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The Minimum Monopoly Energy of a Graph

Research Article

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Abstract: In a graph $G(V, E)$, a subset $M \subseteq V(G)$ is called a monopoly set of G if every vertex $v \in V - M$ has at least $\frac{d(v)}{2}$ neighbors in M. The monopoly size of G is the minimum cardinality of a monopoly set among all monopoly sets in G, denoted by $mo(G)$. In this paper, we introduce minimum monopoly energy, denoted $E_M(G)$, of a graph G and computed minimum monopoly energies of some standard graphs. Upper and lower bounds for $E_M(G)$ are established.

MSC: 05C50, 05C99.

Keywords: Monopoly Set, Monopoly Size, Minimum Monopoly Matrix, Minimum Monopoly Eigenvalues, Minimum Monopoly Energy of a Graph.

1. Introduction

In this paper, by a graph $G(V, E)$ we mean a simple graph, that is nonempty, finite, having no loops, no multiple and directed edges. Let n and m be the number of vertices and edges, respectively, of G. The degree of a vertex v in a graph G, denoted by $d(v)$, is the number of vertices adjacent to v. For any vertex v of a graph G, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E(G)\}$. For a subset $S \subseteq V(G)$ the degree of a vertex $v \in V(G)$ with respect to a subset S is $d_S(v) = |N(v) \cap S|$. For graph theortic terminology we refer to Harary book [9].

A subset $M \subseteq G$ is called a monopoly set if for every vertex $v \in V - M$ has at least $\frac{d(v)}{2}$ neighbors in M, the monopoly size of a graph G, denoted by $mo(G)$, is a minimum cardinality of a monopoly set in G. In particular, monopolies are dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg [15]. For more details in dynamos in graphs we refer to [4–6, 13]. In [10], the author defined a monopoly set of a graph G, proved that the $mo(G)$ for general graph is at least $\frac{n}{2}$, discussed the relationship between matchings and monopolies and he showed that any graph G admits a monopoly with at most $\alpha^{'}(G)$ vertices.

The concept energy of a graph introduced by I. Gutman [7] in the year 1978. Let G be a graph with n vertices and m edges and let $A(G) = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of a matrix A(G), assumed in nonincreasing order, are the eigenvalues of the graph G. Let $\lambda_1, \lambda_2, ..., \lambda_t$ for $t \leq n$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, ..., m_t$, respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_t \\ m_1 & m_2 & ... & m_t \end{pmatrix}.$$

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As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e. $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For more details on the mathematical aspects of the theory of graph energy we refer to [2, 8, 12]. Recently C. Adiga et al. [1] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Motivated by this paper, we introduce minimum monopoly energy, denoted by $E_M(G)$, of a graph G, and computed minimum monopoly energies of some standard graphs. Upper and lower bounds for $E_M(G)$ are established. It is possible that the minimum monopoly energy that we are considering in this paper may be have some applications in chemistry as well as in other areas.

2. The Minimum Monopoly Energy of Graphs

Let G be a graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. any monopoly set M in a graph G with minimum cardinality is called a minimum monopoly set. Let M be a minimum monopoly set of G. The minimum monopoly matrix of G is the $n \times n$ matrix, denoted $A_M(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in M; \\ 0, & \text{othewise.} \end{cases}$$

The characteristic polynomial of $A_M(G)$, denoted by $f_n(G, \lambda)$, is defined as $f_n(G, \lambda) = det(\lambda I - A_M(G))$. The monopoly eigenvalues of G are the eigenvalues of $A_M(G)$. Since $A_M(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The minimum monopoly energy of G is defined as $E_M(G) = \sum_{i=1}^{n} |\lambda_i|$. We first compute the minimum monopoly energy of a graph G in Fig. 1, to illustrious this concept

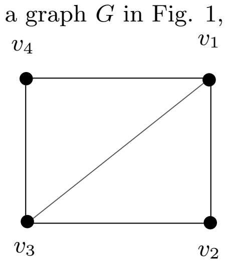

Figure 1 : A graph G

Example 2.1. *Let G be a graph in Figure 1, with vertices v_1, v_2, v_3, v_4 and let its minimum monopoly set be $M_1 = \{v_1, v_3\}$. Then minimum monopoly matrix of G is*

$$A_{M_1}(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{M_1}(G)$ is $f_n(G, \lambda) = \lambda^4 - 2\lambda^3 - 4\lambda^2$. Then the minimum monopoly eigenvalues are $\lambda_1 = 3.2361, \quad \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = -1.2361$. Therefore, the minimum monopoly energy of G is $E_{M_1}(G) = 4.4722$. If we take another minimum monopoly set in G, namely $M_2 = \{v_1, v_4\}$, then

$$A_{M_2}(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of $A_{M_2}(G)$ *is* $f_n(G,\lambda) = \lambda^4 - 2\lambda^3 - 4\lambda^2 + \lambda + 1$. *The minimum monopoly eigenvalues are* $\lambda_1 = 3.1401$, $\lambda_2 = 0.57117$, $\lambda_3 = -0.43783$, $\lambda_4 = -1.2735$. *Therefore, the minimum monopoly energy of* G *is* $E_{M_2}(G) = 5.4226$.

The examples above illustrate that the minimum monopoly energy of a graph G depends on the choice of the minimum monopoly set. i.e. the minimum monopoly energy is not a graph invariant.

3. Some Properties of Minimum Monopoly Energy of Graphs

In this section, we introduce some properties of characteristic polynomials of minimum monopoly matrix of a graph G and some properties of minimum monopoly eigenvalues.

Theorem 3.1. *Let* G *be a graph of order* n*, size* m *and monopoly size* $mo(G)$*. Let* $f_n(G,\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_n$ *be the characteristic polynomial of minimum monopoly matrix of* G*. Then*

1. $c_0 = 1$.
2. $c_1 = -mo(G)$.
3. $c_2 = \binom{mo(G)}{2} - m$.

Proof.

1. From the definition of $f_n(G,\lambda)$.
2. Since the sum of diagonal elements of $A_M(G)$ is equal to $|M| = mo(G)$, where M is a minimum monopoly set in G. The sum of determinants of all 1×1 principal submatrices of $A_M(G)$ is the trace of $A_M(G)$, which evidently is equal to $mo(G)$. Thus, $(-1)^1 c_1 = mo(G)$.
3. $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of $A_M(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} a_{ij}^2 \\ &= \binom{mo(G)}{2} - m. \end{aligned}$$

□

Theorem 3.2. *Let* $\lambda_1, \lambda_2, ..., \lambda_n$ *be the eigenvalues of* $A_M(G)$*. Then*

(i) $\sum_{i}^{n} \lambda_i = mo(G)$.

(ii) $\sum_{i}^{n} \lambda_i^2 = mo(G) + 2m$.

Proof.

(i) Since the sum of the eigenvalues of $A_M(G)$ is the trace of $A_M(G)$, it follows that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |M| = mo(G)$$

.

(ii) Similarly the sum of squares of the eigenvalues of $A_M(G)$ is the trace of $(A_M(G))^2$. Then

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i\neq j}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}^{n} a_{ij}^2 \\ &= mo(G) + 2m.\end{aligned}$$

□

Theorem 3.3. *Let G be a graph of order n and size m and let $\lambda_1(G)$ be the largest eigenvalue of $A_M(G)$. Then*

$$\lambda_1(G) \geq \frac{2m + mo(G)}{n}.$$

Proof. Let G be a graph of order n and let λ_1 be the largest minimum monopoly eigenvalue of $A_M(G)$. Then from [2] we have $\lambda_1 = \max_{X\neq 0}\left\{\frac{X^tAX}{X^tX}\right\}$, where X is any nonzero vector and X^t is its transpose and A is a matrix . If we tack $X = J = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Then we have $\lambda_1 \geq \frac{J^tA_M(G)J}{J^tJ} = \frac{2m+mo(G)}{n}$. □

Theorem 3.4. *Let G be a graph with a minimum monopoly set M. If the minimum monopoly energy $E_M(G)$ of G is a rational number, then $E_M(G) \equiv mo(G) \pmod 2$.*

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the minimum monopoly eigenvalues of G of which $\lambda_1, \lambda_2, ..., \lambda_r$ are positive and the rest are non-positive, then

$$\begin{aligned}\sum_{i=1}^{n} |\lambda_i| &= (\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_r + 1 + ... + \lambda_n) \\ &= 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_1 + \lambda_2 + ... + \lambda_n).\end{aligned}$$

Hence, By Theorem 3.2 we have $E_M(G) = 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - mo(G)$. Since $\lambda_1, \lambda_2, ..., \lambda_r$ are algebraic integers, so is their sum. Therefore, $(\lambda_1 + \lambda_2 + ... + \lambda_r)$ must be integer if $E_M(G)$ is rational. Hence, the Theorem. □

4. Minimum Monopoly Energy of Some Standard Graphs

In this section, we investigate the exact values of the minimum monopoly energy of some standard graphs.

Theorem 4.1. *For the complete graph K_n, $n \geq 2$, the minimum monopoly energy is*

$$E_M(K_n) = \begin{cases} \frac{n-1}{2} + \sqrt{n^2-1}, & \textit{if} \quad n \textit{ is odd;} \\ \frac{n-2}{2} + \sqrt{n^2-1}, & \textit{if} \quad n \textit{ is even.} \end{cases}$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Then the minimum monopoly size is

$$mo(K_n) = \lfloor \frac{n}{2} \rfloor = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Hence, the minimum monopoly set either $\{v_1, v_2, \cdots, v_{\frac{n-1}{2}}\}$ if n is odd or $\{v_1, v_2, \cdots, v_{\frac{n}{2}}\}$ if n is even. Then, we consider the following two cases:

Case 1: If n is odd,

$$A_M(K_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}$$

The respective characteristic polynomial is

$$f_n(K_n, \lambda) = \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & \lambda & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{n \times n}$$

$$= \lambda^{\frac{n-3}{2}} (\lambda+1)^{\frac{n-1}{2}} (\lambda^2 - (n-1)\lambda - \frac{n-1}{2}).$$

The minimum monopoly spectrum of K_n will be written as

$$MM\ Spec(K_n) = \begin{pmatrix} 0 & -1 & \frac{(n-1)+\sqrt{n^2-1}}{2} & \frac{(n-1)-\sqrt{n^2-1}}{2} \\ \frac{n-3}{2} & \frac{n-1}{2} & 1 & 1 \end{pmatrix}.$$

Hence, the minimum monopoly energy of a complete graph in this case is

$$E_M(K_n) = \frac{n-1}{2} + \sqrt{n^2-1}.$$

Case 2: If n is even,

$$A_M(K_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n\times n}$$

The respective characteristic polynomial is

$$f_n(K_n,\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda-1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{n\times n}$$

$$= \lambda^{\frac{n-2}{2}}(\lambda+1)^{\frac{n-2}{2}}(\lambda^2-(n-1)\lambda-\frac{n}{2}).$$

The minimum monopoly spectrum of K_n is

$$MM\ Spec(K_n) = \begin{pmatrix} 0 & -1 & \frac{(n-1)+\sqrt{n^2+1}}{2} & \frac{(n-1)-\sqrt{n^2+1}}{2} \\ \frac{n-2}{2} & \frac{n-2}{2} & 1 & 1 \end{pmatrix}.$$

Hence, the minimum monopoly energy of a complete graph in this case is $E_M(K_n) = \frac{n-2}{2} + \sqrt{n^2+1}$. □

Theorem 4.2. *For the complete bipartite graph $K_{r,s}$, for $r \leq s$, the minimum monopoly energy is equal to $(r-1)+\sqrt{4rs+1}$.*

Proof. For the complete bipartite graph $K_{r,s}$, for $r \leq s$ with vertex set $V = \{v_1, v_2, \cdots, v_r, v_1, v_2, \cdots, v_s\}$. The minimum monopoly set is $M = \{v_1, v_2, \cdots, v_r\}$.

Then

$$A_M(K_{r,s}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(r+s)\times(r+s)}$$

The characteristic polynomial of $A_M(K_{r,s})$ is

$$f_n(K_{r,s},\lambda) = \begin{vmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & \lambda-1 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & \lambda-1 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{(r+s)\times(r+s)}$$

$$= \lambda^{r-1}(\lambda-1)^{s-1}(\lambda^2-\lambda-rs).$$

and

$$MM\ Spec(K_{r,s}) = \begin{pmatrix} 0 & -1 & \frac{1+\sqrt{4rs+1}}{2} & \frac{1-\sqrt{4rs+1}}{2} \\ r-1 & s-1 & 1 & 1 \end{pmatrix}$$

Hence, $E_M(K_{r,s}) = (r-1)+\sqrt{4rs+1}$. □

Theorem 4.3. *For a star graph $K_{1,n-1}$, $n \geq 2$, the minimum monopoly energy is equal to $\sqrt{4n-3}$.*

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \cdots, v_n\}$, where v_0 is the center vertex. The minimum monopoly set of $k_{1,n-1}$ is $M = \{v_0\}$. Then

$$A_M(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

The characteristic polynomial of $A_M(K_{1,n-1})$ is

$$f_n(K_{n,m}),\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \lambda^{n-2}(\lambda^2 - \lambda - (n-1)).$$

and

$$MM\ Spec(K_{1,n-1}) = \begin{pmatrix} 0 & \frac{1+\sqrt{4n-3}}{2} & \frac{1-\sqrt{4n-3}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Therefore, the minimum monopoly energy of a star graph is $E_M(K_{1,n-1}) = \sqrt{4n-3}$. □

Definition 4.4. *The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n-1}$ and $K_{1,m-1}$ by join whose centers v_0 with u_0. Then $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, ..., v_{n-1}, u_0, u_1, ..., u_{m-1}\}$ and edge set $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j \mid 1 \le i \le n-1, 1 \le j \le m-1\}$.*

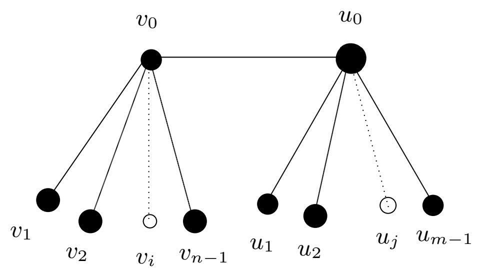

Figure 2 : Double Star Graph $S_{n,m}$

Theorem 4.5. *For the double star graph $S_{t,t}$, with $t \ge 3$, the minimum monopoly energy is equal to $2(\sqrt{t-1} + \sqrt{t})$.*

Proof. For the double star graph $S_{t,t}$ with $V = \{v_0, v_1, ..., v_t - 1, u_0, u_1, ..., u_t - 1\}$ the minimum monopoly set is $M = \{v_0, u_0\}$. Then

$$A_M(S_{t,t}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}_{2t\times 2t}$$

The characteristic polynomial of $A_M(S_{t,t})$ is

$$f_n(S_{t,t}),\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda-1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & \lambda \end{vmatrix}_{2t\times 2t}$$

$$= \lambda^{2t-4}(\lambda^2-(t-1))(\lambda^2-2\lambda-(t-1)).$$

Then the minimum monopoly spectrum of $S_{t,t}$ is

$$MM\ Spec(S_{t,t}) = \begin{pmatrix} 0 & \sqrt{t-1} & -\sqrt{t-1} & 1+\sqrt{t} & 1-\sqrt{t} \\ 2t-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Hence, the minimum monopoly energy of $S_{t,t}$ is

$$E_M(S_{t,t}) = 2(\sqrt{t-1}+\sqrt{t}).$$

□

Definition 4.6. *The crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $\{u_iv_i : 1 \leq i, j \leq n,\ i \neq j\}$. Therefore S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Theorem 4.7. *For $n \geq 3$, the minimum monopoly energy of the crown graph S_n^0 is equal to $\sqrt{5}(n-1)+\sqrt{4n^2-8n+5}$.*

Proof. For the crown graph S_n^0 with vertex set $V = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$, the subset $M = \{u_1, u_2, ..., u_n\}$ is a minimum monopoly set in S_n^0 . Then

$$A_M(S_n^0) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n}.$$

The Characteristic polynomial of $A_M(S_n^0)$ is

$$f_n(S_n^0,\lambda)=\begin{vmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda-1 & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ 0 & 0 & \lambda-1 & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda-1 & -1 & -1 & -1 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}$$

$$=(\lambda^2-\lambda-1)^{n-1}(\lambda^2-\lambda-(n-1)^2).$$

Then, the minimum monopoly spectrum of S_n^0 is

$$MM\ Spec(S_n^0)=\begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{4n^2-8n+5}}{2} & \frac{1+\sqrt{4n^2-8n+5}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore, the minimum monopoly energy of S_n^0 is $E_M(S_n^0)=(n-1)\sqrt{5}+\sqrt{4n^2-8n+5}$. □

5. Bounds on Minimum Monopoly Energy of Graphs

Theorem 5.1. *Let G be a connected graph of order n and size m. Then*

$$\sqrt{2m+mo(G)}\leq E_M(G)\leq\sqrt{n(2m+mo(G))}$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\leq\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right).$$

By choose $a_i=1$ and $b_i=|\lambda_i|$ and by Theorem 3.2, we get

$$(E_M(G))^2=\left(\sum_{i=1}^{n}|\lambda_i|\right)^2\leq\left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\lambda_i^2\right)\leq n\left(2m+mo(G)\right).$$

Therefore, the upper bound is hold. Now, since $\left(\sum_{i=1}^{n}|\lambda_i|\right)^2\geq\sum_{i=1}^{n}\lambda_i^2$, it follows by Theorem 3.2 that $(E_M(G))^2\geq 2m+mo(G)$. Therefore, the lower bound is hold. □

Theorem 5.2. *For a connected graph G of order n and size m.*

$$\sqrt{n+1}\leq E_M(G)\leq n\sqrt{n}.$$

Proof. Since for any graph $mo(G) \leq \frac{n}{2}$ (see [10]), it follows that by using Theorem 5.1 and well-known result $2m < n^2 - n$, we have

$$E_M(G) \leq \sqrt{n\left(2m + mo(G)\right)} \leq \sqrt{n\left[(n^2 - n) + \frac{n}{2}\right]} \leq n\sqrt{n}.$$

For the lower bound, Since for any connected graph $n \leq 2m$ and $mo(G) \geq 1$ (see [14]), it follows by Theorem 5.1 that

$$E_M(G) \geq \sqrt{2m + mo(G)} \geq \sqrt{n+1}.$$

□

Similar to Koolen and Moultons [11], upper bound for $E_M(G)$ is given in the following theorem.

Theorem 5.3. *Let G be a graph of order n and size m. Then*

$$E_M(G) \leq \frac{2m + mo(G)}{n} + \sqrt{(n-1)\left[2m + mo(G) - (\frac{2m + mo(G)}{n})^2\right]}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$, we have

$$\left(\sum_{i=2}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=2}^{n} 1\right)\left(\sum_{i=2}^{n} \lambda_i^2\right).$$

Hence, by Theorem 3.2 we have

$$(E_M(G) - |\lambda_1|)^2 \leq (n-1)(2m + mo(G) - \lambda_1^2).$$

Therefore,

$$E_M(G) \leq \lambda_1 + \sqrt{(n-1)(2m + mo(G) - \lambda_1^2)}.$$

From Theorem 3.3 we have $\lambda_1 \geq \frac{2m+mo(G)}{n}$.

Since $f(x) = x + \sqrt{(n-1)(2m + mo(G) - x^2)}$ is a decreasing function, we have

$$f(\lambda_1) \leq f(\frac{2m + mo(G)}{n}).$$

Thus,

$$E_M(G) \leq f(\lambda_1) \leq f(\frac{2m + mo(G)}{n}).$$

Therefore,

$$E_M(G) \leq \frac{2m + mo(G)}{n} + \sqrt{(n-1)\left[2m + mo(G) - (\frac{2m + mo(G)}{n})^2\right]}.$$

□

Theorem 5.4. *Let G be a connected graph of order n and size m. If $D = det(A_M(G))$, then*

$$E_M(G) \geq \sqrt{2m + mo(G) = n(n-1)D^{2/n}}.$$

Proof. Since

$$(E_M(G))^2 = (\sum_{i=1}^{n} |\lambda_i|)^2 = (\sum_{i=1}^{n} |\lambda_i|)(\sum_{i=1}^{n} |\lambda_i|) = \sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{i<j} |\lambda_i||\lambda_j|.$$

Using the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| \geq (\prod_{i \neq j} |\lambda_i||\lambda_j|)^{1/[n(n-1)]}.$$

Hence, by this and Theorem 3.2 we get

$$\begin{aligned}(E_M(G))^2 &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)(\prod_{i \neq j} |\lambda_i||\lambda_j|)^{1/[n(n-1)]} \\ &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)(\prod_{i=j} |\lambda_i|^{2(n-1)})^{1/[n(n-1)]} \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)|\prod_{i=j} \lambda_i|^{2/n} \\ &= 2m + mo(G) + n(n-1)D^{2/n}.\end{aligned}$$

□

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