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Oscillatory Properties of Third-Order Neutral Delay Difference Equations

Research Article

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Abstract: The objective of this paper is to examine oscillatory properties of the third order neutral delay difference equations of the form

$$\Delta\left(a(n)\Delta\left(b(n)\Delta\left(x(n)+p(n)x(\sigma(n))\right)\right)\right)+q(n)x(\tau(n))=0.$$

Riccati transformation technique is used to obtain some new oscillation criteria.

MSC: 39A10, 39A12, 39A21

Keywords: oscillation, third order, difference equations.

1. Introduction

The aim of this paper is to study the oscillation of solutions of the third order neutral delay difference equation of the form

$$\Delta\left(a(n)\Delta\left(b(n)\Delta\left(x(n)+p(n)x(\sigma(n))\right)\right)\right)+q(n)x(\tau(n))=0. \tag{1}$$

where $a(n), b(n), p(n), q(n)$ are positive sequences and $n \in N(n_0) = \{n_0, n_0+1...\}$, n_0 is a nonnegative integer.

In this paper, we consider the following three cases

$$\sum_{s=n}^{\infty}\frac{1}{a(s)}=\infty, \quad \sum_{s=n}^{\infty}\frac{1}{b(s)}=\infty, \tag{2}$$

$$\sum_{s=n}^{\infty}\frac{1}{a(s)}<\infty, \quad \sum_{s=n}^{\infty}\frac{1}{b(s)}=\infty, \tag{3}$$

$$\sum_{s=n}^{\infty}\frac{1}{a(s)}<\infty, \quad \sum_{s=n}^{\infty}\frac{1}{b(s)}<\infty, \tag{4}$$

Let $\theta = \max\{\sigma(n), \tau(n)\}$. By a solution of equation (1) we mean a real sequence $x(n)$ is defined for all $n \geq n_0 - \theta$ satisfies (1) for all $n \geq n_0$. A nontrivial solution $x(n)$ is said to be oscillatory if it is neither eventually positive or eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

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There has been considerable interest in studying the oscillation of solutions of neutral delay difference equations in the last two decades. Although the oscillation of third-order equations has received less attentions relatively comparing with those of second-order, however there is an increasing interest in studying the oscillation of neutral delay third-order equations [3–7, 9–12]. For general theory of oscillation of difference equations, we refer to [1, 2]. Compared to the second-order difference equations, the study of third-order difference equations has received considerably less attention in the literature even though such equations arise in economics, mathematical biology, and other areas of mathematics. There are many papers dealing with the oscillatory and asymptotic behavior of solutions of difference and differential equations, see. The following lemma is useful in establishing oscillation criteria for equation (1).

Lemma 1.1 ([8]). *If X and Y are nonnegative, then*

$$mXY^{m-1} - X^m \leq (m-1)Y^m \quad \textit{for} \quad m > 1 \tag{5}$$

where equality holds if and only if $X = Y$.

2. Main Results

In this section, we provide some important theorems regarding the oscillation of the equation (1). In the following theorems, we set $z(n) = x(n) + p(n)x(\sigma(n))$.

Theorem 2.1. *Assume that (2) holds, $0 \leq p(n) \leq p_1 < 1$, for all sufficiently large $n_1 \geq n_0$ and for $n_3 > n_2 > n_1$, one has*

$$\limsup_{n\to\infty} \sum_{s=n_3}^{n-1} \rho(s) \frac{q(s)(1-p(\tau(s))) \sum\limits_{t=n_2}^{\tau(s)-1} \frac{\sum\limits_{u=n_1}^{s-1} \frac{1}{a(u)}}{b(t)}}{\sum\limits_{t=n_1}^{s} \frac{1}{a(t)}} - \frac{(\Delta\rho(s))^2 a(s)}{4\rho^2(s+1)\rho(s)} = \infty \tag{6}$$

then (1) is almost oscillatory.

Proof. Assume that $x(n)$ is a positive solution of (1) on $n \geq n_0$. Based on the condition (2) there exists two possible cases:

(1) $z(n) > 0, \Delta z(n) > 0, \Delta(b(n)\Delta z(n)) > 0, \Delta(a(n)\Delta(b(n)\Delta z(n))) < 0$,

(2) $z(n) > 0, \Delta z(n) < 0, \Delta(b(n)\Delta z(n)) > 0, \Delta(a(n)\Delta(b(n)\Delta z(n))) < 0$ for $n \geq n_1, n_1$ large enough.

Assume that case (1) holds. We define the function w by

$$w(n) = \rho(n)\frac{a(n)\Delta(b(n)\Delta z(n))}{b(n)\Delta z(n)},\ n \geq n_1. \tag{7}$$

Then $w(n) > 0$ for $n \geq n_1$. Using $\Delta z(n) > 0$, we have

$$x(n) \geq (1-p(n))z(n). \tag{8}$$

Since

$$b(n)\Delta z(n) \geq \sum_{s=n_1}^{n-1} \frac{a(s)\Delta(b(s)\Delta z(s))}{a(s)} \geq a(n)\Delta(b(n)\Delta z(n)) \sum_{s=n_1}^{n-1} \frac{1}{a(s)}, \tag{9}$$

we have that

$$\Delta\left(\frac{b(n)\Delta z(n)}{\sum\limits_{s=n_1}^{n-1}\frac{1}{a(s)}}\right) \leq 0. \tag{10}$$

Thus, we get

$$
\begin{aligned}
z(n) &= z(n_2) + \sum_{s=n_2}^{n-1} \frac{b(s)\Delta z(s)}{\sum_{t=n_1}^{s-1}\frac{1}{a(t)}} \frac{\sum_{t=n_1}^{s-1}\frac{1}{a(t)}}{b(s)} \\
&\geq \frac{b(n+1)\Delta z(n+1)}{\sum_{s=n_1}^{n}\frac{1}{a(s)}} \sum_{s=n_2}^{n-1} \frac{\sum_{t=n_1}^{s-1}\frac{1}{a(t)}}{b(s)},
\end{aligned}
\tag{11}
$$

for $n > n_2 > n_1$, we obtain

$$
\begin{aligned}
\Delta w(n) &= \rho(n)\left(\frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{b(n+1)\Delta z(n+1)} - \frac{a(n)\Delta(b(n)\Delta z(n))\Delta(b(n)\Delta z(n))}{b(n+1)\Delta z(n+1)b(n)\Delta z(n)}\right) + \Delta\rho(n)\frac{a(n+1)\Delta(b(n+1)\Delta z(n+1))}{b(n+1)\Delta z(n+1)} \\
&\leq -\rho(n)\frac{q(n)(1-p(\tau(n)))\frac{b(\tau(n)+1)\Delta z(\tau(n)+1)}{\sum_{s=n_1}^{\tau(n)}\frac{1}{a(s)}}\sum_{s=n_2}^{\tau(n)-1}\frac{\sum_{t=n_1}^{s-1}\frac{1}{a(t)}}{b(s)}}{b(n+1)\Delta z(n+1)} - \frac{\rho(n)w^2(n+1)}{a(n)\rho^2(n+1)} + \Delta\rho(n)\frac{w(n+1)}{\rho(n+1)} \\
&\leq -\rho(n)\frac{q(n)(1-p(\tau(n)))\sum_{s=n_2}^{\tau(n)-1}\frac{\sum_{t=n_1}^{s-1}\frac{1}{a(t)}}{b(s)}}{\sum_{s=n_1}^{n}\frac{1}{a(s)}} + \Delta\rho(n)\frac{w(n+1)}{\rho(n+1)} - \frac{\rho(n)w^2(n+1)}{a(n)\rho^2(n+1)}
\end{aligned}
$$

Using the inequality (5), we get

let $m = 2$ $\quad X = \frac{\sqrt{\rho(n)}w(n+1)}{\sqrt{a(n)}\rho(n+1)}, \quad Y = \frac{\Delta\rho(n)\sqrt{a(n)}}{2\sqrt{\rho(n)}\rho(n+1)}$

$$
\Delta w(n) \leq -\rho(n)\frac{q(n)(1-p(\tau(n)))\sum_{s=n_2}^{\tau(n)-1}\frac{\sum_{t=n_1}^{s-1}\frac{1}{a(t)}}{b(s)}}{\sum_{s=n_1}^{n}\frac{1}{a(s)}} + \frac{(\Delta\rho(n))^2 a(n)}{4\rho^2(n+1)\rho(n)}.
$$

Summing the last inequality from n_3 to $n-1$

$$
\sum_{s=n_3}^{n-1}\rho(s)\frac{q(s)(1-p(\tau(s)))\sum_{t=n_2}^{\tau(s)-1}\frac{\sum_{u=n_1}^{s-1}\frac{1}{a(u)}}{b(t)}}{\sum_{t=n_1}^{s}\frac{1}{a(t)}} - \frac{(\Delta\rho(s))^2 a(s)}{4\rho^2(s+1)\rho(s)} \leq w(n_3).
$$

which contradicts (6). ☐

Theorem 2.2. *Assume that (3) holds,* $0 \leq p(n) \leq p_1 < 1$*, for all sufficiently large* $n_1 \geq n_0$ *and for* $n_3 > n_2 > n_1$*, one has*

$$
\limsup_{n\to\infty}\sum_{s=n_2}^{n-1}\delta(s)q(s)(1-p(\tau(s)))\sum_{t=n_1}^{\tau(s)-1}\frac{1}{b(t)} - \frac{a(s)(\Delta\delta(s))^2}{4\delta(s)} = \infty, \tag{12}
$$

where $\delta(n) = \sum_{s=n}^{\infty}\frac{1}{a(s)}$ *then (1) is almost oscillatory.*

Proof. Assume that $x(n)$ is a positive solution of (1) on $n \geq n_0$. Based on the condition (2) there exists two possible cases and

(3) $z(n) > 0, \Delta z(n) > 0, \Delta(b(n)\Delta z(n)) < 0, \Delta(a(n)\Delta(b(n)\Delta z(n))) < 0$, for $n \geq n_1, n_1$ large enough.

Assume that case(3) holds.

$$
a(n+1)\Delta(b(n+1)\Delta z(n+1)) \leq a(n)\Delta(b(n)\Delta z(n)). \tag{13}
$$

Divide the above inequality by $a(n+1)$ and summing it from n to ∞, we obtain that is

$$0 \le b(n)\Delta z(n) + a(n)\Delta(b(n)\Delta z(n)) \sum_{s=n}^{\infty} \frac{1}{a(s+1)}. \tag{14}$$

$$-\sum_{s=t}^{\infty} \frac{a(n)\Delta(b(n)\Delta z(n))}{a(s+1)b(n)\Delta z(n)} \le 1 \tag{15}$$

We define the function w by

$$w(n) = \frac{a(n)\Delta(b(n)\Delta z(n))}{b(n)\Delta z(n)}, \ n \ge n_1. \tag{16}$$

Then, $w(n) < 0$ for $n \ge n_1$. Hence by (15) and (16), we obtain

$$-\delta(n)w(n) \le 1. \tag{17}$$

For $n > n_2 > n_1$, from (7) we obtain

$$\begin{aligned} \Delta w(n) &= \left(\frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{b(n+1)\Delta z(n+1)} - \frac{a(n)\Delta(b(n)\Delta z(n))\Delta(b(n)\Delta z(n))}{b(n+1)\Delta z(n+1)b(n)\Delta z(n)} \right) \\ &\le -\frac{q(n)(1-p(\tau(n)))z(\tau(n))}{b(n+1)\Delta z(n+1)} - \frac{(a(n+1)\Delta(b(n+1)\Delta z(n+1)))^2}{a(n)\left(b(n+1)\Delta z(n+1)\right)^2} \\ &\le -\frac{q(n)(1-p(\tau(n)))z(\tau(n))}{b(n+1)\Delta z(n+1)} - \frac{w^2(n+1)}{a(n)}. \end{aligned} \tag{18}$$

In view of (3), we see that

$$z(n) \ge b(n)\Delta z(n) \sum_{s=n_1}^{n-1} \frac{1}{b(s)}. \tag{19}$$

$$\Delta \left(\frac{z(n)}{\sum_{s=n_1}^{n-1} \frac{1}{b(s)}} \right) \le 0.. \tag{20}$$

which implies that

$$\frac{z(\tau(n))}{z(n)} \ge \frac{\sum_{s=n_1}^{\tau(n)-1} \frac{1}{b(s)}}{\sum_{s=n_1}^{n-1} \frac{1}{b(s)}} \tag{21}$$

By (17) and (19), (20) and (22), we obtain

$$\Delta w(n) \le -q(n)(1-p(\tau(n))) \sum_{s=n_1}^{\tau(n)-1} \frac{1}{b(s)} - \frac{w^2(n+1)}{a(n)}. \tag{22}$$

Multiplying the last inequality (22) by $\delta(n)$ and summing it from $n_2 > n_1$ to $n-1$, we have

$$\sum_{s=n_2}^{n-1} \delta(s)w(s) + \sum_{s=n_2}^{n-1} \delta(s)q(s)(1-p(\tau(s))) \sum_{t=n_1}^{\tau(s)-1} \frac{1}{b(t)} + \sum_{s=n_2}^{n-1} \frac{\delta(s)}{a(s)} w^2(s+1) \le 0,$$

$$\begin{aligned} & w(n)\delta(n) - w(n_2)\delta(n_2) - \sum_{s=n_2}^{n-1} \Delta\delta(s)w(s+1) \\ & \quad + \sum_{s=n_2}^{n-1} \delta(s)q(s)(1-p(\tau(s))) \sum_{t=n_1}^{\tau(s)-1} \frac{1}{b(t)} + \sum_{s=n_2}^{n-1} \frac{\delta(s)}{a(s)} w^2(s+1) \le 0, \end{aligned}$$

$$\sum_{s=n_2}^{n-1}\delta(s)q(s)(1-p(\tau(s)))\sum_{t=n_1}^{\tau(s)-1}\frac{1}{b(t)}-\sum_{s=n_2}^{n-1}\Delta\delta(s)w(s+1)-\frac{\delta(s)}{a(s)}w^2(s+1)$$
$$\leq 1+w(n_2)\delta(n_2).$$

Let

$$m=2,\quad X=\sqrt{\frac{\delta(n)}{a(n)}}w(n+1),\quad Y=\frac{\Delta\delta(n)}{2\sqrt{\frac{\delta(n)}{a(n)}}}$$

which follows that

$$\sum_{s=n_2}^{n-1}\delta(s)q(s)(1-p(\tau(s)))\sum_{t=n_1}^{\tau(s)-1}\frac{1}{b(t)}$$
$$-\sum_{s=n_2}^{n-1}2\sqrt{\frac{\delta(s)}{a(s)}}w(s+1)\left(\frac{\Delta\delta(s)}{2\sqrt{\frac{\delta(s)}{a(s)}}}\right)^{2-1}-\left(\sqrt{\frac{\delta(s)}{a(s)}}w(s+1)\right)^2\leq 1+w(n_2)\delta(n_2).$$

$$\sum_{s=n_2}^{n-1}\delta(s)q(s)(1-p(\tau(s)))\sum_{t=n_1}^{\tau(s)-1}\frac{1}{b(t)}-\frac{a(s)(\Delta\delta(s))^2}{4\delta(s)}\leq 1+w(n_2)\delta(n_2).$$

which contradicts (14). This completes the proof. □

References

[1] R.P.Agarwal, *Difference Equations and Inequalities*, 2nd Edition, Marcel Dekker, New York, (2000).

[2] R.P.Agarwal, M.Bohner, S.R.Grace and D.O.Regan, *Discrete Oscillation Theory*, Hindawi Publication Corporation, (2005).

[3] R.P.Agarwal, S.R.Grace and D.O.Regan, *On the oscillation of certain third order difference equations*, Adv. Differ. Equ., 3(2005), 345-367.

[4] A.George Maria Selvam, M.Paul Loganathan, K.R.Rajkumar, *Oscillation of Third-Order Neutral Difference Equations*, International Journal of Mathematics and Computer Research, 3(7)(2015). 1071-1080.

[5] A.George Maria Selvam, M.Paul Loganathan, K.R.Rajkumar, *Some Oscillation Properties of Third Order Linear Neutral Delay Difference Equations*, International Journal of Computational Engineering Research, 5(7)(2015), 1-5.

[6] J.R.Graef and E.Thandapani, *Oscillatory and asymptotic behavior of solutions of third order delay difference equations*, Funkcial. Ekvac., 42(1999), 355-369.

[7] S.R.Grace, R.P.Agarwal and J.R.Graef, *Oscillation criteria for certain third order non-linear difference equations*, Appl. Anal. Discrete Math., 3(2009), 27-38.

[8] G.H.Hardy, J.E.Littlewood and G.Pólya, *Inequalities*, Cambridge University Press, Cambridge (1959).

[9] Mustafa Fahri Aktas, Aydn Tiryaki, Agack Zafer, *Oscillation of third-order nonlinear delay difference equations*, Turk J Math 36(2012), 422-436.

[10] P.Mohankumar, V.Ananthan and A.Ramesh, *Oscillation Solutions Of Third Order Nonlinear Difference Equations With Delay*, International Journal of Mathematics and Computer Research, 2(8)(2014), 581-586.

[11] S.H.Saker, *Oscillation and asymptotic behavior of third order nonlinear neutral delay difference equations*, Dyna.Sys.Appl.,15(2006), 549-568.

[12] S.H.Saker, *Oscillation theory of delay differential and difference equations second and third orders*, Lambert Academic Publishing (2010).