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An Exact solution of Diffusion Equation with boundary conditions by Padé-Laplace Differential Transform Method

Research Article

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Abstract: In this paper, we solve Diffusion equation with boundary conditions analytically by using a combined form of Laplace transform method and differential transform method (DTM) and Padé approximation. The aim of using the Laplace transform is to overcome the deficiency that is caused by unsatisfied boundary conditions in using differential transform method. The combined method is capable to handle diffusion equation with boundary conditions. The solutions obtained by proposed method are compared with the known exact solutions and found that our solutions are in good agreement with the known exact solutions. The obtained results show the simplicity of proposed method and massive reduction in calculations.

MSC: 41A58, 44A10, 41A28, 35G30.

Keywords: Diffusion equation, Boundary condition, Laplace transforms, DTM, LDTM, Padé approximation.

1. Introduction

The concept of diffusion was first studied by the French sociologist Gabriel Trade in late 19th century and by German and Austrian anthropologists such as Friedrich Ratzel and Leo Frobenius.

Ovsiannikov (1959) investigated the solution of the non-linear diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(g(u)\frac{\partial u}{\partial x}\right). \tag{1}$$

By symmetry method, where $u(x,t)$ represent mass concentration. In many cases, $g(u)$ is approximated as:

$$g(u) = u^n. \tag{2}$$

In this paper, we are interested in the LDTM to solve diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right). \tag{3}$$

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Chen and Liu [4] have presented the solution of two point boundary value problems by using the differential transformation method. Wazwaz [2] has presented the approximate solution of higher order boundary value problems for first order linear and second order nonlinear equations by the modified decomposition method. Arikoglu and zkol [1] have used DTM to solve the boundary value problems for integro-differential equations. Ertrk and Momani [13] have compared numerical methods for solving fourth-order boundary value problems. Chun et al. [3] have investigated the homotopy perturbation method for the wave and nonlinear diffusion equations. Jafari and Seifii [5] have performed the homotopy analysis method for solving linear and nonlinear fraction diffusion- wave equation. Iagar and Sanchez [11] have introduced radial equivalence between the porous medium equation and the evolution p-Laplacian equation and studied the self-similarity for two very fast diffusion equations. The researchers found that the self-similar solutions for the very fast p-Laplacian equation that have finite mass. Rashidi and Keimanesh [8] used DTM- Padé approximation method which is a combination of differential transform method and Pade approximant to solve Magnetro hydro dynamics (MHD) flow in a laminar liquid film from a horizontal stretching surface. Madani et al. [7] have presented the combination of the homotopy perturbation method and Laplace transformation. Gupta [9] has performed the homotopy analysis method for approximate analytical solution of non-linear fractional diffusion equation. Gupta [10] has solved the fractional BenneyLin equation by reduced differential transform method and the homotopy perturbation method for finding the approximate analytical solutions. Das et al. [12] have applied homotopy analysis method for solving fractional diffusion equation; they accurately compute the solutions in a series or in an exact form. Alquran et al. [6] have investigated the coupling of the Laplace transform method and the differential transformation method to find the exact solution of linear non- homogeneous PDEs.

Padé approximation is the ratio of two polynomials which are constructed with the co-efficient of the Taylor's series expansion of a function. Padé approximation is a topic in mathematical approximation theory and analytical function theory. It has widely applicable in the area of knowledge that involves analytical techniques. It is used in computer calculations often gives better approximation of the function than truncating its power series and where the power series does not converge it may still works. By using symbolic programming language such as mathematica, it can be easily computed. A Padé approximation of on is the quotient of two polynomials, say and of degree and respectively. In order to obtain better numerical results, the diagonal approximants and differential transform method will be used.

Now, a day BVPs appears more and more frequently in different research areas and engineering applications. Solving the non-linear singular BVPs accurately and efficiently is considered a very important issue. In general, classical numerical methods fail to produce good approximations for the BVPs. In this paper, we apply LDTM- Padé approximation to solve diffusion equations with boundary conditions.

2. Differential Transformation Method

The one variable differential transform [10] of a function $u(x,t)$, is defined as:

$$U_k(t) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0}; k \geq 0 \tag{4}$$

where $u(x,t)$ is the original function and $U_k(t)$ is the transformed function.The inverse differential transform of $U_k(t)$ is defined as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)(x-x_0)^k, \tag{5}$$

where x_0 is the initial point for the given initial condition.Then the function $u(x,t)$ can be written as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t) x^k. \tag{6}$$

3. Solution of the Problem by LDTM

Let us consider the following one-dimensional time dependent diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right); \quad x \in \mathbb{R},\ \mathrm{t} \in \mathbb{R}^+, \tag{7}$$

subject to the initial conditions

$$u(x,0) = f(x); x \in \mathbb{R} \tag{8}$$

and the Dirichlet boundary conditions

$$u(0,t) = g(t), \quad u(1,t) = m(t); t \in \mathbb{R}+ \tag{9}$$

or the Neumann boundary conditions

$$u(0,t) = g(t), \quad u_x(1,t) = n(t); t \in \mathbb{R}^+ \tag{10}$$

Taking the Laplace transform on (7), can be obtain

$$sL\Big[u(x,t)\Big] - u(x,0) = L\left[\frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right)\right].$$

Using initial condition from equation (8), we get

$$L\Big[u(x,t)\Big] = \frac{f(x)}{s} + \frac{1}{s} L\left[\frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right)\right].$$

Applying Inverse Laplace transform on both sides

$$u(x,t) = f(x) + L^{-1}\left[\frac{1}{s} L\left[\frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right)\right]\right].$$

Now taking Differential transform on both sides w.r.to x, we get

$$\begin{aligned} U_k(t) &= L^{-1}\left[\frac{1}{s} L\left[\sum_{r=0}^{k} (k-r+2)(k-r+1) U_{k-r+2}(t) \sum_{r_1=0}^{r} \sum_{r_2=0}^{r_1} \dots \sum_{r_{n-1}=0}^{n-2} U_{r_{n-1}}(t) U_{r-r_1}(t) \left[\prod_{i=1}^{n-2} U_{r_i - r_{i+1}}(t)\right]\right]\right] \\ &+ nL^{-1}\left[\frac{1}{s} L\left[\sum_{r=0}^{k} (k-r+1)(r-p+1) U_{k-r+1}(t) U_{r-p+1}(t) \sum_{p_1=0}^{p} \sum_{p_2=0}^{p_1} \dots \sum_{p_{n-2}=0}^{n-3} U_{p_{n-2}}(t) U_{p-p_1}(t) \left[\prod_{i=1}^{n-3} U_{p_i - p_{i+1}}(t)\right]\right]\right] \end{aligned} \tag{11}$$

Now, apply the differential transform method on the given Dirihlet and Neumann boundary conditions (9) and (10), we get

$$U_0(t) = g(t). \tag{12}$$

Let us assume

$$U_1(t) = az(t). \tag{13}$$

By the definition of DTM, we take

$$u(1,t) = \sum_{i=0}^{\infty} U_i(t),\ u_x(1,t) = \sum_{i=0}^{\infty} iU_i(t). \tag{14}$$

By equation (14), we calculate the value of a.

Now by substituing equations (12) and (13) in (11), and by straightforward iterative steps, we obtain $U_k(t)(k = 2, 3, ...)$. Then the inverse transformation of the set of values $U_k(t)(k = 2, 3, ...)$ gives approximate solution as,

$$u(x,t) = \sum_{k=0}^{\infty} U(k,t)x^k. \tag{15}$$

which is the closed form of the solution.

4. Numerical Examples

To illustrate the applicability of LDTM, we have applied it to diffusion equation. In this section, we are considering two linear and non-linear cases of equation (3), to demonstrate the reliability of the method LDTM.

Case I. If $n = 0$, $f(x) = x^2$, $g(t) = 2t$ and $m(t) = 1 + 2t$, then equation (11) becomes

$$U_k(t) = \delta(k-2,t) + L^{-1}\left[\frac{1}{s}L\left[(k+2)(k+1)U_{k+2}(t)\right]\right]; \delta(k,t) = \begin{cases} 1,\ k = 0 \\ 0,\ k \neq 0 \end{cases} \tag{16}$$

with

$$U_0(t) = 2t, \tag{17}$$

And let us assume

$$U_1(t) = at^n, \tag{18}$$

Substituting (17) and (18) into (16) and by straightforward iterative steps, we obtain

$$U_2(t) = 1, U_3(t) = \frac{ant^{n-1}}{3!}, U_4(t) = 0, U_3(t) = \frac{an(n-1)t^{n-2}}{5!}, ... \tag{19}$$

From equation (14), we get

$$u(1,t) = 1 + 2t = \sum_{i=0}^{\infty} U_i(t).$$

And find

$$a = 0.$$

Now putting the value of a in equation (18) and (19), and we get the component $U_k(t)$, $k \geq 0$ of the DTM can be obtained. When we substitute all values of $U_k(t)$ into equation (15), them the series solution can be formed as

$$u(x,t) = x^2 + 2t.$$

which is the exact solution.

Case II. If $n = 1$, $f(x) = x^2$, $g(t) = 0$ and $n(t) = \dfrac{1}{1-6t}$, then equation (11) becomes

$$U_k(t) = \delta(k-2,t) + L^{-1}\left[\frac{1}{s}L\left[\sum_{r=0}^{k}(r+2)(r+1)U_{r+2}(t)U_{k-r}(t) + \sum_{r=0}^{k}(r+1)(k-r+1)U_{r+1}(t)U_{k-r+1}(t)\right]\right], \tag{20}$$

with

$$U_0(t) = 0, \tag{21}$$

And let us assume

$$U_1(t) = \frac{a}{1-6t}, \tag{22}$$

Substituting (21) and (22) into (20) and by straightforward iterative steps, we obtain

$$U_2(t) = \frac{1}{1-6t}, U_3(t) = 0, U_4(t) = 0, ... \tag{23}$$

From equation (14), we get

$$u_x(1,t) = \frac{1}{1-6t} = \sum_{i=0}^{\infty} iU_i(t).$$

And find

$$a = 0.$$

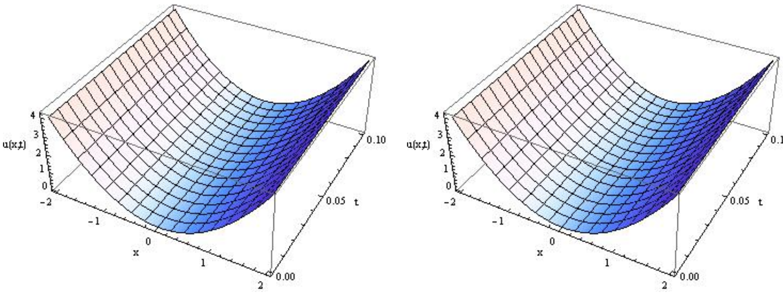
Now putting the value of a in equation (22) and (23), and we get the component $U_k(t)$, $k \geq 0$ of the DTM can be obtained. When we substitute all values of $U_k(t)$ into equation (15), them the series solution can be formed as

$$u(x,t) = \frac{x^2}{1-6t}.$$

which is the exact solution.

5. Numerical Results

In this section, the numerical results of $u(x,t)$ for the different boundary conditions for various values of t and x are obtained. Fig. 1(b), Fig. 3(c) and Fig. 3(b) represents the LDTM solution and LDTM- Padé solution of $u(x,t)$ respectively for both cases-I & II and it is increases when the values of space x is increases or decreases from zero. It is also seen that $u(x,t)$ is slightly increases with time t compare to x in both cases. In Fig. (2) for case-I, we compare the LDTM solution to the exact solution and the result shows that the value of exact solution and LDTM solution are same. In Fig. (4), we compare the LDTM solution and LDTM-Padé solution with the exact solution. Then we find out that initially the value of exact solution, LDTM solution and LDTM- Padé solution are same, later on the LDTM- Padé solution and exact solution are same but there is a slightly change in the LDTM solution. In Fig. (5), we find the absolute error, we analyze that initially the error is constant but after some time it increases drastically.



(a) (b)

Figure 1. The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to 'x' and 't', when $n = 0$, $f(x) = x^2$, $g(t) = 2t$ and $m(t) = 1 + 2t$, for Case-I.

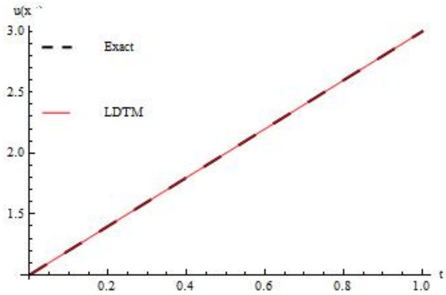
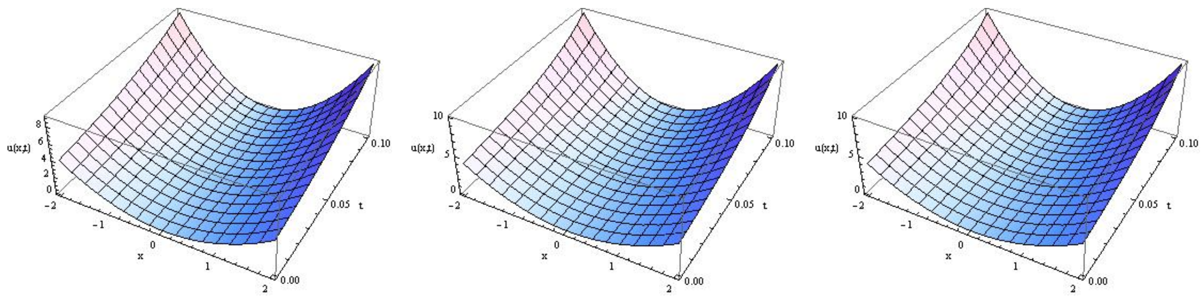


Figure 2. Plot of the Exact solution (dashed) and LDTM solution (line) vs. 't', when $n = 0$, $f(x) = x^2$, $g(t) = 2t$ and $m(t) = 1 + 2t$, for Case-I.



(a) (b) (c)

Figure 3. The behavior of the (a) Exact solution, (b) LDTM-Padé solution, (c) LDTM solution w.r.to x and t, when $n = 1$, $f(x) = x^2$, $g(t) = 0$ and $n(t) = \dfrac{1}{1-6t}$, for Case II.

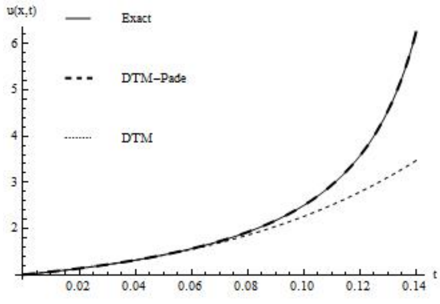


Figure 4. Plot of the Exact solution (black), LDTM-Padé solution (Thick black dashed) and LDTM solution (Thin black dashed) vs. 't', when $n = 1$, $f(x) = x^2$, $g(t) = 0$ and $n(t) = \dfrac{1}{1-6t}$, for Case II.

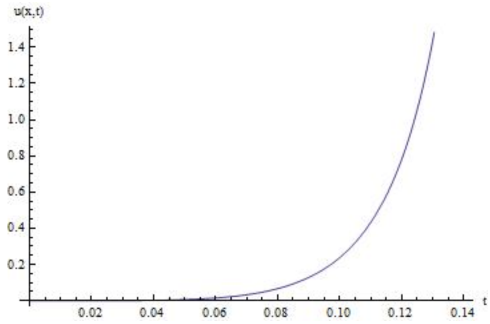


Figure 5. Plot of absolute error $\mid u_{exact}(1,t) - u_{LDTM}(1,t) \mid$ vs. 't', when $n = 1$, $f(x) = x^2$, $g(t) = 0$ and $n(t) = \dfrac{1}{1-6t}$, for Case II.

6. Conclusion

In this paper, we solved linear and nonlinear diffusion equation with boundary conditions by Padé-Laplace Differential transform method. We find the exact solution by applying this method and obtained a simple iterative process. The present method reduces the computational work and calculations can be made simple manipulations. The computations of this paper have been carried out using Mathematica 8.

References

[1] A. Arikoglu and I. zkol, *Solution of boundary value problems for integro-differential equations by using differential transform method*, Applied Mathematics and Computation, 168(2005), 1145-1158.

[2] A.M. Wazwaz, *Approximate solution to boundary value problems of higher order by the modified decomposition method*, Computer Mathematics with Applications, 40(2000), 679-691.

[3] C. Chun, H. Jafari and Y. Kim, *Numerical method for the wave and nonlinear diffusion equations with the homotopy perturbation method*, Computer Mathematics with Applications, 57(2009), 1226-1231.

[4] C.L. Chen and Y.C. Liu, *Solution of two point boundary value problems using the differential transformation method*, Journal of Optimization Theory and Applications, 99(1998), 23-35.

[5] H. Jafari and S. Seifii, *Homotopy analysis method for solving linear and nonlinear fraction diffusion- wave equation*, Communications in Nonlinear Science and Numerical Simulation, 14(2009), 2006-2012.

[6] M. Alquran, K.A. Khaled, M. Ali and A. Taany, *The combined Laplace transform- differential transformation method for solving linear non-homogeneous PDEs*, Journal of Mathematical and Computational Science, 2, 3(2012), 690-701.

[7] M. Madani, M. Fathizadeh, Y. Khan and A. Yildirim, *On the coupling of the homotopy perturbation method and Laplace transformation*, Mathematical and Computer Modelling, 53, 9-10(2011), 1937-45.

[8] M.M. Rashidi and M. Keimanesh, *Using Differential Transform Method and Pad Approximant for Solving MHD Flow in a Laminar Liquid Film from a Horizontal Stretching Surface*, Mathematical Problems in Engineering, Article Id 491319,(2010), 1-14.

[9] P.K. Gupta, *An approximate analytical solution of non-linear fractional diffusion equation by homotopy analysis method*, International Journal of the Physical Sciences, 6, 34(2011), 7721-7728.

[10] P.K. Gupta, *Approximate analytical solutions of fractional BenneyLin equation by reduced differential transform method and the homotopy perturbation method*, Computers & Mathematics with Applications, 61, 9(2011), 2829-2842.

[11] R.G. Iagar and A. Sanchez, *Radial equivalence and study of self-similarity for two very fast diffusion equations*, Journal

of Mathematical Analysis and Applications, 351, 2(2009), 635-652.

[12] S. Das, K. Vishal and P.K. Gupta, S.S. Ray, *Homotopy analysis method for solving fractional diffusion equation*, International Journal of Applied Mathematics and Mechanics, 7, 9(2011), 28-37.

[13] V.S. Ertrk and S. Momani, *Comparing numerical methods for solving fourth-order boundary value problems*, Applied Mathematics and Computation, 188, 2(2007), 1963-1968.