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Co-maximal Filters in $(\mathcal{Z}^+, \leq_C)$

Research Article

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Abstract: A convolution is a mapping $\mathcal{C}$ of the set Z^+ of positive integers into the set $\mathcal{P}(Z^+)$ of all subsets of Z^+ such that, for any $n \in Z^+$, each member of $C(n)$ is a divisor of n. If $D(n)$ is the set of all divisors of n, for any n, then D is called the Dirichlet's convolution[2]. If $U(n)$ is the set of all Unitary(square free) divisors of n , for any n, then U is called unitary(square free) convolution. Corresponding to any general convolution C, we can define a binary relation $\leq_C$ on Z^+ by ' $m \leq_C n$ if and only if $m \in C(n)$ '. In this paper, we discuss co-maximal filters in $(\mathcal{Z}^+, \leq_{\mathcal{C}})$, where $\leq_{\mathcal{C}}$ is the binary relation induced by the convolution $\mathcal{C}$.

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1. Introduction

A Convolution is a mapping $\mathcal{C}$ of the set $\mathcal{Z}^+$of positive integers into the set $\mathcal{P}(\mathcal{Z}^+)$ of subsets of $\mathcal{Z}^+$ such that, for any $n \in \mathcal{Z}^+$, $\mathcal{C}(n)$ is a nonempty set of divisors of n. If $\mathcal{C}(n)$ is the set of all divisors of n, for each $n \in \mathcal{Z}^+$, then $\mathcal{C}$ is the classical Dirichlet convolution[2]. If $\mathcal{C}(n)$= $\{d \ / \ d|n$ and $(d, \frac{n}{d}) = 1\}$,Then $\mathcal{C}$ is the Unitary convolution[1]. As another example if $\mathcal{C}(n)$=$\{d \ / \ d|n$ and m^k doesnot divide d for any $m \in \mathcal{Z}^+\}$ then $\mathcal{C}$ is the k-free convolution.

$$\mathcal{C}(n) = \{d \ | \ d/n \ \text{ and } \ (d, \frac{n}{d}) = 1\}$$

Corresponding to any convolution $\mathcal{C}$, we can define a binary relation $\leq_{\mathcal{C}}$ in a natural way by

$$m \leq_{\mathcal{C}} n \text{ if and only if } m \in \mathcal{C}(n).$$

$\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ and is called partial order induced by the convolution $\mathcal{C}$[5, 6]. In this paper, we discuss co-maximality of filters in $(\mathcal{Z}^+, \leq_C)$.

2. Preliminaries

Let us recall that a partial order on a non-empty set X is defined as a binary relation $\leq$ on X which is reflexive $(a \leq a)$, transitive $(a \leq b, b \leq c \Longrightarrow a \leq c)$ and antisymmetric $(a \leq b, b \leq a \Longrightarrow a = b)$ and that a pair $(X, \leq)$ is called a partially ordered set(poset) if X is a non-empty set and $\leq$ is a partial order on X. For any $A \subseteq X$ and $x \in X$, x is called a

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lower(upper) bound of A if $x \leq a$(respectively $a \leq x$) for all $a \in A$. We have the usual notations of the greatest lower bound(glb) and least upper bound(lub) of A in X. If A is a finite subset $\{a_1, a_2, \cdots, a_n\}$, the glb of A(lub of A) is denoted by $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ or $\bigwedge\limits_{i=1}^{n} a_i$ (respectively by $a_1 \vee a_2 \vee \cdots \vee a_n$ or $\bigvee\limits_{i=1}^{n} a_i$). A partially ordered set $(X, \leq)$ is called a meet semi lattice if $a \wedge b$ (=glb$\{a, b\}$) exists for all a and $b \in X$. $(X, \leq)$ is called a join semi lattice if $a \vee b$ (=lub$\{a, b\}$) exists for all a and $b \in X$. A poset $(X, \leq)$ is called a lattice if it is both a meet and join semi lattice. Equivalently, lattice can also be defined as an algebraic system $(X, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations which are associative, commutative and idempotent and satisfying the absorption laws, namely $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ for all $a, b \in X$; in this case the partial order $\leq$ on X is such that $a \wedge b$ and $a \vee b$ are respectively the glb and lub of $\{a, b\}$. The algebraic operations $\wedge$ and $\vee$ and the partial order $\leq$ are related by

$$a = a \wedge b \iff a \leq b \iff a \vee b = b.$$

Throughout the paper, $\mathcal{Z}^+$ and $\mathcal{N}$ denote the set of positive integers and the set of non-negative integers respectively.

Definition 2.1. *A mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ is called a convolution if the following are satisfied for any $n \in \mathcal{Z}^+$.*

1. *$\mathcal{C}(n)$ is a set of positive divisors of n*
2. *$n \in \mathcal{C}(n)$*
3. *$\mathcal{C}(n) = \bigcup\limits_{m \in \mathcal{C}(n)} \mathcal{C}(m)$.*

Definition 2.2. *For any convolution $\mathcal{C}$ and m and $n \in \mathcal{Z}^+$, we define*

$$m \leq n \text{ if and only if } m \in \mathcal{C}(n)$$

Then $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ and is called the partial order induced by $\mathcal{C}$ on $\mathcal{Z}^+$. In fact, for any mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ such that each member of $\mathcal{C}(n)$ is a divisor of n, $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ if and only if $\mathcal{C}$ is a convolution [6], as defined above.

Definition 2.3. *A poset is said to satisfy the Descending Chain Condition (D.C.C) if every non-empty subset has a minimal element.*

Definition 2.4. *A chain is a totally ordered subset of the partially ordered set $(X, \leq)$ and a maximal chain is one that is not a proper subset of another chain.*

Definition 2.5. *A partially ordered set $(X, \leq)$ is said to be a disjoint union of maximal chains if there is a class $\{Y_i\}_{i \in I}$ of subsets of X satisfying the following properties [4].*

1. *Each $Y_i, i \in I$ is a maximal chain in $(X, \leq)$*
2. *$Y_i \cap Y_j = \phi$ for all $i \neq j \in I$*
3. *x and y are incomparable (we express this by $x \| y$) for any $x \in X_i$ and $y \in Y_j$ with $i \neq j$.*
4. *$X = \bigcup\limits_{i \in I} Y_i$*

Example 2.6. *Any chain (totally ordered set) is a disjoint union of maximal chains.*

Example 2.7. *Let $X = \mathcal{Z}^+ \times \mathcal{Z}^+$ and, for any (a, b) and $(c, d) \in X$, define*

$$(a, b) \leq (c, d) \text{ if and only if } a = c \text{ and } b \leq d$$

where $\leq$ is the usual ordering in $\mathcal{Z}^+$. Then, for any $a \in \mathcal{Z}^+$, $\{a\} \times \mathcal{Z}^+$ is a maximal chain in X. X is the disjoint union of $(\{a\} \times \mathcal{Z}^+)'s$.

Definition 2.8. *Two filters F and G of a meet semi lattice $(S, \wedge)$ are said to be co-maximal if no proper filter of S contains both F and G (or, equivalently, S is the only filter of S containing $F \cup G$). In this case, we write $F \wedge G = S$.*

Example 2.9. *For any m and $n \in \mathcal{Z}^+$ with $(m, n) = 1$, $[m)$ and $[n)$ are co-maximal in $(\mathcal{Z}^+, \leq_C)$ for any convolution C.*

Definition 2.10. *Let $(S, \wedge)$ be a meet semi lattice. A proper filter F of S is called a prime filter if, for any a and b in S,*

$$a \vee b \text{ exists in } S \text{ and } a \vee b \in F \quad \Longrightarrow \quad a \in F \text{ or } b \in F.$$

3. Co-maximality in $(\mathcal{Z}^+, \leq_C)$

First we have the following theorem on prime filters.

Theorem 3.1. *Let $(S, \wedge)$ be any meet semi lattice. Then every proper filter of $(S, \wedge)$ is prime if and only if, for any x and y in S,*

$$x \vee y \text{ exists in } S \Leftrightarrow x \text{ and } y \text{ are comparable.}$$

Proof. Suppose that every proper filter of $(S, \wedge)$ is prime. Let x and $y \in S$. If x and y are comparable, then clearly $x \vee y$ exists in S.On the other hand, suppose $x \vee y$ exists and $x \vee y = z$. If $[z) = S$, then x and $y \in [z)$ and hence $x = z = y$. If $[z) \neq S$, then by hypothesis, $[z)$ is a prime filter and $x \vee y \in [z)$ and hence $x \in [z)$ or $y \in [z)$ so that $x = z$ or $y = z$. Therefore $x = x \vee y$ or $y = x \vee y$, which imply that x and y are comparable. The converse is trivial. □

Theorem 3.2. *Let $(S, \wedge)$ be any meet semi lattice with smallest element 0 satisfying the Descending Chain Condition(DCC) . Also, suppose that every proper filter of S is prime. Then the following are equivalent to each other.*

1. *For any x and $y \in S$, $x \| y \Longrightarrow x \wedge y = 0$*
2. *$S - \{0\}$ is a disjoint union of maximal chains*
3. *Any two incomparable filters of S are co-maximal.*

Proof. (1) ⇒ (2) : Let M be the set of all minimal elements in $S - \{0\}$ and, for any $m \in M$, et

$$X_m = [m) = \{x \in S \mid m \leq x\}.$$

If x and $y \in X_m$ and $x \| y$, then, by (1), $x \wedge y = 0$ which is not true since $m \leq x$ and $m \leq y$ and hence $0 < m \leq x \wedge y$. Therefore, any two elements of X_m are comparable. That is, X_m is a chain for each $m \in M$. We shall prove that each X_m is a maximal chain and $S - \{0\}$ is the disjoint union of X_m's. If $0 \neq x \in S$ and $X_m \cup \{x\}$ is a chain, then x must be comparable with m and hence $m \leq x$ (since m is minimal, $0 < x < m$ is not possible), so that $x \in X_m$. This shows that X_m is a maximal chain in $S - \{0\}$, for each $m \in M$. Next, suppose $m \neq n \in M$. Then m and n are incomparable (since both are minimal) and hence, by Theorem 1, $m \vee n$ does not exist in S. This implies that m and n have no common upper bounds in S (since S is a meet semi lattice satisfying the descending chain condition) and hence $X_m \cap X_n = \phi$. Further, let $m \neq n \in M$, $x \in M_m$ and $y \in M_n$. Then

$$\begin{aligned} x \leq y &\Longrightarrow m \leq x \leq y \\ &\Longrightarrow y \in X_m \cap X_n. \end{aligned}$$

But, since $X_m \cap X_n = \phi$, it follows that x and y are incomparable. Finally, for any $x \in S - \{0\}$, there exists a minimal element in $\{y \in S - \{0\} \mid y \leq x\}$ (since S satisfies descending chain condition), say m. Then m is minimal in the whole of $S - \{0\}$ and hence $m \in M$ and $x \in [m) = X_m$. Therefore $S - \{0\}$ is the disjoint union maximal chains X_m's, $m \in M$.

$(2) \Rightarrow (3)$: Suppose that $\{Y_i\}_{i \in I}$ is a class of maximal chains in $S - \{0\}$ such that $S - \{0\}$ is the disjoint union of Y_i's. Let F and G be two incomparable filters of S. Then $F = [x)$ and $G = [y)$ for some $x, y \in S$. Since F and G are incomparable, x and y are also incomparable. Since x and $y \in S - \{0\} = \bigcup_{i \in I} Y_i$, there exist $i \neq j \in I$ such that $x \in Y_i$ and $y \in Y_j$. Then $x \wedge y = 0$ (otherwise, by (3) of Definition 5, $x \wedge y \in Y_i \cap Y_j$, which is a contradiction to (2) of Definition 5). Therefore, if H is any filter containing both F and G, then x and $y \in H$ and hence $0 = x \wedge y \in H$, so that $H = S$. Thus F and G are co-maximal.

$(3) \Rightarrow (1)$: Let x and $y \in S$ such that $x \| y$. Then $[x)$ and $[y)$ are incomparable filters of S and, by (3), $[x)$ and $[y)$ are co-maximal. Since $[x) \subseteq [x \wedge y)$ and $[y) \subseteq [x \wedge y)$, it follows that $[x \wedge y) = S$ and hence $x \wedge y = 0$. □

Theorem 3.3. *Let C be any multiplicative convolution such that $(\mathcal{Z}^+, \leq_C)$ is a meet semi lattice. Then any two incomparable prime filters of $(\mathcal{Z}^+, \leq_C)$ are co-maximal if and only if any two incomparable prime filters of $(\mathcal{N}, \leq_C^p)$ are co-maximal, for each $p \in \mathcal{P}$.*

Proof. This follows from the fact that F is a prime filter of $(\mathcal{Z}^+, \leq_C)$ if and only if $F = [p^a)$ for some $p \in \mathcal{P}$ and $a \in \mathcal{N}$ such that $[a)$ is a prime filter in $(\mathcal{N}, \leq_C^p)$ and that $a \wedge b = 0$ in $(\mathcal{N}, \leq_C^p)$ if and only if $p^a \wedge p^b = 1$ in $(\mathcal{Z}^+, \leq_C)$. Note that $[x)$ and $[y)$ are co-maximal if and only if $x \wedge y = 0$. □

Theorem 3.4. *Let p be a prime number. Then every proper filter in $(\mathcal{N}, \leq_C^p)$ is prime if and only if $[p^a)$ is a prime filter in $(\mathcal{Z}^+, \leq_C)$ for all $n > 0$.*

Proof. By Theorem 3.1, every proper filter in $(\mathcal{N}, \leq_C^p)$ is prime if and only if, for any a and $b \in \mathcal{N}$, $a \vee b$ exists in $(\mathcal{N}, \leq_C^p)$ only when a and b are comparable in $(\mathcal{N}, \leq_C^p)$ and we shall prove that this is equivalent to saying that $[p^a)$ is a prime filter in $(\mathcal{Z}^+, \leq_C)$ for all $n > 0$. Suppose that $a \vee b$ exists in $(\mathcal{N}, \leq_C^p)$. Let $n > 0$ and $F = [p^a)$. Let m and $k \in \mathcal{Z}^+$ such that $m \vee k$ exists and belong to F. Let $m = p^a.u$ and $k = p^b.v$, where u and $v \in \mathcal{Z}^+$ such that $(p, u) = 1 = (p, v)$. Then $\theta(m)(p) = a$, $\theta(k)(p) = b$ and $\theta(m)(p) \vee \theta(k)(p)$ exists and is equal to $\theta(m \vee k)(p)$ in $(\mathcal{N}, \leq_C^p)$. Therefore $a \vee b$ exists in $(\mathcal{N}, \leq_C^p)$. From our hypothesis, $a \leq_C^p b$ or $b \leq_C^p a$ and hence $p^a \leq_C p^b$ or $p^b \leq_C p^a$ and therefore $n = \theta(p^n)(p) \leq_C^p \theta(m \vee k)(p) = a \vee b = a$ or b, so that $p^n \leq_C p^a.u = m$ or $p^n \leq_C p^b.v = k$. Therefore $m \in F$ or $k \in F$. Thus F is a prime filter in $(\mathcal{Z}^+, \leq_C)$. Converse can be proved by a similar technique. □

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