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Voronovskaja Type Asymptotic Approximation by General Gamma Type Operators

Research Article

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Abstract: In the present paper, we studied the voronovskaja type theorem for general Gamma type operators. Also, we obtain an error estimate for general Gamma type operators.

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1. Introduction

In [11], Lupas and Müller defined and studied the Gamma operators $G_n(f;x)$ as

$$G_n(f;x) = \int_0^\infty g_n(x,u) f\left(\frac{n}{u}\right) du,$$

where

$$g_n(x,u) = \frac{x^{n+1}}{n!} e^{-xu} u^n, \qquad x > 0.$$

In [12], Mazhar gives an important modifications of the Gamma operators using the same $g_n(x,u)$

$$\begin{aligned} F_n(f;x) &= \int_0^\infty \int_0^\infty g_n(x,u) g_{n-1}(u,t) f(t)\,du\,dt \\ &= \frac{(2n)!x^{n+1}}{n!(n-1)!} \int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}} f(t)dt, \ n>1, \ x>0. \end{aligned}$$

Recently, Karsli [7] considered a modification and obtain the rate of convergence of these operators for functions with derivatives of bounded variation.

$$\begin{aligned} L_n(f;x) &= \int_0^\infty \int_0^\infty g_{n+2}(x,u) g_n(u,t) f(t)\,du\,dt \\ &= \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t)dt, \ x>0. \end{aligned}$$

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Karsli and Özarslan obtained local and global approximation results for $L_n(f;x)$ in [10]. Also, Voronovskaja type asymptotic formula for $L_n(f;x)$ were proved in [3] and [5].

In the year 2007, Mao [13] define the following generalized Gamma type linear and positive operators

$$\begin{aligned} M_{n,k}(f;x) &= \int_0^\infty \int_0^\infty g_n(x,u) g_{n-k}(u,t) f(t) du dt \\ &= \frac{(2n-k+1)! x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} f(t) dt, \; x > 0, \end{aligned}$$

which includes the operators $F_n(f;x)$ for $k = 1$ and $L_{n-2}(f;x)$ for $k = 2$. Some approximation properties of $M_{n,k}$ were studied in [6] and [8].

We can rewrite the operators $M_{n,k}(f;x)$ as

$$M_{n,k}(f;x) = \int_0^\infty K_{n,k}(x,t) f(t) dt, \tag{1}$$

where

$$K_{n,k}(x,t) = \frac{(2n-k+1)! x^{n+1}}{n!(n-k)!} \frac{t^{n-k}}{(x+t)^{2n-k+2}}, \quad x, t \in (0,\infty).$$

The main goal of this paper is to obtain a Voronovskaja type asymptotic formula and an error estimates for the operators (1).

2. Auxiliary Results

In this section, we give some lemmas which are necessary to prove our main results.

Lemma 2.1 ([8])**.** *For any $m \in N_0$(the set of non-negative integers), $m \leq n-k$*

$$M_{n,k}(t^m;x) = \frac{[n-k+m]_m}{[n]_m} x^m. \tag{2}$$

where $n, k \in N$ and $[x]_m = x(x-1)...(x-m+1), [x]_0 = 1, x \in R$.

In particular for m = 0, 1, 2... in (2.1) we get

(i) $M_{n,k}(1;x) = 1$;

(ii) $M_{n,k}(t;x) = \frac{n-k+1}{n} x$;

(iii) $M_{n,k}(t^2;x) = \frac{(n-k+2)(n-k+1)}{n(n-1)} x^2$.

Lemma 2.2 ([8])**.** *Following equalities holds:*

(i) $M_{n,k}((t-x);x) = \frac{1-k}{n} x$;

(ii) $M_{n,k}((t-x)^2;x) = \frac{(k^2-5k+2n+4)}{n(n-1)} x^2$;

(iii) $M_{n,k}((t-x)^m;x) = O\left(n^{-[(m+1)/2]}\right)$.

For simplicity, put $\beta_n = \frac{(2n-k+1)!}{n!(n-k)!}$.

Lemma 2.3. *If r^{th} derivative $f^{(r)}(r = 0, 1, 2...)$ exists continuously, then we get*

$$M_{n,k}^{(r)}(f;x) = \beta_n x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} f^{(r)}(t)dt, \; x > 0.$$

Proof. Using the substitution $t = vx$ in (1), we obtain

$$M_{n,k}(f;x) = \beta_n \int_0^\infty \frac{v^{n-k}}{(1+v)^{2n-k+2}} f(vx)dv.$$

Using Leibniz's rule $r(r = 0, 1, 2...)$ times, we obtain

$$\begin{aligned} M_{n,k}^{(r)}(f;x) &= \beta_n \frac{d^r}{dx^r} \int_0^\infty \frac{v^{n-k}}{(1+v)^{2n-k+2}} f(vx)dv \\ &= \beta_n \int_0^\infty \frac{v^{n-k}}{(1+v)^{2n-k+2}} \frac{\partial^r}{\partial x^r} f(vx)dv \\ &= \beta_n \int_0^\infty \frac{v^{n-k+r}}{(1+v)^{2n-k+2}} f^{(r)}(vx)dv. \end{aligned}$$

Using $v = \frac{t}{x}$, we get

$$M_{n,k}^{(r)}(f;x) = \beta_n x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} f^{(r)}(t)dt.$$

□

Next, we define

$$M_{n,k,r}^{*}(g;x) = \frac{\beta_n x^{n+1-r}}{b(n,k,r)} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} g(t)dt,$$

where

$$b(n,k,r) = \beta_n x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} dt = \frac{(n-r)!(n-k+r)!}{n!(n-k)!}.$$

Let us define

$$e_m(t) = t^m, \;\; \varphi_{x,m}(t) = (t-x)^m, \;\; m \in N_0, \;\; x, t \in (0, \infty).$$

Lemma 2.4. *For any $m \in N_0$, $n \geq r + m$ and $r \leq n$*

$$M_{n,k,r}^{*}(e_m;x) = \frac{(n-r-m)!(n-k+r+m)!}{(n-r)!(n-k+r)!} x^m; \tag{3}$$

and

$$M_{n,k,r}^{*}(\varphi_{x,m};x) = \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(n-r-m+j)!(n-k+r+m-j)!}{(n-r)!(n-k+r)!} \right) x^m, \tag{4}$$

for each $x \in (0, \infty)$.

Proof. The proof of (3) is follows from Lemma 2.1. On the other hand, we have the following identity,

$$(t-x)^m = \sum_{j=0}^{m}(-1)^j \binom{m}{j} x^j t^{m-j}.$$

Then, we have

$$\begin{aligned} M^*_{n,k,r}((t-x)^m;x) &= \int_0^\infty K_{n,k}(x,t)(t-x)^m dt \\ &= \int_0^\infty K_{n,k}(x,t)\sum_{j=0}^{m}(-1)^j \binom{m}{j} x^j t^{m-j} dt \\ &= \sum_{j=0}^{m}(-1)^j \binom{m}{j} x^j M^*_{n,k,r}(t^{m-j};x). \end{aligned}$$

Using (3), we get (4). □

Lemma 2.5. *For $m = 0,1,2,3,4$, one has*

(i) $M^*_{n,k,r}(\varphi_{x,0};x) = 1$,

(ii) $M^*_{n,k,r}(\varphi_{x,1};x) = \dfrac{2r-k+1}{n-r}x$,

(iii) $M^*_{n,k,r}(\varphi_{x,2};x) = \dfrac{4r^2+4r(2-k)+2n+k^2-5k+4}{(n-r)(n-r-1)}x^2$,

(iv) $M^*_{n,k,r}(\varphi_{x,3};x) = \dfrac{c_{n,k,r}}{(n-r)(n-r-1)(n-r-2)}x^3$,

(v) $M^*_{n,k,r}(\varphi_{x,4};x) = \dfrac{d_{n,k,r}}{(n-r)(n-r-1)(n-r-2)(n-r-3)}x^4$,

where $c_{n,k,r} = 8r^3 + r^2(36-2k) + r(51+14n-42k+6k^2) - k^3 + 12k^2 - 34k - n^2 + n(17-6k-6k^2+2kr) + 21$ *and* $d_{n,k,r} = 16r^4 + r^3(128-32k) + r^2(348+48n-216k+24k^2) + r(366+177n+k(6n^2-54n-440)+120k^2-8k^3) + k^4 + k^3(4n-22) + 139k^2 - k(245+116n) + 24n^2 + 131n + 100$.

3. Voronovskaja Type Theorem

In this section we obtain the Voronovskaja type theorem for the operators $M^{(r)}_{n,k}$.

Let $C_B[0,\infty)$ be the space of all real valued continuous and bounded functions on $[0,\infty)$ endowed with the usual supremum norm. By $C^{(r+2)}_B[0,\infty)(r\in N_0)$, we denote the space of all functions $f\in C_B[0,\infty)$ such that $f', f'', ... f^{(r+2)} \in C_B[0,\infty)$.

Theorem 3.1. *Let f be integrable in $(0,\infty)$ and admits its $(r+1)^{th}$ and $(r+2)^{th}$ derivatives, which are bounded at a fixed point $x\in(0,\infty)$ and $f^{(r)}(t) = O(t^\alpha)$, as $t\to\infty$ for some $\alpha>0$, then*

$$\lim_{n\to\infty} n\left(\frac{1}{b(n,k,r)}M^{(r)}_{n,k}(f;x) - f^{(r)}(x)\right) = (2r-k+1)xf^{(r+1)}(x) + x^2 f^{(r+2)}(x)$$

holds.

Proof. Using Taylor's theorem, we get

$$f^{(r)}(t) - f^{(r)}(x) = (t-x)f^{(r+1)}(x) + \frac{1}{2}(t-x)^2 f^{(r+2)}(x) + (t-x)^2\xi(t,x),$$

where $\xi(t,x)$ is the peano form of the remainder and $\lim_{t\to x}\xi(t,x)=0$.

Then, we have

$\frac{1}{b(n,k,r)}M_{n,k}^{(r)}(f;x)-f^{(r)}(x)$

$$\begin{aligned}
&= \frac{\beta_n}{b(n,k,r)}x^{n+1-r}\int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}}\Big(f^{(r)}(t)-f^{(r)}(x)\Big)dt\\
&= \frac{\beta_n}{b(n,k,r)}x^{n+1-r}\int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}}\Big((t-x)f^{(r+1)}(x)+\frac{1}{2}(t-x)^2f^{(r+2)}(x)+(t-x)^2\xi(t,x)\Big)\,dt\\
&= f^{(r+1)}(x)M_{n,k,r}^{*}(t-x,x)+\frac{1}{2}f^{(r+2)}(x)M_{n,k,r}^{*}((t-x)^2,x)+M_{n,k,r}^{*}((t-x)^2\xi(t,x);x).
\end{aligned}$$

Using Lemma 2.5, we get

$$n\left(\frac{1}{b(n,k,r)}M_{n,k}^{(r)}(f;x)-f^{(r)}(x)\right)=\frac{n(2r-k+1)}{n-r}xf^{(r+1)}(x)+\frac{n(4r^2+4r(2-k)+2n+k^2-5k+4)}{2(n-r)(n-r-1)}x^2f^{(r+2)}(x)+nM_{n,k,r}^{*}((t-x)^2\xi(t,x);x).$$

By using Cauchy-Schwarz inequality, we have

$$n\left(M_{n,k,r}^{*}((t-x)^2\xi(t,x);x)\right)\le\sqrt{n^2M_{n,k,r}^{*}(\varphi_{x,4};x)}\sqrt{M_{n,k,r}^{*}(\xi^2(t,x);x)}. \tag{5}$$

We observe that $\xi^2(x,x)=0$ and $\xi^2(.,x)$ is continuous at $t\in(0,\infty)$ and bounded as $t\to\infty$. Then from Korovkin theorem that

$$\lim_{n\to\infty}M_{n,k,r}^{*}(\xi^2(t,x);x)=\xi^2(x,x)=0, \tag{6}$$

in view of fact that $M_{n,k,r}^{*}(\varphi_{x,4};x)=O\left(\frac{1}{n^2}\right)$. Now, from (5) and (6) we obtain

$$\lim_{n\to\infty}nM_{n,k,r}^{*}\left((t-x)^2\xi(t,x);x\right)=0. \tag{7}$$

Using (7), we have

$$\lim_{n\to\infty}n\left(\frac{1}{b(n,k,r)}M_{n,k}^{(r)}(f;x)-f^{(r)}(x)\right)=(2r-k+1)xf^{(r+1)}(x)+x^2f^{(r+2)}(x).$$

This completes the proof. □

4. Direct Results

In this section we obtain the rate of convergence of the operators $M_{n,k}^{(r)}$.

Let us consider the following K-functional:

$$K(f,\delta)\;=\;\inf_{g\in C_B^2[0,\infty)}\{\| f-g\|+\delta\| g''\|\}, \tag{8}$$

where $\delta>0$. By, p. 177, Theorem 2.4 in [2], there exists an absolute constant $C>0$ such that

$$K(f,\delta)\le C\omega_2(f,\sqrt{\delta}), \tag{9}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |\, f(x+2h) - 2f(x+h) + f(x) \,| \tag{10}$$

is the second order modulus of smoothness of f. By

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} |\, f(x+h) - f(x) \,|,$$

we denote the first order modulus of continuity of f and satisfies the following property:

$$|f(t) - f(x)| \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega(f,\delta), \tag{11}$$

where $\delta > 0$.

Theorem 4.1. *Let $f \in C_B^r[0,\infty)$ and $r \in N_0$. Then for $n > r$, we have*

$$\left|\frac{1}{b(n,k,r)} M_{n,k}^{(r)}(f;x) - f^{(r)}(x)\right| \leq 2\omega(f^{(r)}, \sqrt{\delta_n}),$$

where

$$\delta_n = \left(\frac{4r^2 + 4r(2-k) + 2n + k^2 - 5k + 4}{(n-r)(n-r-1)}\right) x^2.$$

Proof. By using monotonicity of $M^*_{n,k,r}$, we get
$\left|\frac{1}{b(n,k,r)} M_{n,k}^{(r)}(f;x) - f^{(r)}(x)\right|$

$$\begin{aligned}
&= \left|\frac{\beta_n}{b(n,k,r)} x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} (f^{(r)}(t) - f^{(r)}(x))dt\right| \\
&= |M^*_{n,k,r}((f^{(r)}(t) - f^{(r)}(x));x)| \\
&\leq M^*_{n,k,r}(|f^{(r)}(t) - f^{(r)}(x)|;x) \\
&\leq \omega\left(f^{(r)},\delta\right) \frac{\beta_n}{b(n,k,r)} x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} \left(1 + \frac{|t-x|}{\delta}\right) dt \\
&\leq \omega\left(f^{(r)},\delta\right) \left(1 + \frac{1}{\delta}\frac{\beta_n}{b(n,k,r)} x^{n+1-r} \int_0^\infty \frac{t^{n-k+r}}{(x+t)^{2n-k+2}} |t-x| dt\right).
\end{aligned}$$

Thus, by applying the Cauchy-Schwarz inequality, we have

$$\left|\frac{1}{b(n,k,r)} M_{n,k}^{(r)}(f;x) - f^{(r)}(x)\right| \leq \omega\left(f^{(r)},\delta\right) \left(1 + \frac{1}{\delta}\left(M^*_{n,k,r}((t-x)^2;x)\right)^{1/2}\right).$$

Choosing $\delta = \sqrt{\delta_n}$, we have

$$\left|\frac{1}{b(n,k,r)} M_{n,k}^{(r)}(f;x) - f^{(r)}(x)\right| \leq 2\omega\left(f^{(r)}, \sqrt{\delta_n}\right).$$

Hence, the proof is completed. □

Theorem 4.2. *Let $f \in C_B^r[0,\infty)$ and $r \in N_0$. Then for $n > r$, we have*

$$\left|\frac{1}{b(n,k,r)} M_{n,k}^{(r)}(f;x) - f^{(r)}(x)\right| \leq C\omega_2\left(f^{(r)}, \gamma_n\right) + \omega\left(f^{(r)}, \frac{2r-k+1}{n-r}x\right),$$

where C is an absolute constant and

$$\gamma_n = \left(\frac{4r^2 + 4r(2-k) + 2n + k^2 - 5k + 4}{(n-r)(n-r-1)} x^2 + \left(\frac{2r-k+1}{n-r}x\right)^2\right)^{1/2}.$$

Proof. Let us consider the auxiliary operators $\overline{M^*_{n,k,r}}$ defined by

$$\overline{M^*_{n,k,r}}(f;x) = M^*_{n,k,r}(f;x) - f\left(x + \frac{2r-k+1}{n-r}x\right) + f(x). \tag{12}$$

Using Lemma 2.5, we observe that the operators $\overline{M^*_{n,k,r}}$ are linear and reproduce the linear functions.

Hence

$$\overline{M^*_{n,k,r}}((t-x);x) = 0. \tag{13}$$

Let $g \in C_B^{r+2}[0,\infty)$ and $x \in (0,\infty)$. By Taylor's theorem, we have

$$g^{(r)}(t) - g^{(r)}(x) = (t-x)g^{(r+1)}(x) + \int_x^t (t-v)g^{(r+2)}(v)dv, \;\; t \in (0,\infty).$$

Using (12) and (13), we get

$|\overline{M^*_{n,k,r}}(g^{(r)};x) - g^{(r)}(x)|$

$$\begin{aligned} &= \left| g^{(r+1)}(x)\overline{M^*_{n,k,r}}(t-x;x) + \overline{M^*_{n,k,r}}\left(\int_x^t (t-v)g^{(r+2)}(v)dv;x\right)\right| \\ &\leq \left| M^*_{n,k,r}\left(\int_x^t (t-v)g^{(r+2)}(v)dv;x\right)\right| + \left|\int_x^{x+\frac{2r-k+1}{n-r}x}\left(x+\frac{2r-k+1}{n-r}x - v\right)g^{(r+2)}(v)dv\right|. \end{aligned}$$

Observe that

$$\left| M^*_{n,k,r}\left(\int_x^t (t-v)g^{(r+2)}(v)dv;x\right)\right| \leq ||g^{(r+2)}|| M^*_{n,k,r}((t-x)^2;x)$$

and

$$\left|\int_x^{x+\frac{2r-k+1}{n-r}x}\left(x+\frac{2r-k+1}{n-r}x - v\right)g^{(r+2)}(v)dv\right| \leq ||g^{(r+2)}|| \left(\frac{2r-k+1}{n-r}x\right)^2.$$

Hence by Lemma 2.5, we have

$$|\overline{M^*_{n,k,r}}(g^{(r)};x) - g^{(r)}(x)| \leq ||g^{(r+2)}|| \left(\frac{4r^2+4r(2-k)+2n+k^2-5k+4}{(n-r)(n-r-1)}x^2 + \left(\frac{2r-k+1}{n-r}x\right)^2\right). \tag{14}$$

Now $g \in C_B^{r+2}[0,\infty)$, using (14), we obtain

$$\begin{aligned} \left|\frac{1}{b(n,k,r)}M^{(r)}_{n,k}(f;x) - f^{(r)}(x)\right| &= |M^*_{n,k,r}(f^{(r)};x) - f^{(r)}(x)| \\ &\leq |\overline{M^*_{n,k,r}}(f^{(r)} - g^{(r)};x) - (f^{(r)} - g^{(r)})(x)| + |\overline{M^*_{n,k,r}}(g^{(r)};x) - g^{(r)}(x)| \\ &+ \left| f^{(r)}\left(x + \frac{2r-k+1}{n-r}x\right) - f^{(r)}(x)\right| \\ &\leq 4||f^{(r)} - g^{(r)}|| + \gamma_n^2||g^{(r+2)}|| + \omega\left(f^{(r)}, \frac{2r-k+1}{n-r}x\right). \end{aligned}$$

Taking infimum over all $g \in C_B^{r+2}[0,\infty)$, we obtain

$$\left|\frac{1}{b(n,k,r)}M^{(r)}_{n,k}(f;x) - f^{(r)}(x)\right| \leq K\left(f^{(r)}, \gamma_n^2\right) + \omega\left(f^{(r)}, \frac{2r-k+1}{n-r}x\right).$$

Using (9), we have

$$\left|\frac{1}{b(n,k,r)}M^{(r)}_{n,k}(f;x) - f^{(r)}(x)\right| \leq C\omega_2\left(f^{(r)}, \gamma_n\right) + \omega\left(f^{(r)}, \frac{2r-k+1}{n-r}x\right).$$

Hence, the proof is completed. □

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