tau_k + 0)$ exists for $k = 1, 2, ..., m\}$. Clearly, B is a Banach space with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : t \in [-r, T] \setminus \{\tau_1, \tau_2, ..., \tau_m\}\}$. For any $x \in B$ and $t \in [0, T] \setminus \{\tau_1, \tau_2, ..., \tau_m\}$, we denote x_t the element of C given by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$ and ϕ is a given element of C.

Definition 2.1. *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called strongly continuous cosine family if and only if*

(a) $C(0) = I$ is the identity operator

(b) $C(t+s) + C(t-s) = 2C(t)C(s) \quad \forall \quad t, \quad s \in \mathbb{R}$

(c) The map $t \mapsto C(t)(x)$ is strongly continuous for each $x \in X$.

The associated sine function is the family $\{S(t)\}_{t\in\mathbb{R}}$ of operators defined by $S(t)x = \int_0^t C(s)xds$, for $x \in X, t \in \mathbb{R}$. The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by $Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}$, $x \in D(A)$, where $D(A) = \{x \in X : C(.)x \in C^2(\mathbb{R}, X)\}$.
In this paper, we assume that, there exist positive constant $M \geq 1$ and N such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for every $t \in [0, T]$. Also $k : [0, T] \times [0, T] \to \mathbb{R}$ and since k is continuous on compact set $[0, T] \times [0, T]$, there is constant $L > 0$ such that $|k(t, s)| \leq L$, for $0 \leq s \leq t \leq T$.

Definition 2.2. *A function $x \in B$ satisfying the equations:*

$$x(t) = C(t)[\phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)] + S(t)\eta + \int_0^t S(t-s)f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau)ds$$
$$+ \sum_{0<\tau_k<t} [C(t-\tau_k)I_k x(\tau_k) - S(t-\tau_k)\bar{I}_k x(\tau_k)], \quad t \in (0, T],$$
$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0$$
$$x'(0) = \eta, \quad \eta \in X$$

is said to be the mild solution of the initial value problem (1)-(5).

The following inequality will be useful while proving our result.

Lemma 2.3 ([12]). *Let a nonnegative piecewise continuous function $u(t)$ satisfy for $t \geq t_0$, the inequality*

$$u(t) \leq C + \int_{t_0}^t v(s)u(s)ds + \sum_{t_0<\tau_i<t} \beta_i u(\tau_i)$$

where $C \geq 0$, $\beta_i \geq 0$, $v(t) > 0$, τ_i are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t) \leq C \prod_{t_0<\tau_i<t} (1+\beta_i) exp(\int_{t_o}^t v(s)ds).$$

We list the following hypotheses for our convenience.

(H_1) Let $f : [0, T] \times C \times X \to X$ such that for every $w \in B$, $x \in X$ and $t \in [0, T]$, $f(., w_t, x) \in B$ and there exists a constant $L > 0$ such that

$$\|f(t, \psi, x) - f(t, \phi, y)\| \leq F(\|\psi - \phi\|_C + \|x - y\|), \qquad \phi, \psi \in C, \quad x, y \in X.$$

(H_2) Let $h : [0, T] \times C \to X$ such that for every $w \in B$ and $t \in [0, T]$, $h(., w_t) \in B$ and there exists a constant $H > 0$ such that

$$\|h(t, \psi) - h(t, \phi)\| \leq H\|\psi - \phi\|_C, \qquad \phi, \psi \in C.$$

(H_3) Let $g : C^p \to C$ such that there exists a constant $G \geq 0$ satisfying

$$\|(g(x_{t_1}, x_{t_2}, ..., x_{t_p}))(t) - (g(y_{t_1}, y_{t_2}, ..., y_{t_p}))(t)\| \leq G\|x - y\|_B, \quad t \in [-r, 0].$$

(H_4) Let $I_k, \bar{I_k} : X \to X$ are functions such that there exists constants $L_k, \bar{L_K}$ satisfying

$$\|I_k(v)\| \leq L_k\|v\|, \quad v \in X, \quad k = 1, 2, ..., m.$$
$$\|\bar{I_k}(v)\| \leq \bar{L_k}\|v\|, \quad v \in X, \quad k = 1, 2, ..., m.$$
$$L_k{}^* = Max(L_k, \bar{L_k})$$

3. Existence and Uniqueness

Theorem 3.1. *Suppose that the hypotheses* (H_1) - (H_4) *are satisfied and* $\Gamma < 1$*, where,*

$$\Gamma = MG + NF[1 + LHT]T + \sum_{0<\tau_k<t} [(M + N)L_k{}^*].$$

Then the initial-value problem (1)-(5) *has a unique mild solution* x *on* $[-r, T]$.

Proof. We introduce an operator $\mathcal{F}$ on a Banach space B by the formula,

$$(\mathcal{F}x)(t) = \begin{cases} \phi(t) - (g(x_{t_1}, ..., x_{t_p}))(t) \quad \text{if} \quad -r \leq t \leq 0 \\ C(t)[\phi(0) - g(x_{t_1}, ..., x_{t_p})(0)] + S(t)\eta \\ + \int_0^t S(t-s)f\left(s, x_s, \int_0^s k(s,\tau)h(\tau, s_\tau)d\tau\right) ds \\ + \sum_{0<\tau_k<t}[C(t-\tau_k)I_k x(\tau_k) - S(t-\tau_k)\bar{I_k}x(\tau_k)] \quad \text{if } t \in (0, T] \end{cases}$$

It is easy to see that $\mathcal{F} : B \to B$. Now we will show that $\mathcal{F}$ is a contraction on B. Let $x, y \in B$. Then for $t \in [-r, 0]$,

$$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| = \|g(x_{t_1}, ..., x_{t_p})(t) - g(y_{t_1}, ..., y_{t_p})(t)\| \leq G\|x - y\|_B \tag{6}$$

and for $t \in (0, T]$,

$$\begin{aligned} \|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| = \|C(t)\left[g(x_{t_1}, ..., x_{t_p})(0) - g(y_{t_1}, ..., y_{t_p})(0)\right] + \int_0^t S(t-s)[f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau) \\ - f(s, y_s, \int_0^s k(s,\tau)h(\tau, y_\tau)d\tau)]ds + \sum_{0<\tau_k<t} [C(t-\tau_k)I_k x(\tau_k) - S(t-\tau_k)\bar{I_k}x(\tau_k)]\| \\ \leq MG\|x - y\|_B + J_1 + J_2 \end{aligned} \tag{7}$$

where

$$\begin{aligned} J_1 &= \int_0^t \|S(t-s)\|\|f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau) - f(s, y_s, \int_0^s k(s,\tau)h(\tau, y_\tau)d\tau)\|ds \\ &\leq N\int_0^t F[\|x_s - y_s\|_C + \int_0^s |k(s,\tau)|\|h(\tau, x_\tau) - h(\tau, y_\tau)\|d\tau]ds \\ &\leq NF\int_0^t [\|x - y\|_B + LHT\|x - y\|_B]ds \\ &\leq NF[1 + LHT]T\|x - y\|_B \end{aligned} \tag{8}$$

$$
\begin{aligned}
J_2 &= \sum_{0<\tau_k<t} [\|C(t-\tau_k)\| \|I_k x(\tau_k) - I_k y(\tau_k)\| + \|S(t-\tau_k)\| \|\bar{I}_k x(\tau_k) - \bar{I}_k y(\tau_k)\|] \\
&\leq \sum_{0<\tau_k<t} [M\|I_k x(\tau_k) - I_k y(\tau_k)\| + N\|\bar{I}_k x(\tau_k) - \bar{I}_k y(\tau_k)\|] \\
&\leq K \sum_{0<\tau_k<t} (M+N){L_k}^* \|x-y\|_B \qquad (9)
\end{aligned}
$$

Using (8)-(9), inequality (7) becomes

$$
\|(Fx)(t) - (Fy)(t)\| \leq MG\|x-y\|_B + \left(NF[1+LHT]T + \sum_{0<\tau_k<t} (M+N){L_k}^* \right) \|x-y\|_B \qquad t \in [0,T]. \qquad (10)
$$

Since $M \geq 1$, in view of inequality (6) and (10), we can say that inequality (10) holds good for $t \in [-r,T]$. Therefore, for $t \in [-r,T]$,

$$
\begin{aligned}
\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| &\leq MG\|x-y\|_B + \left(MF[1+LHT]T + \sum_{0<\tau_k<t} (M+N){L_k}^* \right) \|x-y\|_B \\
&\leq \left(MG + NF[1+LHT]T + \sum_{0<\tau_k<t} (M+N){L_k}^* \right) \|x-y\|_B
\end{aligned}
$$

which implies

$$
\|\mathcal{F}x - \mathcal{F}y\|_B \leq \Gamma \|x-y\|_B,
$$

where

$$
\Gamma = MG + NF[1+LHT]T + \sum_{0<\tau_k<t} [(M+N){L_k}^*]
$$

Since, $\Gamma < 1$, the operator $\mathcal{F}$ satisfies all the assumptions of Banach contraction theorem and therefore $\mathcal{F}$ has unique fixed point in the space B and clearly it is the mild solution of nonlocal IVP problem (1)-(5) with impulse effect. This completes the proof of the theorem. □

4. Continuous Dependence on Initial Data

Theorem 4.1. *Suppose that hypotheses (H_1)-(H_4) are satisfied and $\Gamma < 1$. Then for each $\phi_1, \phi_2 \in C, \eta_1, \eta_2 \in X$ and for the corresponding mild solutions x_1, x_2 of the problems,*

$$
x'(t) = Ax(t) + f(t, x_t, \int_0^t k(t,s)h(s,x_s)ds), \quad t \in (0,T], \qquad (11)
$$

$$
x(t) + g(x_{t_1}, ..., x_{t_p})(t) = \phi_i(t), \quad i = 1,2, \quad , t \in [-r,0] \qquad (12)
$$

$$
x'(0) = \eta_i, \quad i = 1,2, \qquad (13)
$$

$$
\Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1,2,...,m, \qquad (14)
$$

$$
\Delta x'(\tau_k) = \bar{I}_k x(\tau_k), \quad k = 1,2,...,m, \qquad (15)
$$

the following inequality holds

$$
\|x_1 - x_2\|_B \leq \frac{M \prod_{0<\tau_k<t} (1+(M+N){L_k}^*) exp(NFT)}{[1 - \Lambda \prod_{0<\tau_k<t} (1+(M+N){L_k}^*) exp(NFT)]} \times \|\phi_1 - \phi_2\|_C \qquad (16)
$$

where

$$
\Lambda = GM + NFLHT^2
$$

Proof. Let $\phi_1, \phi_2 \in B$ be arbitrary functions and let x_1, x_2 be the mild solutions of the problem(11)-(15). Then we have,for $t \in (0, T]$,

$$x_1(t) - x_2(t) = C(t)[\phi_1(0) - \phi_2(0)] - C(t)[g(x_{1_{t_1}}, ..., x_{1_{t_p}})(0) - g(x_{2_{t_1}}, ..., x_{2_{t_p}})(0)] + \int_0^t S(t-s)\Big[f(s, x_{1_s}, \int_0^s k(s,\tau)h(s, x_{1_s})d\tau) - f(s, x_{2_s}, \int_0^s k(s,\tau)h(s, x_{2_s})d\tau)\Big]ds + \sum_{0<\tau_k<t} C(t-\tau_k)\big[I_k x_1(\tau_k) - I_k x_2(\tau_k)\big] + S(t-\tau_k)\big[\bar{I}_k x_1(\tau_k) - \bar{I}_k x_2(\tau_k)\big], \tag{17}$$

and for $t \in [-r, 0]$,

$$x_1(t) - x_2(t) = \phi_1(t) - \phi_2(t) - [g(x_{1_{t_1}}, ..., x_{1_{t_p}})(t) - g(x_{2_{t_1}}, ..., x_{2_{t_p}})(t)] \tag{18}$$

From (17) and using hypothesis (H_1)-(H_4), we get,

$$\begin{aligned}\|x_1(t) - x_2(t)\| &\le M\|\phi_1 - \phi_2\|_C + GM\|x_1 - x_2\|_B + N\int_0^t F\Big[\|x_{1s} - x_{2s}\|_C ds \\ &+ HL\int_0^s \|x_{1\tau} - x_{2\tau}\|_C d\tau\Big]ds + \sum_{0<\tau_k<t}(M+N)L_k{}^*\|x_1(\tau_k) - x_2(\tau_k)\| \\ &\le M\|\phi_1 - \phi_2\|_C + GM\|x_1 - x_2\|_B + N\int_0^t F\|x_{1s} - x_{2s}\|_C ds \\ &+ FNLHT^2\|x_1 - x_2\|_B + \sum_{0<\tau_k<t}(M+N)L_k{}^*\|x_1(\tau_k) - x_2(\tau_k)\| \\ &\le M\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B + NF\int_0^t \|x_{1s} - x_{2s}\|_C ds + \sum_{0<\tau_k<t}(M+N)L_k{}^*\|x_1(\tau_k) - x_2(\tau_k)\|\end{aligned} \tag{19}$$

Simultaneously, by (18) and hypothesis (H_3),we get,

$$\|x_1(t) - x_2(t)\| \le \|\phi_1 - \phi_2\|_C + G\|x_1 - x_2\|_B, \quad t \in [-r, 0]. \tag{20}$$

Since $M \ge 1$, the inequalities (19) and(20) imply, for $t \in [-r, T]$

$$\|x_1(t) - x_2(t)\| \le M\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B + NF\int_0^t \|x_{1s} - x_{2s}\|_C ds + \sum_{0<\tau_k<t}(M+N)L_k{}^*\|x_1(\tau_k) - x_2(\tau_k)\| \tag{21}$$

Define the function $z : [-r, T] \to \mathbb{R}$ by $z(t) = \sup\{\|x_1(s) - x_2(s)\| : -r \le s \le t\}, t \in [0, T]$. Let $t^* \in [-r, t]$ be such that $z(t) = \|x_1(t^*) - x_2(t^*)\|$. If $t^* \in [0, t]$, then from inequality (21), we have

$$\begin{aligned}z(t) = \|x_1(t^*) - x_2(t^*)\| &\le M\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B + NF\int_0^{t^*} \|x_{1s} - x_{2s}\|_C ds \\ &+ \sum_{0<\tau_k<t}(M+N)L_k{}^*\|x_1(\tau_k) - x_2(\tau_k)\|, \quad 0 < \tau_k \le t \\ z(t) &\le K\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B + NF\int_0^t z(s)ds + \sum_{0<\tau_k<t}(M+N)L_k{}^* z(\tau_k)\end{aligned} \tag{22}$$

Now applying Lemma 2.3 to the inequality (22), we get,

$$z(t) \le \Big(M\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B\Big)\prod_{0<\tau_k<t}[1 + (M+N)L_k{}^*]exp(NFT)$$

Hence, we get,

$$\|x_1 - x_2\|_B \le \Big(M\|\phi_1 - \phi_2\|_C + \Lambda\|x_1 - x_2\|_B\Big)\prod_{0<\tau_k<t}(1 + (M+N)L_k{}^*)exp(NFT).$$

The inequality given by (16), is the immediate consequence of the above inequality. This completes the proof. □

5. Application

To illustrate the application of our result proved in section 3, consider the following semilinear partial functional integro-differential equation of the form

$$\frac{\partial^2}{\partial t^2}w(u,t) = \frac{\partial^2}{\partial u^2}w(u,t) + H\left(t, w(u,t-r), \int_0^t k(t,s)P(s,w(s-r))ds\right), 0 \le u \le \pi, t \in [0,T] \tag{23}$$

$$w(0,t) = w(\pi,t) = 0, \quad 0 \le t \le T, \tag{24}$$

$$w(u,t) + \sum_{i=1}^{p} w(u,t_i+t) = \phi(u,t), \quad 0 \le u \le \pi, \quad -r \le t \le 0, \tag{25}$$

$$\frac{\partial}{\partial t}w(u,0) = \eta(u) \quad 0 \le u \le \pi \tag{26}$$

$$\Delta w(u,\tau_k) = I_k(w(u,\tau_k)), \quad k = 1,2,...,m. \tag{27}$$

$$\Delta w'(u,\tau_k) = \bar{I}_k(w(u,\tau_k)), \quad k = 1,2,...,m. \tag{28}$$

where $0 < t_1 \le t_2 \le t_p \le T$, the functions $H : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $P : [0,T] \times \mathbb{R} \to \mathbb{R}$, $I_k, \bar{I}_k : \mathbb{R} \to \mathbb{R}$ are continuous. We assume that the functions H, P and $I_k, \bar{I}_k$ satisfy the following conditions:

For every $t \in [0,T]$ and $u, v, x, y \in \mathbb{R}$, there exists positive constants l, p, c_k , $\bar{c_k}$such that

$$|H(t,u,x) - H(t,v,y)| \le l(|u-v| + |x-y|)$$

$$|P(t,u) - P(t,v)| \le p(|u-v|)$$

$$|I_k(x)| \le c_k|x|, \quad k = 1,2,...,m.$$

$$|\bar{I}_k(x)| \le \bar{c_k}|x|$$

Let us take $X = L^2[0,\pi]$. Define the operator $A : X \to X$ by $Az = z_{uu}$ with domain $D(A) = \{z \in X : z, \ \ z_u$ are absolutely continuous, $z_{uu} \in X$ and $z(0) = z(\pi) = 0\}$. Then the operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t \in \mathbb{R}}$ on X. More over A has a discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding eigenvectors $z_n(u) = (\sqrt{2/\pi})\sin(nu)$.

The set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and the following properties hold:

(a) If $z \in D(A)$, then $Az = -\sum_{n=1}^{\infty} n^2(z,z_n)z_n$

(b) For every $z \in X$, $C(t)z = \sum_{n=1}^{\infty} \cos nt(z,z_n)z_n$

(c) For every $z \in X$, $S(t)z = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(z,z_n)z_n$.

Consequently, $\|C(t)\| = \|S(t)\| \le 1$ and $S(t)$ is compact for $t \in \mathbb{R}$. Define the functions $f : [0,T] \times C \times X \to X$,$h :$ $[0,T] \times C \to X$, $I_k, \bar{I}_k : X \to X$ as follows

$$f(t,\psi,x)(u) = H(t,\psi(-r)u, x(u)),$$

$$h(t,\phi)(u) = P(t,\phi(-r)u)$$

for $t \in [0,T], \psi, \phi \in C$, $x \in X$ and $0 \le u \le \pi$. With these choices of the functions, the equations (23)-(28) can be formulated as an abstract integro differential equations (1)-(5) in Banach space X.

Since all the hypotheses of the Theorem 3.1 are satisfied, the Theorem 3.1, can be applied to guarantee the existence of mild solution $w(u,t) = x(t)u, \quad t \in [0,T], u \in [0,\pi]$, of the semilinear partial differential equation (23)-(28).

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