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Ulam Stabilities of K - AC - Mixed Type Functional Equations in Three Variables

Research Article

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Abstract: In this paper, we obtain the general solution and generalized Ulam - Hyers stability of a 3 - variable $k-$ AC - mixed type functional equation

$$\begin{aligned} &f(kx+y,kz+w,ku+v)-f(kx-y,kz-w,ku-v) \\ &= k^2[f(x+y,z+w,u+v)-f(x-y,z-w,u-v)]-2(k^2-1)f(y,w,v) \end{aligned}$$

where $k \geq 2$, in Banach space using direct and fixed point methods.

MSC: 39B52, 32B72, 32B82

Keywords: Additive functional equations, cubic functional equation, Mixed type AC functional equation, Ulam - Hyers stability, Ulam - TRassias stability, Ulam - Gavruta - Rassias stability, Ulam - JRassias stability, generalized Ulam - Hyers stability, fixed point.

1. Introduction

The history of the stability of functional equations dates back to 1925 when a stability result appeared in the celebrated book by George Pólya and Gábor Szegö [26]. In 1940, S.M. Ulam [40] posed the famous Ulam stability problem which was partially solved by D.H. Hyers [18] in the framework of Banach spaces. Later, T. Aoki [2] considered the stability problem with unbounded Cauchy differences. In 1978, Th. M. Rassias [33] provided a generalization of the Hyers theorem by proving the existence of unique linear mappings near approximate additive mappings. P. Gavruta [14] obtained a generalized result of Th.M. Rassias theorem which allows the Cauchy difference to be controlled by a general unbounded function. On the other hand, J.M. Rassias [28] considered the Cauchy difference controlled by a product of different powered of norms. However, there was a singular case; for this singularity, a counter example was given by P. Gavruta [15]. In 2008, J.M.Rassias [37] introduced an orthogonally Euler-Lagrange type quadratic functional equation controlled by his mixed type product-sum function. Then Ravi et. al., [37] investigated the Ulam-JRassias stability. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 12, 19, 20, 23, 34]) and references cited there in.

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1.1. Additive Functional Equations

The solution and stability of the following additive functional equations

$$f(x+y) = f(x) + f(y), \tag{1}$$

$$f(2x-y) + f(x-2y) = 3f(x) - 3f(y), \tag{2}$$

$$f(x+y-2z) + f(2x+2y-z) = 3f(x) + 3f(y) - 3f(z), \tag{3}$$

$$f(m(x+y)-2mz) + f(2m(x+y)-mz) = 3m[f(x)+f(y)-f(z)], m \geq 1, \tag{4}$$

$$f(2x \pm y \pm z) = f(x \pm y) + f(x \pm z), \tag{5}$$

$$f(qx \pm y \pm z) = f(x \pm y) + f(x \pm z) + (q-2)f(x), \quad q \geq 2 \tag{6}$$

were investigated by J. Aczel [1], D.O. Lee [13], K. Ravi, M. Arunkumar [35, 39], M. Arunkumar [3, 4].

1.2. Cubic Functional Equations

Also, the solution and stability of the following cubic functional equations

$$C(x+2y) + 3C(x) = 3C(x+y) + C(x-y) + 6C(y), \tag{7}$$

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{8}$$

$$f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) = 2[f(x+y) + 2f(x+z) + 2f(y+z) + 2f(x-z) + 2f(y-z)], \tag{9}$$

$$g(2x-y) + g(x-2y) = 6g(x-y) + 3g(x) - 3g(y) \tag{10}$$

were discussed by J.M. Rassias [29], K.W. Jun, H.M. Kim [21], M.Arunkumar [5].

1.3. Additive - Cubic Functional Equations

Finally, the solution and stability of the following additive - cubic functional equations

$$\begin{aligned} 3f(x+y+z) + f(-x+y+z) + f(x-y+z) + f(x+y-z) + 4[f(x) + f(y) + f(z)] \\ = 4[f(x+y) + f(x+z) + f(y+z)], \end{aligned} \tag{11}$$

$$f(x+ky) + f(x-ky) = k^2 [f(x+y) + f(x-y)] + 2\left(1-k^2\right) f(x), \tag{12}$$

$$f(kx+y) + f(kx-y) = kf(x+y) + kf(x-y) + 2f(kx) - 2kf(x), \tag{13}$$

$$f(2x+y) - f(2x-y) = 4[f(x+y) - f(x-y)] - 6f(y) \tag{14}$$

were discussed by J.M. Rassias [30], M. Eshaghi Gordji, H. Khodaie [16], T.Z. Xu et. al., [41], M. Arunkumar [8].

J.H. Bae and W.G. Park [10] proved the general solution of the 2- variable quadratic functional equation

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x,z) + 2f(y,w) \tag{15}$$

and investigated the generalized Hyers-Ulam-Rassias stability of (15). The above functional equation have solution

$$f(x,y) = ax^2 + bxy + cy^2. \tag{16}$$

The stability of the functional equation (15) in fuzzy normed space was investigated by M. Arunkumar et. al., [6]. Using the ideas in [6], the general solution and generalized Hyers-Ulam-Rassias stability of a 3-variable quadratic functional equation

$$f(x+y, z+w, u+v) + f(x-y, z-w, u-v) = 2f(x, z, u) + 2f(y, w, v). \tag{17}$$

was discussed by K. Ravi and M. Arunkumar [36]. The solution of the functional equation (17) is of the form

$$f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx. \tag{18}$$

Also, M. Arunkumar, S. Hema Latha [9] established the general solution and generalized Ulam - Hyers stability of a 2 - variable additive quadratic functional equation

$$f(x+y, u+v) + f(x-y, u-v) = 2f(x, u) + f(y, v) + f(-y, -v) \tag{19}$$

having solutions

$$f(x, y) = ax + by \tag{20}$$

and

$$f(x, y) = ax^2 + bxy + cy^2 \tag{21}$$

using Banach and Non Archimedean Fuzzy spaces respectively. In fact, M. Arunkumar et. al., [7] first time introduced and investigated a 2 - variable AC - mixed type functional equation

$$f(2x+y, 2z+w) - f(2x-y, 2z-w) = 4[f(x+y, z+w) - f(x-y, z-w)] - 6f(y, w) \tag{22}$$

having solutions

$$f(x, y) = ax + by \tag{23}$$

and

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3. \tag{24}$$

In this paper, we obtain the general solution and generalized Ulam - Hyers stability of a 3 - variable $k-$ AC - mixed type functional equation of the form

$$\begin{aligned} &f(kx+y, kz+w, ku+v) - f(kx-y, kz-w, ku-v) \\ &\qquad = k^2[f(x+y, z+w, u+v) - f(x-y, z-w, u-v)] - 2\left(k^2-1\right) f(y, w, v) \end{aligned} \tag{25}$$

where $k \geq 2$, having solutions

$$f(x, y, z) = ax + by + cz \tag{26}$$

and

$$f(x, y, z) = a_1x^3 + a_2y^3 + a_3z^3 + a_4(x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2) + a_5xyz. \tag{27}$$

In Section 2, we present the general solution of the functional equation (25). The generalized Ulam-Hyers stability in Banach space using direct and fixed point method are discussed in Section 3 and Section 4, respectively.

2. General Solution

In this section, we present the solution of the functional equation (25). Through out this section let U and V be real vector spaces.

Lemma 2.1. *If $f : U^3 \to V$ be a mapping satisfying (25) and let $g : U^3 \to V$ be a mapping given by*

$$g(u, u, u) = f(2u, 2u, 2u) - 8f(u, u, u) \tag{28}$$

for all $u \in U$ then

$$g(2u, 2u, 2u) = 2g(u, u, u) \tag{29}$$

for all $u \in U$ such that g is additive.

Proof. Letting (x, y, z, w, u, v) by $(0, 0, 0, 0, 0, 0)$ in (25), we get

$$f(0, 0, 0) = 0. \tag{30}$$

Setting (x, y, z, w, u, v) by $(0, v, 0, v, 0, v)$ in (25), we obtain

$$f(-v, -v, -v) = -f(v, v, v) \tag{31}$$

for all $v \in U$. Replacing (x, y, z, w, u, v) by (y, x, w, z, v, u) in (25) and using (31), we arrive

$$\begin{aligned} &f(x + ky, z + kw, u + kv) + f(x - ky, z - kw, u - kv) \\ &\qquad = k^2[f(x + y, z + w, u + v) + f(x - y, z - w, u - v)] - 2\left(k^2 - 1\right) f(x, z, u) \end{aligned} \tag{32}$$

for all $x, y, z, w, u, v \in U$. Letting (x, y, z, w, u, v) by (u, u, u, u, u, u) in (32) and using (30), we get

$$f((1 + k)u, (1 + k)u, (1 + k)u) + f((1 - k)u, (1 - k)u, (1 - k)u)k^2 f(2u, 2u, 2u) - 2\left(k^2 - 1\right) f(u, u, u) \tag{33}$$

for all $u \in U$. Replacing u by $2u$ in (33), we obtain

$$f(2(1 + k)u, 2(1 + k)u, 2(1 + k)u) + f(2(1 - k)u, 2(1 - k)u, 2(1 - k)u) = k^2 f(4u, 4u, 4u) - 2\left(k^2 - 1\right) f(2u, 2u, 2u) \tag{34}$$

for all $u \in U$. Substituting (x, y, z, w, u, v) by $(2u, u, 2u, u, 2u, u)$ in (32), we have

$$f((2 + k)u, (2 + k)u, (2 + k)u) + f((2 - k)u, (2 - k)u, (2 - k)u) = k^2[f(3u, 3u, 3u) + f(u, u, u)] - 2\left(k^2 - 1\right) f(2u, 2u, 2u) \tag{35}$$

for all $u \in U$. Again substituting (x, y, z, w, u, v) by $(u, 2u, u, 2u, u, 2u)$ in (32) and using (31), we get

$$\begin{aligned} &f((1 + 2k)u, (1 + 2k)u, (1 + 2k)u) + f((1 - 2k)u, (1 - 2k)u, (1 - 2k)u) \\ &\qquad = k^2[f(3u, 3u, 3u) - f(u, u, u)] - 2\left(k^2 - 1\right) f(u, u, u) \end{aligned} \tag{36}$$

for all $u \in U$. Putting (x, y, z, w, u, v) by $(u, 3u, u, 3u, u, 3u)$ in (32) and using (31), we obtain

$$f((1+3k)u,(1+3k)u,(1+3k)u)+f((1-3k)u,(1-3k)u,(1-3k)u) nonumber \tag{37}$$

$$= k^2[f(4u,4u,4u)-f(2u,2u,2u)]-2\left(k^2-1\right)f(u,u,u) \tag{38}$$

for all $u \in U$. Again putting (x, y, z, w, u, v) by $((1+k)u, u, (1+k)u, u, (1+k)u, u)$ in (32), we have

$$f((1+2k)u,(1+2k)u,(1+2k)u)+f(u,u,u) = k^2[f((2+k)u,(2+k)u,(2+k)u)+f(ku,ku,ku)] - 2\left(k^2-1\right)f((1+k)u,(1+k)u,(1+k)u) \tag{39}$$

for all $u \in U$. Replacing (x, y, z, w, u, v) by $((1-k)u, u, (1-k)u, u, (1-k)u, u)$ in (32) and using (31), we get

$$f(u,u,u)+f((1-2k)u,(1-2k)u,(1-2k)u) = k^2[f((2-k)u,(2-k)u,(2-k)u)-f(ku,ku,ku)] - 2\left(k^2-1\right)f((1-k)u,(1-k)u,(1-k)u) \tag{40}$$

for all $u \in U$. Adding (39) and (40), we arrive

$$\begin{aligned} &f((1+2k)u,(1+2k)u,(1+2k)u)+f((1-2k)u,(1-2k)u,(1-2k)u)+2f(u,u,u)\\ &= k^2[f((2+k)u,(2+k)u,(2+k)u)+f((2-k)u,(2-k)u,(2-k)u)]\\ &\quad - 2\left(k^2-1\right)[f((1+k)u,(1+k)u,(1+k)u)+f((1-k)u,(1-k)u,(1-k)u)] \end{aligned} \tag{41}$$

for all $u \in U$. Replacing (x, y, z, w, u, v) by $((1+2k)u, u, (1+2k)u, u, (1+2k)u, u)$ in (32), we get

$$\begin{aligned} &f((1+3k)u,(1+3k)u,(1+3k)u)+f((1+k)u,(1+k)u,(1+k)u)\\ &= k^2[f(2(1+k)u,2(1+k)u,2(1+k)u)+f(2ku,2ku,2ku)]-2\left(k^2-1\right)f((1+2k)u,(1+2k)u,(1+2k)u) \end{aligned} \tag{42}$$

for all $u \in U$. Again replacing (x, y, z, w, u, v) by $((1-2k)u, u, (1-2k)u, u, (1-2k)u, u)$ in (32) and using (31), we obtain

$$\begin{aligned} &f((1-k)u,(1-k)u,(1-k)u)+f((1-3k)u,(1-3k)u,(1-3k)u)\\ &= k^2[f(2(1-k)u,2(1-k)u,2(1-k)u)-f(2ku,2ku,2ku)]-2\left(k^2-1\right)f((1-2k)u,(1-2k)u,(1-2k)u) \end{aligned} \tag{43}$$

for all $u \in U$. Adding (42) and (43), we arrive

$$\begin{aligned} &f((1+3k)u,(1+3k)u,(1+3k)u)+f((1-3k)u,(1-3k)u,(1-3k)u)\\ &f((1+k)u,(1+k)u,(1+k)u)+f((1-k)u,(1-k)u,(1-k)u)\\ &= k^2[f(2(1+k)u,2(1+k)u,2(1+k)u)+f(2(1-k)u,2(1-k)u,2(1-k)u)]\\ &\quad - 2\left(k^2-1\right)[f((1+2k)u,(1+2k)u,(1+2k)u)+f((1-2k)u,(1-2k)u,(1-2k)u)] \end{aligned} \tag{44}$$

for all $u \in U$. Multiplying (33) by $2(k^2-1)$, (35) by $-k^2$ and adding the resulting value to (36), (41), one can get

$$f(3u,3u,3u) = 4f(2u,2u,2u)-5f(u,u,u) \tag{45}$$

for all $u \in U$. Similarly, multiplying (34) by k^2, (35) by $-2(k^2-1)$, adding to (44) and subtracting (33), (38) from the resulting value, one can get

$$f(4u,4u,4u) = 2f(3u,3u,3u) + 2f(2u,2u,2u) - 6f(u,u,u) \tag{46}$$

for all $u \in U$. Using (45) in (46), we have

$$f(4u,4u,4u) = 10f(2u,2u,2u) - 16f(u,u,u) \tag{47}$$

for all $u \in U$. From (28), we establish

$$g(2u,2u,2u) - 2g(u,u,u) = f(4u,4u,4u) - 10f(2u,2u,2u) + 16f(u,u,u) \tag{48}$$

for all $x \in U$. Using (47) in (48), we desired our result. □

Lemma 2.2. *If $f : U^3 \to V$ be a mapping satisfying (25) and let $h : U^3 \to V$ be a mapping given by*

$$h(u,u,u) = f(2u,2u,2u) - 2f(u,u,u) \tag{49}$$

for all $u \in U$ then

$$h(2u,2u,2u) = 8h(u,u,u) \tag{50}$$

for all $u \in U$ such that h is cubic.

Proof. It follows from (49) that

$$h(2u,2u,2u) - 8h(u,u,u) = f(4u,4u,4u) - 10f(2u,2u,2u) + 16f(u,u,u) \tag{51}$$

for all $x \in U$. Using (47) in (51), we desired our result. □

Remark 2.3. *If $f : U^3 \to V$ be a mapping satisfying (25) and let $g, h : U^3 \to V$ be a mapping defined in (28) and (49) then*

$$f(u,u,u) = \frac{1}{6}(h(u,u,u) - g(u,u,u)) \tag{52}$$

for all $u \in U$.

Lemma 2.4. *If $f : U^3 \to V$ be a mapping satisfying (25) and let $t : U \to V$ be a mapping given by*

$$t(u) = f(u,u,u) \tag{53}$$

for all $u \in U$, then t satisfies

$$t(ku+v) - t(ku-v) = k^2[t(u+v) - t(u-v)] - 2(k^2-1)t(v) \tag{54}$$

for all $u, v \in U$.

Proof. From (25) and (53), we get

$$\begin{aligned} t(ku+v)-t(ku-v) &= f(ku+v,ku+v,ku+v)-f(ku-v,ku-v,ku-v) \\ &= k^2[f(u+v,u+vu+v)-f(u-v,u-v,u-v)]-2(k^2-1)f(v,v,v) \\ &= k^2[t(u+v)-t(u-v)]-2(k^2-1)t(v) \end{aligned}$$

for all $u,v\in U$. □

Hereafter through out this paper, we define a mapping $F:U^3\to V$ by

$$\begin{aligned} F(x,y,z,w,u,v) =& f(kx+y,kz+w,ku+v)-f(kx-y,kz-w,ku-v) \\ & -k^2[f(x+y,z+w,u+v)-f(x-y,z-w,u-v)]+2(k^2-1)f(y,w,v) \end{aligned}$$

for all $x,y,z,w,u,v\in U$.

3. Stability Results: Direct Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (25) using direct method. Through out this section, let U be a normed space and V be a Banach space.

Theorem 3.1. *Let $j=\pm1$. Let $f:U^3\to V$ be a mapping for which there exist a function $\phi:U^6\to[0,\infty)$ with the condition*

$$\lim_{n\to\infty}\frac{1}{2^{nj}}\phi(2^{nj}x,2^{nj}y,2^{nj}z,2^{nj}w,2^{nj}u,2^{nj}v)=0 \tag{55}$$

such that the functional inequality

$$\|F(x,y,z,w,u,v)\|\le\phi(x,y,z,w,u,v) \tag{56}$$

for all $x,y,z,w,u,v\in U$. Then there exists a unique 3-variable additive mapping $A:U^3\to V$ satisfying the functional equation (25) and

$$\|f(2u,2u,2u)-8f(u,u,u)-A(u,u,u)\|\le\frac{1}{2}\sum_{m=\frac{1-j}{2}}^{\infty}\frac{\Phi(2^{mj}u)}{2^{mj}} \tag{57}$$

where $\Phi(2^{mj}u)$ and $A(u,u,u)$ are defined by

$$\begin{aligned} \Phi(2^{mj}u) =& (4k^2-1)\phi(2^{mj}u,2^{mj}u,2^{mj}u,2^{mj}u,2^{mj}u,2^{mj}u)+(-4k^2+2)\phi(2^{(m+1)j}u,2^{mj}u,2^{(m+1)j}u,2^{mj}u,2^{(m+1)j}u,2^{mj}u) \\ & +2\phi(2^{mj}u,2^{(m+1)j}u,2^{mj}u,2^{(m+1)j}u,2^{mj}u,2^{(m+1)j}u) \\ & +k^2\phi(2^{(m+1)j}u,2^{(m+1)j}u,2^{(m+1)j}u,2^{(m+1)j}u,2^{(m+1)j}u,2^{(m+1)j}u)+\phi(2^{mj}u,2^{mj}3u,2^{mj}u,2^{mj}3u,2^{mj}u,2^{mj}3u) \\ & +2\phi(2^{mj}(1+k)u,2^{mj}u,2^{mj}(1+k)u,2^{mj}u,2^{mj}(1+k)u,2^{mj}u) \\ & +2\phi(2^{mj}(1-k)u,2^{mj}u,2^{mj}(1-k)u,2^{mj}u,2^{mj}(1-k)u,2^{mj}u) \\ & +\phi(2^{mj}(1+2k)u,2^{mj}u,2^{mj}(1+2k)u,2^{mj}u,2^{mj}(1+2k)u,2^{mj}u) \\ & +\phi(2^{kj}(1-2k)u,2^{kj}u,2^{kj}(1-2k)u,2^{kj}u,2^{kj}(1-2k)u,2^{kj}u) \end{aligned} \tag{58}$$

$$A(u,u,u)=\lim_{n\to\infty}\frac{1}{2^{nj}}(f(2^{(n+1)j}u,2^{(n+1)j}u,2^{(n+1)j}u)-8f(2^{nj}u,2^{nj}u,2^{nj}u)) \tag{59}$$

for all $u\in U$.

Proof. Assume $j=1$. Replacing (x,y,z,w,u,v) by (y,x,w,z,v,u) in (56) and using (31), we arrive

$$\begin{aligned}&\big\|f(x+ky,z+kw,u+kv)+f(x-ky,z-kw,u-kv)-k^2f(x+y,z+w,u+v)\\&\qquad -k^2f(x-y,z-w,u-v)+2\left(k^2-1\right)f(x,z,u)\big\|\le\phi(y,x,w,z,v,u)\end{aligned}\tag{60}$$

for all $x,y,z,w,u,v\in U$. Letting (x,y,z,w,u,v) by (u,u,u,u,u,u) in (60) and using (30), we get

$$\big\|f((1+k)u,(1+k)u,(1+k)u)+f((1-k)u,(1-k)u,(1-k)u)-k^2f(2u,2u,2u)+2\left(k^2-1\right)f(u,u,u)\big\|\le\phi(u,u,u,u,u,u)\tag{61}$$

for all $u\in U$. Replacing u by $2u$ in (61), we obtain

$$\begin{aligned}&\|f(2(1+k)u,2(1+k)u,2(1+k)u)+f(2(1-k)u,2(1-k)u,2(1-k)u)\\&\qquad -k^2f(4u,4u,4u)+2\left(k^2-1\right)f(2u,2u,2u)\|\le\phi(2u,2u,2u,2u,2u,2u)\end{aligned}\tag{62}$$

for all $u\in U$. Substituting (x,y,z,w,u,v) by $(2u,u,2u,u,2u,u)$ in (60), we have

$$\begin{aligned}&\|f((2+k)u,(2+k)u,(2+k)u)+f((2-k)u,(2-k)u,(2-k)u)\\&\qquad -k^2[f(3u,3u,3u)+f(u,u,u)]+2\left(k^2-1\right)f(2u,2u,2u)\|\le\phi(2u,u,2u,u,2u,u)\end{aligned}\tag{63}$$

for all $u\in U$. Again substituting (x,y,z,w,u,v) by $(u,2u,u,2u,u,2u)$ in (60) and using (31), we get

$$\begin{aligned}&\|f((1+2k)u,(1+2k)u,(1+2k)u)+f((1-2k)u,(1-2k)u,(1-2k)u)\\&\qquad -k^2[f(3u,3u,3u)-f(u,u,u)]+2\left(k^2-1\right)f(u,u,u)\|\le\phi(u,2u,u,2u,u,2u)\end{aligned}\tag{64}$$

for all $u\in U$. Putting (x,y,z,w,u,v) by $(u,3u,u,3u,u,3u)$ in (60) and using (31), we obtain

$$\begin{aligned}&\|f((1+3k)u,(1+3k)u,(1+3k)u)+f((1-3k)u,(1-3k)u,(1-3k)u)\\&\qquad -k^2[f(4u,4u,4u)-f(2u,2u,2u)]-2\left(k^2-1\right)f(u,u,u)\|\le\phi(u,3u,u,3u,u,3u)\end{aligned}\tag{65}$$

for all $u\in U$. Again putting (x,y,z,w,u,v) by $((1+k)u,u,(1+k)u,u,(1+k)u,u)$ in (60), we have

$$\begin{aligned}&\big\|((1+2k)u,(1+2k)u,(1+2k)u)+f(u,u,u)-k^2[f((2+k)u,(2+k)u,(2+k)u)+f(ku,ku,ku)]\\&\qquad -2\left(k^2-1\right)f((1+k)u,(1+k)u,(1+k)u)\big\|\le\phi((1+k)u,u,(1+k)u,u,(1+k)u,u)\end{aligned}\tag{66}$$

for all $u\in U$. Replacing (x,y,z,w,u,v) by $((1-k)u,u,(1-k)u,u,(1-k)u,u)$ in (60) and using (31), we get

$$\begin{aligned}&\big\|f(u,u,u)+f((1-2k)u,(1-2k)u,(1-2k)u)-k^2[f((2-k)u,(2-k)u,(2-k)u)-f(ku,ku,ku)]\\&\qquad -2\left(k^2-1\right)f((1-k)u,(1-k)u,(1-k)u)\big\|\le\phi((1-k)u,u,(1-k)u,u,(1-k)u,u)\end{aligned}\tag{67}$$

for all $u\in U$. It follows from (66) and (67), we arrive

$$\begin{aligned}&\|f((1+2k)u,(1+2k)u,(1+2k)u)+f((1-2k)u,(1-2k)u,(1-2k)u)+2f(u,u,u)\\&\quad -k^2[f((2+k)u,(2+k)u,(2+k)u)+f((2-k)u,(2-k)u,(2-k)u)]\\&\qquad +2\left(k^2-1\right)[f((1+k)u,(1+k)u,(1+k)u)+f((1-k)u,(1-k)u,(1-k)u)]\|\\&\qquad\quad \le\phi((1+k)u,u,(1+k)u,u,(1+k)u,u)+\phi((1-k)u,u,(1-k)u,u,(1-k)u,u)\end{aligned}\tag{68}$$

for all $u \in U$. Replacing (x, y, z, w, u, v) by $((1+2k)u, u, (1+2k)u, u, (1+2k)u, u)$ in (60), we get

$$\begin{aligned}
&\|f((1+3k)u, (1+3k)u, (1+3k)u) + f((1+k)u, (1+k)u, (1+k)u) \\
&\quad - k^2[f(2(1+k)u, 2(1+k)u, 2(1+k)u) + f(2ku, 2ku, 2ku)] \\
&\qquad +2\left(k^2-1\right) f((1+2k)u, (1+2k)u, (1+2k)u)\| \leq \phi((1+2k)u, u, (1+2k)u, u, (1+2k)u, u)
\end{aligned} \tag{69}$$

for all $u \in U$. Again replacing (x, y, z, w, u, v) by $((1-2k)u, u, (1-2k)u, u, (1-2k)u, u)$ in (60) and using (57), we obtain

$$\begin{aligned}
&\|f((1-k)u, (1-k)u, (1-k)u) + f((1-3k)u, (1-3k)u, (1-3k)u) \\
&\quad - k^2[f(2(1-k)u, 2(1-k)u, 2(1-k)u) - f(2ku, 2ku, 2ku)] \\
&\qquad +2\left(k^2-1\right) f((1-2k)u, (1-2k)u, (1-2k)u)\| \leq \phi((1-2k)u, u, (1-2k)u, u, (1-2k)u, u)
\end{aligned} \tag{70}$$

for all $u \in U$. It follows from (69) and (70), we arrive

$$\begin{aligned}
&\|f((1+3k)u, (1+3k)u, (1+3k)u) + f((1-3k)u, (1-3k)u, (1-3k)u) \\
&\quad + f((1+k)u, (1+k)u, (1+k)u) + f((1-k)u, (1-k)u, (1-k)u) \\
&\qquad - k^2[f(2(1+k)u, 2(1+k)u, 2(1+k)u) + f(2(1-k)u, 2(1-k)u, 2(1-k)u)] \\
&\qquad\quad +2\left(k^2-1\right)[f((1+2k)u, (1+2k)u, (1+2k)u) + f((1-2k)u, (1-2k)u, (1-2k)u)]\| \\
&\qquad\qquad \leq \phi((1+2k)u, u, (1+2k)u, u, (1+2k)u, u) + \phi((1-2k)u, u, (1-2k)u, u, (1-2k)u, u)
\end{aligned} \tag{71}$$

for all $u \in U$. Multiplying (61) by $2(k^2-1)$, (63) by $-k^2$ and adding the resulting value to (64), (68), one can get

$$\begin{aligned}
&\|f(3u, 3u, 3u) - 4f(2u, 2u, 2u) + 5f(u, u, u)\| \\
&\leq 2(k^2-1)\left\|f((1+k)u, (1+k)u, (1+k)u) + f((1-k)u, (1-k)u, (1-k)u) - k^2 f(2u, 2u, 2u) + 2\left(k^2-1\right) f(u, u, u)\right\| \\
&- k^2\left\|f((2+k)u, (2+k)u, (2+k)u) + f((2-k)u, (2-k)u, (2-k)u) - k^2[f(3u, 3u, 3u) + f(u, u, u)] + 2\left(k^2-1\right) f(2u, 2u, 2u)\right\| \\
&+ \left\|f((1+2k)u, (1+2k)u, (1+2k)u) + f((1-2k)u, (1-2k)u, (1-2k)u) - k^2[f(3u, 3u, 3u) - f(u, u, u)] + 2\left(k^2-1\right) f(u, u, u)\right\| \\
&+ \|f((1+2k)u, (1+2k)u, (1+2k)u) + f((1-2k)u, (1-2k)u, (1-2k)u) + 2f(u, u, u) \\
&\qquad - k^2[f((2+k)u, (2+k)u, (2+k)u) + f((2-k)u, (2-k)u, (2-k)u)] \\
&\qquad\quad +2\left(k^2-1\right)[f((1+k)u, (1+k)u, (1+k)u) + f((1-k)u, (1-k)u, (1-k)u)]\| \\
&\leq 2(k^2-1)\phi(u, u, u, u, u, u) - k^2\phi(2u, u, 2u, u, 2u, u) + \phi(u, 2u, u, 2u, u, 2u) \\
&\qquad + \phi((1+k)u, u, (1+k)u, u, (1+k)u, u) + \phi((1-k)u, u, (1-k)u, u, (1-k)u, u)
\end{aligned} \tag{72}$$

for all $u \in U$. Similarly, multiplying (62) by k^2, (63) by $-2(k^2-1)$, adding to (71) and subtracting (61), (65) from the resulting value, one can get

$$
\begin{aligned}
&\|f(4u,4u,4u)-2f(3u,3u,3u)-2f(2u,2u,2u)+6f(u,u,u)\| \\
&\le k^2\left\|f(2(1+k)u,2(1+k)u,2(1+k)u)+f(2(1-k)u,2(1-k)u,2(1-k)u)\right. \\
&\qquad\qquad \left.-k^2f(4u,4u,4u)+2\left(k^2-1\right)f(2u,2u,2u)\right\| \\
&-2(k^2-1)\left\|f((2+k)u,(2+k)u,(2+k)u)+f((2-k)u,(2-k)u,(2-k)u)\right. \\
&\qquad\qquad \left.-k^2[f(3u,3u,3u)+f(u,u,u)]+2\left(k^2-1\right)f(2u,2u,2u)\right\| \\
&+\left\|f((1+3k)u,(1+3k)u,(1+3k)u)+f((1-3k)u,(1-3k)u,(1-3k)u)\right. \\
&\quad +-f((1+k)u,(1+k)u,(1+k)u)+f((1-k)u,(1-k)u,(1-k)u) \\
&\quad -k^2[f(2(1+k)u,2(1+k)u,2(1+k)u)+f(2(1-k)u,2(1-k)u,2(1-k)u)] \\
&\quad \left.+2\left(k^2-1\right)[f((1+2k)u,(1+2k)u,(1+2k)u)+f((1-2k)u,(1-2k)u,(1-2k)u)]\right\| \\
&-\left\|f((1+k)u,(1+k)u,(1+k)u)+f((1-k)u,(1-k)u,(1-k)u)\right. \\
&\qquad\qquad\qquad \left.-k^2f(2u,2u,2u)+2\left(k^2-1\right)f(u,u,u)\right\| \\
&-\left\|f((1+3k)u,(1+3k)u,(1+3k)u)+f((1-3k)u,(1-3k)u,(1-3k)u)\right. \\
&\qquad\qquad\qquad \left.-k^2[f(4u,4u,4u)-f(2u,2u,2u)]-2\left(k^2-1\right)f(u,u,u)\right\| \\
&\le k^2\phi(2u,2u,2u,2u,2u,2u)-2(k^2-1)\phi(2u,u,2u,u,2u,u) \\
&\quad +\phi((1+2k)u,u,(1+2k)u,u,(1+2k)u,u)+\phi((1-2k)u,u,(1-2k)u,u,(1-2k)u,u) \\
&\qquad +\phi(u,u,u,u,u,u)+\phi(u,3u,u,3u,u,3u)
\end{aligned}
\tag{73}
$$

for all $u \in U$. Now, from (72) and (73), we have

$$
\begin{aligned}
&\|f(4u,4u,4u)-10f(2u,2u,2u)+16f(u,u,u)\| \\
&\le \|f(4u,4u,4u)-2f(3u,3u,3u)-2f(2u,2u,2u)+6f(u,u,u)\|+2\|f(3u,3u,3u)-4f(2u,2u,2u)+5f(u,u,u)\| \\
&\le (4k^2-1)\phi(u,u,u,u,u,u)+(-4k^2+2)\phi(2u,u,2u,u,2u,u) \\
&\qquad +2\phi(u,2u,u,2u,u,2u)+k^2\phi(2u,2u,2u,2u,2u,2u)+\phi(u,3u,u,3u,u,3u) \\
&\qquad\quad +2\phi((1+k)u,u,(1+k)u,u,(1+k)u,u)+2\phi((1-k)u,u,(1-k)u,u,(1-k)u,u) \\
&\qquad\qquad +\phi((1+2k)u,u,(1+2k)u,u,(1+2k)u,u)+\phi((1-2k)u,u,(1-2k)u,u,(1-2k)u,u)
\end{aligned}
\tag{74}
$$

for all $u \in U$. It follows from (74) that

$$
\|f(4u,4u,4u)-10f(2u,2u,2u)+16f(u,u,u)\| \le \Phi(u) \tag{75}
$$

where

$$
\begin{aligned}
\Phi(u) &= (4k^2-1)\phi(u,u,u,u,u,u)+(-4k^2+2)\phi(2u,u,2u,u,2u,u) \\
&\quad +2\phi(u,2u,u,2u,u,2u)+k^2\phi(2u,2u,2u,2u,2u,2u)+\phi(u,3u,u,3u,u,3u) \\
&\qquad +2\phi((1+k)u,u,(1+k)u,u,(1+k)u,u)+2\phi((1-k)u,u,(1-k)u,u,(1-k)u,u) \\
&\qquad\quad +\phi((1+2k)u,u,(1+2k)u,u,(1+2k)u,u)+\phi((1-2k)u,u,(1-2k)u,u,(1-2k)u,u)
\end{aligned}
\tag{76}
$$

for all $u \in U$. It is easy to see from (75) that

$$\|f(4u,4u,4u) - 8f(2u,2u,2u) - 2(f(2u,2u,2u) - 8f(u,u,u))\| \leq \Phi(u) \tag{77}$$

for all $u \in U$. Using (28) in (77), we obtain

$$\|g(2u,2u,2u) - 2g(u,u,u)\| \leq \Phi(u) \tag{78}$$

for all $u \in U$. From (78), we arrive

$$\left\| \frac{g(2u,2u,2u)}{2} - g(u,u,u) \right\| \leq \frac{\Phi(u)}{2} \tag{79}$$

for all $u \in U$. Now replacing u by $2u$ and dividing by 2 in (79), we get

$$\left\| \frac{g(2^2u,2^2u,2^2u)}{2^2} - \frac{g(2u,2u,2u)}{2} \right\| \leq \frac{\Phi(2u)}{2^2} \tag{80}$$

for all $x \in U$. From (79) and (80), we obtain

$$\begin{aligned}\left\| \frac{g(2^2u,2^2u,2^2u)}{2^2} - g(u,u,u) \right\| &\leq \left\| \frac{g(2u,2u,2u)}{2} - g(u,u,u) \right\| + \left\| \frac{g(2^2u,2^2u,2^2u)}{2^2} - \frac{g(2u,2u,2u)}{2} \right\| \\ &\leq \frac{1}{2}\left[\Phi(u) + \frac{\Phi(2u)}{2}\right] \end{aligned} \tag{81}$$

for all $x \in U$. Proceeding further and using induction on a positive integer n , we get

$$\begin{aligned}\left\| \frac{g(2^nu,2^nu,2^nu)}{2^n} - g(u,u,u) \right\| &\leq \frac{1}{2}\sum_{m=0}^{n-1} \frac{\Phi(2^mu)}{2^m} \\ &\leq \frac{1}{2}\sum_{m=0}^{\infty} \frac{\Phi(2^mu)}{2^m}\end{aligned} \tag{82}$$

for all $u \in U$. In order to prove the convergence of the sequence

$$\left\{ \frac{g(2^nu,2^nu,2^nu)}{2^n} \right\},$$

replacing u by $2^l u$ and dividing by 2^l in (82), for any $l, n > 0$, we deduce

$$\begin{aligned}\left\| \frac{g(2^{n+l}u,2^{n+l}u,2^{n+l}u)}{2^{(n+l)}} - \frac{g(2^lu,2^lu,2^lu)}{2^l} \right\| &= \frac{1}{2^l} \left\| \frac{g(2^n\cdot 2^lu,2^n\cdot 2^lu,2^n\cdot 2^lu)}{2^n} - g(2^lu,2^lu,2^lu) \right\| \\ &\leq \frac{1}{2}\sum_{m=0}^{n-1} \frac{\Phi(2^{m+l}u)}{2^{m+l}} \\ &\leq \frac{1}{2}\sum_{m=0}^{\infty} \frac{\Phi(2^{m+l}u)}{2^{m+l}} \\ &\to 0 \quad as \ \ l \to \infty\end{aligned}$$

for all $u \in U$. This shows that the sequence $\left\{ \frac{g(2^nu,2^nu,2^nu)}{2^n} \right\}$ is a Cauchy sequence. Since V is complete, there exists a mapping $A(u,u,u) : U^3 \to V$ such that

$$A(u,u,u) = \lim_{n\to\infty} \frac{g(2^nu,2^nu,2^nu)}{2^n}, \quad \forall \ u \in U.$$

Letting $n \to \infty$ in (82) and using (28), we see that (57) holds for all $u \in U$.

To show that A satisfies (25), replacing (x, y, z, w, u, v) by $(2^n x, 2^n y, 2^n z, 2^n w, 2^n u, 2^n v)$ and dividing by 2^n in (56), we obtain

$$\frac{1}{2^n}\left\|F(2^n x, 2^n y, 2^n z, 2^n w, 2^n u, 2^n v)\right\| \leq \frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z, 2^n w, 2^n u, 2^n v)$$

for all $x, y, z, w, u, v \in U$. Letting $n \to \infty$ in the above inequality and using the definition of $A(u, u, u)$, we see that

$$A(kx+y, kz+w, ku+v) - A(kx-y, kz-w, ku-v) = k^2[A(x+y, z+w, u+v) - A(x-y, z-w, u-v)] - 2\left(k^2-1\right)A(y, w, v).$$

Hence A satisfies (25) for all $x, y, z, w, u, v \in U$.

To prove $A(u, u, u)$ is unique 3-variable additive function satisfying (25), we let $B(u, u, u)$ be another 3-variable additive mapping satisfying (25) and (57), then

$$\begin{aligned}
\|A(u,u,u) - B(u,u,u)\| &= \frac{1}{2^n}\|A(2^n u, 2^n u, 2^n u) - B(2^n u, 2^n u, 2^n u)\| \\
&\leq \frac{1}{2^n}\left\{\left\|A(2^n u, 2^n u, 2^n u) - f(2^{n+1}u, 2^{n+1}u, 2^{n+1}u) + 8f(2^n u, 2^n u, 2^n u)\right\|\right. \\
&\qquad \left. + \left\|f(2^{n+1}u, 2^{n+1}u, 2^{n+1}u) - 8f(2^n u, 2^n u, 2^n u) - B(2^n u, 2^n u, 2^n u)\right\|\right\} \\
&\leq \sum_{m=0}^{\infty} \frac{\Phi(2^{m+n}u)}{2^{(m+n)}} \\
&\to 0 \quad as \ n \to \infty
\end{aligned}$$

for all $u \in U$. Hence A is unique. For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (25).

Corollary 3.2. *Let $F : U^3 \to V$ be a mapping and there exists real numbers λ and s such that*

$$\|F(x,y,z,w,u,v)\| \leq \begin{cases} \rho, \\ \rho\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s + ||u||^s + ||v||^s\right\}, \\ \rho\ ||x||^s||y||^s||z||^s||w||^s||u||^s||v||^s, \\ \rho\left\{||x||^s||y||^s||z||^s||w||^s + ||u||^s||v||^s \right. \\ \qquad \left. + \left\{||x||^{6s} + ||y||^{6s} + ||z||^{6s} + ||w||^{6s} + ||u||^{6s} + ||v||^{6s}\right\}\right\}, \end{cases} \tag{83}$$

for all $x, y, z, w, u, v \in U$, then there exists a unique 3- variable additive function $A : U^3 \to V$ such that

$$\|f(2u,2u,2u) - 8f(u,u,u) - A(u,u,u)\| \leq \begin{cases} \rho_1, \\ \dfrac{\rho_2||u||^s}{|2-2^s|}, & s \neq 1; \\ \dfrac{\rho_3||u||^{6s}}{|2-2^{6s}|}, & 6s \neq 1; \\ \dfrac{\rho_4||u||^{6s}}{|2-2^{6s}|}, & 6s \neq 1; \end{cases} \tag{84}$$

where

$$\begin{aligned}
\rho_1 &= \rho(k^2+10) \\
\rho_2 &= \rho\left[12k^2 - 3k^2\cdot 2^{s+1} + 3\left(3^s + 2(1+k)^s + 2(1-k)^s + (1+2k)^s + (1-2k)^s\right) + 27\right] \\
\rho_3 &= \rho\left[4k^2(1-2^{3s}) + 2^{3s+2} + k^2\cdot 2^{6s} + 3^{3s} + 2(1+k)^{3s} + 2(1-k)^{3s} + (1+2k)^{3s} + (1-2k)^{3s} - 1\right] \\
\rho_4 &= \rho_3 + \rho\left[12k^2 - 3k^2\cdot 2^{6s+1} + 3\left(3^{6s} + 2(1+k)^{6s} + 2(1-k)^{6s} + (1+2k)^{6s} + (1-2k)^{6s}\right) + 27\right]
\end{aligned} \tag{85}$$

for all $u \in U$.

Now, we will provide an example to illustrate that the functional equation (25) is not stable for $s = 1$ in condition (ii) of Corollary 3.2.

Example 3.3. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a function defined by*

$$\phi(x) = \begin{cases} \rho u, & \text{if } |u| < 1 \\ \rho, & \text{otherwise} \end{cases}$$

where $\rho > 0$ *is a constant, and define a function* $f : \mathbb{R}^3 \to \mathbb{R}$ *by*

$$f(u, u, u) = \sum_{n=0}^{\infty} \frac{\phi(2^n u)}{2^n}, \qquad \text{for all} \quad u \in \mathbb{R}.$$

Then F *satisfies the functional inequality*

$$|F(x, y, z, w, u, v)| \leq 8k^2 \ \rho\left(|x| + |y| + |z| + |w| + |u| + |v|\right) \tag{86}$$

for all $x, y, z, w, u, v \in \mathbb{R}$*. Then there do not exist a additive mapping* $A : \mathbb{R}^3 \to \mathbb{R}$ *and a constant* $\tau > 0$ *such that*

$$|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)| \leq \tau|u|, \qquad \text{for all} \quad u \in \mathbb{R}. \tag{87}$$

Proof. Now

$$|f(u, u, u)| \leq \sum_{n=0}^{\infty} \frac{|\phi(2^n u)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\rho}{2^n} = 2\ \rho.$$

Therefore, we see that f is bounded. We are going to prove that f satisfies (86).

If $x = y = z = w = u = v = 0$ then (86) is trivial. If $|x| + |y| + |z| + |w| + |u| + |v| \geq \frac{1}{2}$ then the left hand side of (86) is less than $8k^2\rho$. Now suppose that $0 < |x| + |y| + |z| + |w| + |u| + |v| < \frac{1}{2}$. Then there exists a positive integer m such that

$$\frac{1}{2^m} \leq |x| + |y| + |z| + |w| + |u| + |v| < \frac{1}{2^{m-1}}, \tag{88}$$

so that $2^{m-1}x < \frac{1}{2}$, $2^{m-1}y < \frac{1}{2}$, $2^{m-1}z < \frac{1}{2}$, $2^{m-1}w < \frac{1}{2}$, $2^{m-1}u < \frac{1}{2}$, $2^{m-1}v < \frac{1}{2}$ and consequently

$$\begin{aligned} & 2^{m-1}(y, w, v), 2^{m-1}(x + y, z + w, u + v), 2^{m-1}(x - y, z - w, u - v), \\ & \qquad 2^{m-1}(kx + y, kz + w, ku + v), 2^{m-1}(kx - y, kz - w, ku - v), \in (-1, 1). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, m - 1$, we have

$$\begin{aligned} & 2^n(y, w, v), 2^n(x + y, z + w, u + v), 2^n(x - y, z - w, u - v), \\ & \qquad 2^n(kx + y, kz + w, ku + v), 2^n(kx - y, kz - w, ku - v), \in (-1, 1). \end{aligned}$$

and

$$\begin{aligned} & \phi(2^n(kx + y, kz + w, ku + v)) + \phi(2^n(kx - y, kz - w, ku - v) - k^2\phi(2^n(x + y, z + w, u + v)) \\ & \quad + k^2\phi(2^n(x - y, z - w, u - v)) + 2\left(k^2 - 1\right)\phi(2^n(y, w, v)) = 0 \end{aligned}$$

for $n = 0, 1, \dots, m-1$. From the definition of f and (88), we obtain that

$$\begin{aligned}
&\Big| f(kx+y, kz+w, ku+v) - f(kx-y, kz-w, ku-v) - k^2 f(x+y, z+w, u+v) \\
&\qquad + k^2 f(x-y, z-w, u-v) + 2(k^2-1) f(y, w, v) \Big| \\
&\le \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \phi(2^n(kx+y, kz+w, ku+v)) + \phi(2^n(kx-y, kz-w, ku-v) \\
&\qquad - k^2 \phi(2^n(x+y, z+w, u+v)) + k^2 \phi(2^n(x-y, z-w, u-v)) + 2\left(k^2-1\right) \phi(2^n(y, w, v)) \Big| \\
&\le \sum_{n=m}^{\infty} \frac{1}{2^n} \Big| \phi(2^n(kx+y, kz+w, ku+v)) + \phi(2^n(kx-y, kz-w, ku-v) \\
&\qquad - k^2 \phi(2^n(x+y, z+w, u+v)) + k^2 \phi(2^n(x-y, z-w, u-v)) + 2\left(k^2-1\right) \phi(2^n(y, w, v)) \Big| \\
&\le \sum_{n=m}^{\infty} \frac{1}{2^n} 4k^2 \rho = 4k^2 \ \rho \times \frac{2}{2^m} = 8k^2 \ \rho \left(|x|+|y|+|z|+|w|+|u|+|v|\right).
\end{aligned}$$

Thus f satisfies (86) for all $x, y, z, w, u, v \in \mathbb{R}$ with $0 < |x|+|y|+|z|+|w|+|u|+|v| < \frac{1}{2}$.
We claim that the additive functional equation (25) is not stable for $s = 1$ in condition (ii) of Corollary 3.2. Suppose on the contrary that there exist a additive mapping $A : \mathbb{R}^3 \to \mathbb{R}$ and a constant $\tau > 0$ satisfying (87). Since f is bounded and continuous for all $u \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(u, u, u) = cu$ for any u in $\mathbb{R}$. Thus, we obtain that

$$|f(2u, 2u, 2u) - 8f(u, u, u)| \le (\tau + |c|)\, |u|. \tag{89}$$

But we can choose a positive integer ℓ with $\ell\rho > \tau + |c|$.
If $u \in \left(0, \frac{1}{2^{\ell-1}}\right)$, then $2^n u \in (0, 1)$ for all $n = 0, 1, \dots, \ell-1$. For this u, we get

$$f(2u, 2u, 2u) - 8f(u, u, u) = \sum_{n=0}^{\infty} \frac{\phi(2^n u)}{2^n} \ge \sum_{n=0}^{\ell-1} \frac{\rho(2^n u)}{2^n} = \ell\rho u > (\tau + |c|)\, u$$

which contradicts (89). Therefore the additive functional equation (25) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality condition (ii) of (84). □

A counter example to illustrate the non stability in condition (iii) of Corollary 3.2 is given in the following example.

Example 3.4. *Let s be such that $0 < s < \frac{1}{6}$. Then there is a function $F : \mathbb{R}^3 \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying*

$$|F(x, y, z, w, u, v)| \le \lambda |x|^{\frac{s}{6}} \ |y|^{\frac{s}{6}} \ |z|^{\frac{s}{6}} \ |w|^{\frac{s}{6}} \ |u|^{\frac{s}{6}} \ |v|^{\frac{1-5s}{6}} \tag{90}$$

for all $x, y, z, w, u, v \in \mathbb{R}$ and

$$\sup_{u \ne 0} \frac{|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)|}{|u|} = +\infty \tag{91}$$

for every additive mapping $A(u, u, u) : \mathbb{R}^3 \to \mathbb{R}$.

Proof. If we take

$$f(u, u, u) = \begin{cases} (u, u, u) \ln |u, u, u|, & if \quad u \ne 0, \\ 0, & if \quad u = 0. \end{cases}$$

Then from the relation (91), it follows that

$$\begin{aligned}\sup_{u\neq 0}\frac{|f(2u,2u,2u)-8f(u,u,u)-A(u,u,u)|}{|u|} &\geq \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}\frac{|f(2n,2n,2n)-8f(n,n,n)-A(n,n,n)|}{|n|}\\ &= \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}\frac{|n(2,2,2)\ln|2n,2n,2n|-8n(1,1,1)\ln|n,n,n|-n\ A(1,1,1)|}{|n|}\\ &= \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}}|(2,2,2)\ln|2n,2n,2n|-8(1,1,1)\ln|n,n,n|-A(1,1,1)|=\infty.\end{aligned}$$

We have to prove (90) is true.

Case (i): If $x,y,z,w,u,v>0$ in (90) then,

$$\begin{aligned}&\Big|f(kx+y,kz+w,ku+v)-f(kx-y,kz-w,ku-v)\\ &\qquad -k^2[f(x+y,z+w,u+v)-f(x-y,z-w,u-v)]+2\left(k^2-1\right)f(y,w,v)\Big|\\ &\quad=\Big|(kx+y,kz+w,ku+v)\ln|kx+y,kz+w,ku+v|-(kx-y,kz-w,ku-v)\ln|kx-y,kz-w,ku-v|\\ &\qquad\quad -k^2(x+y,z+w,u+v)\ln|x+y,z+w,u+v|+k^2(x-y,z-w,u-v)\ln|x-y,z-w,u-v|\\ &\qquad\qquad +2(k^2-1)(y,w,v)\ln|y,w,v|\Big|.\end{aligned}$$

Set $x=t_1, y=t_2, z=t_3, w=t_4, u=t_5, v=t_6$ it follows that

$$\begin{aligned}&\Big|f(kx+y,kz+w,ku+v)-f(kx-y,kz-w,ku-v)\\ &\qquad -k^2[f(x+y,z+w,u+v)-f(x-y,z-w,u-v)]+2\left(k^2-1\right)f(y,w,v)\Big|\\ &=\Big|(kt_1+t_2,kt_3+t_4,kt_5+t_6)\ln|kt_1+t_2,kt_3+t_4,kt_5+t_6|-(kt_1-t_2,kt_3-t_4,kt_5-t_6)\ln|kt_1-t_2,kt_3-t_4,kt_5-t_6|\\ &\qquad -k^2(t_1+t_2,t_3+t_4,t_5+t_6)\ln|t_1+t_2,t_3+t_4,t_5+t_6|+k^2(t_1-t_2,t_3-t_4,t_5-t_6)\ln|t_1-t_2,t_3-t_4,t_5-t_6|\\ &\qquad\qquad +2(k^2-1)(t_2,t_4,t_6)\ln|t_2,t_4,t_6|\Big|\\ &=\Big|f(kt_1+t_2,kt_3+t_4,kt_5+t_6)-f(kt_1-t_2,kt_3-t_4,kt_5-t_6)\\ &\qquad -k^2[f(t_1+t_2,t_3+t_4,t_5+t_6)-f(t_1-t_2,t_3-t_4,t_5-t_6)]+2\left(k^2-1\right)f(t_2,t_4,t_6)\Big|\\ &\leq\lambda|t_1|^{\frac{s}{6}}\ |t_2|^{\frac{s}{6}}\ |t_3|^{\frac{s}{6}}\ |t_4|^{\frac{s}{6}}\ |t_5|^{\frac{s}{6}}\ |t_6|^{\frac{1-5s}{6}}\\ &=\lambda|x|^{\frac{s}{6}}\ |y|^{\frac{s}{6}}\ |z|^{\frac{s}{6}}\ |w|^{\frac{s}{6}}\ |u|^{\frac{s}{6}}\ |v|^{\frac{1-5s}{6}}.\end{aligned}$$

For cases

Case (ii): If $x,y,z,w<0$,

Case (iii): If $x,z,u>0, y,w,v<0$

then $kx+y,kz+w,ku+vx+y,z+w,u+v>0$,

Case (iv): If $x,z,u>0, y,w,v<0$

then $kx+y,kz+w,ku+vx+y,z+w,u+v<0$,

the proof is similar lines to that of Case (i).

Case (v): If $x=y=z=w=u=v=0$ in (90) then it is trivial. □

Now, we will provide an example to illustrate that the functional equation (25) is not stable for $s = \frac{1}{6}$ in condition (iv) of Corollary 3.2.

Example 3.5. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\phi(x) = \begin{cases} \rho u, & \text{if } |u| < \frac{1}{6} \\ \frac{\rho}{6}, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(u,u,u) = \sum_{n=0}^{\infty} \frac{\phi(2^n u)}{2^n}, \qquad for\ all \quad u \in \mathbb{R}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w,u,v)| \leq \frac{8k^2\ \rho}{3} \left(|x|^{\frac{1}{6}}\ |y|^{\frac{1}{6}}\ |z|^{\frac{1}{6}}\ |w|^{\frac{1}{6}} |u|^{\frac{1}{6}}\ |v|^{\frac{1}{6}} + \{|x| + |y| + |w| + |z| + |u| + |v|\} \right) \tag{92}$$

for all $x, y, z, w, u, v \in \mathbb{R}$. Then there do not exist a additive mapping $A : \mathbb{R}^3 \to \mathbb{R}$ and a constant $\tau > 0$ such that

$$|f(2u,2u,2u) - 8f(u,u,u) - A(u,u,u)| \leq \tau |u|, \qquad for\ all \quad u \in \mathbb{R}. \tag{93}$$

Proof. Now

$$|f(u,u,u)| \leq \sum_{n=0}^{\infty} \frac{|\phi(2^n u)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\rho}{2^n} \times \frac{\rho}{6} = \frac{\rho}{3}.$$

Therefore, we see that f is bounded. We are going to prove that f satisfies (92).

If $x = y = z = w = u = v = 0$ then (92) is trivial.

If $|x|^{\frac{1}{6}}\ |y|^{\frac{1}{6}}\ |z|^{\frac{1}{6}}\ |w|^{\frac{1}{6}} |u|^{\frac{1}{6}}\ |v|^{\frac{1}{6}} + \{|x| + |y| + |w| + |z| + |u| + |v|\} \geq \frac{1}{2}$ then the left hand side of (92) is less than $\frac{4k^2\rho}{3}$. Now, suppose that $0 < |x|^{\frac{1}{6}}\ |y|^{\frac{1}{6}}\ |z|^{\frac{1}{6}}\ |w|^{\frac{1}{6}} |u|^{\frac{1}{6}}\ |v|^{\frac{1}{6}} + \{|x| + |y| + |w| + |z| + |u| + |v|\} < \frac{1}{2}$. Then there exists a positive integer m such that

$$\frac{1}{2^m} \leq |x|^{\frac{1}{6}}\ |y|^{\frac{1}{6}}\ |z|^{\frac{1}{6}}\ |w|^{\frac{1}{6}} |u|^{\frac{1}{6}}\ |v|^{\frac{1}{6}} + \{|x| + |y| + |w| + |z| + |u| + |v|\} < \frac{1}{2^{m-1}}, \tag{94}$$

so that $2^{m-1}|x|^{\frac{1}{6}}\ 2^{m-1}|y|^{\frac{1}{6}}\ 2^{m-1}|z|^{\frac{1}{6}}\ 2^{m-1}|w|^{\frac{1}{6}}\ 2^{m-1}|u|^{\frac{1}{6}}\ 2^{m-1}|v|^{\frac{1}{6}} < \frac{1}{2}$,
$2^{m-1}|x| < \frac{1}{2}, 2^{m-1}|y| < \frac{1}{2}, 2^{m-1}|w| < \frac{1}{2}, 2^{m-1}|z| < \frac{1}{2}, 2^{m-1}|u| < \frac{1}{2}, 2^{m-1}|v| < \frac{1}{2}$, and consequently

$$\begin{aligned} &2^{m-1}(y,w,v), 2^{m-1}(x+y, z+w, u+v), 2^{m-1}(x-y, z-w, u-v), \\ &\qquad 2^{m-1}(kx+y, kz+w, ku+v), 2^{m-1}(kx-y, kz-w, ku-v), \in \left(-\frac{1}{4}, \frac{1}{4}\right). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, m-1$, we have

$$\begin{aligned} &2^{n}(y,w,v), 2^{n}(x+y, z+w, u+v), 2^{n}(x-y, z-w, u-v), \\ &\qquad 2^{n}(kx+y, kz+w, ku+v), 2^{n}(kx-y, kz-w, ku-v), \in \left(-\frac{1}{4}, \frac{1}{4}\right) \end{aligned}$$

and

$$\begin{aligned}
&\Big| f(kx+y,kz+w,ku+v) - f(kx-y,kz-w,ku-v) - k^2 f(x+y,z+w,u+v) \\
&\qquad + k^2 f(x-y,z-w,u-v) + 2(k^2-1) f(y,w,v) \Big| \\
&\le \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \phi(2^n(kx+y,kz+w,ku+v)) + \phi(2^n(kx-y,kz-w,ku-v) \\
&\qquad - k^2\phi(2^n(x+y,z+w,u+v)) + k^2\phi(2^n(x-y,z-w,u-v)) + 2\left(k^2-1\right)\phi(2^n(y,w,v)) \Big| \\
&\le \sum_{n=m}^{\infty} \frac{1}{2^n} \Big| \phi(2^n(kx+y,kz+w,ku+v)) + \phi(2^n(kx-y,kz-w,ku-v) \\
&\qquad - k^2\phi(2^n(x+y,z+w,u+v)) + k^2\phi(2^n(x-y,z-w,u-v)) + 2\left(k^2-1\right)\phi(2^n(y,w,v)) \Big| \\
&\le \sum_{n=m}^{\infty} \frac{4k^2\rho}{3} \times \frac{1}{2^n} = \frac{4k^2\rho}{3} \times \frac{2}{2^m} \\
&= \frac{8k^2\,\rho}{3} \left(|x|^{\frac{1}{6}}\ |y|^{\frac{1}{6}}\ |z|^{\frac{1}{6}}\ |w|^{\frac{1}{6}}\ |u|^{\frac{1}{6}}\ |v|^{\frac{1}{6}} + \{|x|+|y|+|w|+|z|+|v|+|v|\} \right).
\end{aligned}$$

Thus f satisfies (92) for all $x,y,z,w,u,v \in \mathbb{R}$ with
$0 < |x|^{\frac{1}{6}}\ |y|^{\frac{1}{6}}\ |z|^{\frac{1}{6}}\ |w|^{\frac{1}{6}}\ |u|^{\frac{1}{6}}\ |v|^{\frac{1}{6}} + \{|x|+|y|+|w|+|z|+|v|+|v|\} < \frac{1}{2}$.
We claim that the additive functional equation (25) is not stable for $s = \frac{1}{6}$ in condition (iv) of Corollary 3.2. Suppose on the contrary that there exist a additive mapping $A : \mathbb{R}^3 \to \mathbb{R}$ and a constant $\tau > 0$ satisfying (93). Since f is bounded and continuous for all $u \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(u,u,u) = cu$ for any u in $\mathbb{R}$. Thus, we obtain that

$$|f(2u,2u,2u) - 8f(u,u,u)| \le (\tau + |c|)\,|u|. \tag{95}$$

But we can choose a positive integer ℓ with $\ell\rho > \tau + |c|$.
If $u \in \left(0, \frac{1}{2^{\ell-1}}\right)$, then $2^n u \in (0,1)$ for all $n = 0,1,\ldots,\ell-1$. For this x, we get

$$f(2u,2u,2u) - 8f(u,u,u) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n} \ge \sum_{n=0}^{\ell-1} \frac{\rho(2^n x)}{2^n} = \ell\rho x > (\tau + |c|)\,u$$

which contradicts (95). Therefore the additive functional equation (25) is not stable in sense of Ulam, Hyers and Rassias if $s = \frac{1}{6}$, assumed in the inequality condition (iv) of (84). □

Theorem 3.6. *Let $j = \pm 1$. Let $F : U^3 \to V$ be a mapping for which there exist a function $\phi : U^6 \to [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{8^{nj}} \phi(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}w, 2^{nj}u, 2^{nj}v) = 0 \tag{96}$$

such that the functional inequality

$$\|F(x,y,z,w,u,v)\| \le \phi(x,y,z,w,u,v) \tag{97}$$

for all $x,y,z,w,u,v \in U$. Then there exists a unique 3-variable cubic mapping $C : U^3 \to V$ satisfying the functional equation (25) and

$$\|f(2u,2u,2u) - 2f(u,u,u) - C(u,u,u)\| \le \frac{1}{8} \sum_{m=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{mj}u)}{8^{mj}} \tag{98}$$

where $\Phi(2^{mj}u)$ is defined in (58) and $C(u,u,u)$ is defined by

$$C(u,u,u) = \lim_{n\to\infty} \frac{1}{8^{nj}} (f(2^{(n+1)j}u, 2^{(n+1)j}u, 2^{(n+1)j}u) - 2f(2^{nj}u, 2^{nj}u, 2^{nj}u)) \tag{99}$$

for all $u \in U$.

Proof. It is easy to see from (75) that

$$\|f(4u, 4u, 4u) - 2f(2u, 2u, 2u) - 8(f(2u, 2u, 2u) - 2f(u, u, u))\| \leq \Phi(u) \tag{100}$$

for all $u \in U$. Using (35) in (100), we obtain

$$\|h(2u, 2u, 2u) - 8h(u, u, u)\| \leq \Phi(u) \tag{101}$$

for all $u \in U$. From (101), we arrive

$$\left\|\frac{h(2u, 2u, 2u)}{8} - h(u, u, u)\right\| \leq \frac{\Phi(u)}{8} \tag{102}$$

for all $u \in U$. The rest of the proof is similar tracing to that of Theorem 3.1 □

The following Corollary is an immediate consequence of Theorem 3.6 concerning the stability of (25).

Corollary 3.7. *Let $F : U^3 \to V$ be a mapping and there exists real numbers ρ and s such that*

$$\|F(x, y, z, w, u, v)\| \leq \begin{cases} \rho, \\ \rho\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s + ||u||^s + ||v||^s\right\}, \\ \rho\ ||x||^s||y||^s||z||^s||w||^s||u||^s||v||^s, \\ \rho\left\{||x||^s||y||^s||z||^s||w||^s + ||u||^s||v||^s\right. \\ \qquad \left. + \left\{||x||^{6s} + ||y||^{6s} + ||z||^{6s} + ||w||^{6s} + ||u||^{6s} + ||v||^{6s}\right\}\right\}, \end{cases} \tag{103}$$

for all $x, y, z, w, u, v \in U$, then there exists a unique 3- variable cubic function $C : U^3 \to V$ such that

$$\|f(2u, 2u, 2u) - 2f(u, u, u) - C(u, u, u)\| \leq \begin{cases} \frac{\rho_1}{7}, \\ \frac{\rho_2||u||^s}{|8 - 2^s|}, & s \neq 3; \\ \frac{\rho_3||u||^{6s}}{|8 - 2^{6s}|}, & 6s \neq 3; \\ \frac{\rho_4||u||^{6s}}{|8 - 2^{6s}|}, & 6s \neq 3; \end{cases} \tag{104}$$

where $\rho_1, \rho_2, \rho_3, \rho_4$ are defined in (85) for all $u \in U$.

Now, we will provide an example to illustrate that the functional equation (25) is not stable for $s = 3$ in condition (ii) of Corollary 3.7.

Example 3.8. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\phi(u) = \begin{cases} \rho u^3, & \text{if } |x| < 1 \\ \rho, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(u, u, u) = \sum_{n=0}^{\infty} \frac{\phi(2^n u)}{8^n}, \qquad \text{for all} \quad u \in \mathbb{R}.$$

Then F satisfies the functional inequality

$$|F(x, y, z, w, u, v)| \leq \frac{4k^2\ \rho \times 8^3}{7}\left(|x|^3 + |y|^3 + |z|^3 + |w|^3 + |u|^3 + |v|^3\right) \tag{105}$$

for all $x, y, z, w, u, v \in \mathbb{R}$. Then there do not exist a cubic mapping $C : \mathbb{R}^3 \to \mathbb{R}$ and a constant $\tau > 0$ such that

$$|f(2u, 2u, 2u) - 2f(u, u, u) - C(u, u, u)| \leq \tau|u|^3, \qquad \text{for all} \quad u \in \mathbb{R}. \tag{106}$$

A counter example to illustrate the non stability in condition (iii) of Corollary 3.7 is given in the following example.

Example 3.9. *Let s be such that $0 < s < \frac{3}{6}$. Then there is a function $F : \mathbb{R}^3 \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying*

$$|F(x,y,z,w,u,v)| \leq \lambda |x|^{\frac{s}{6}} \ |y|^{\frac{s}{6}} \ |z|^{\frac{s}{6}} \ |w|^{\frac{s}{6}} \ |u|^{\frac{s}{6}} \ |v|^{\frac{3-5s}{6}} \tag{107}$$

for all $x,y,z,w,u,v \in \mathbb{R}$ and

$$\sup_{u \neq 0} \frac{|f(2u,2u,2u) - 2f(u,u,u) - C(u,u,u)|}{|u|^3} = +\infty \tag{108}$$

for every cubic mapping $C : \mathbb{R}^3 \to \mathbb{R}$.

Proof. If we take

$$f(x,x) = \begin{cases} (u,u,u)^3 \ln |u,u,u| \,, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

rest of the proof is similar to that of Example 3.4. □

Now, we will provide an example to illustrate that the functional equation (25) is not stable for $s = \frac{3}{6}$ in condition (iv) of Corollary 3.7.

Example 3.10. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\phi(u) = \begin{cases} \rho u^3, & if \ |u| < \dfrac{3}{6} \\ \dfrac{3\rho}{6}, & otherwise \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(u,u,u) = \sum_{n=0}^{\infty} \frac{\phi(2^n u)}{8^n}, \qquad for \ all \quad u \in \mathbb{R}.$$

Then F satisfies the functional inequality

$$|F(x,y,z,w,u,v)| \leq \frac{2k^2\rho \times 8^3}{7} \left(|x|^{\frac{3}{4}} \ |y|^{\frac{3}{4}} |x|^{\frac{3}{4}} \ |y|^{\frac{3}{4}} \ |z|^{\frac{3}{4}} \ |w|^{\frac{3}{4}} \ |u|^{\frac{3}{4}} \ |v|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |w|^3 + |z|^3 + |u|^3 + |v|^3 \right\} \right) \tag{109}$$

for all $x,y,z,w,u,v \in \mathbb{R}$. Then there do not exist a cubic mapping $C : \mathbb{R}^3 \to \mathbb{R}$ and a constant $\tau > 0$ such that

$$|f(2u,2u) - 2f(u,u,u) - C(u,u,u)| \leq \tau |u|, \qquad for \ all \quad u \in \mathbb{R}. \tag{110}$$

Now, we are ready to prove our main direct stability results.

Theorem 3.11. *Let $j = \pm 1$. Let $F : U^3 \to V$ be a mapping for which there exist a function $\phi : U^6 \to [0,\infty)$ with the conditions given in (55) and (96) respectively, such that the functional inequality*

$$\|F(x,y,z,w,u,v)\| \leq \phi(x,y,z,w,u,v) \tag{111}$$

for all $x,y,z,w,u,v \in U$. Then there exists a unique 3-variable additive mapping $A : U^3 \to V$ and a unique 3-variable cubic mapping $C : U^3 \to V$ satisfying the functional equation (25) and

$$\|f(u,u,u) - A(u,u,u) - C(u,u,u)\| \leq \frac{1}{6} \left\{ \frac{1}{2} \sum_{m=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{mj}u)}{2^{mj}} + \frac{1}{8} \sum_{m=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{mj}u)}{8^{mj}} \right\} \tag{112}$$

for all $u \in U$. The mapping $\Phi(2^{mj}u)$, $A(u,u,u)$ and $C(u,u,u)$ are respectively defined in (58), (59) and (99) for all $x \in U$.

Proof. By Theorems 3.1 and 3.6, there exists a unique 3-variable additive function $A_1 : U^3 \to V$ and a unique 3-variable cubic function $C_1 : U^3 \to V$ such that

$$\|f(2u,2u,2u) - 8f(u,u,u) - A_1(u,u,u)\| \leq \frac{1}{2}\sum_{m=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{mj}u)}{2^{mj}} \tag{113}$$

and

$$\|f(2u,2u,2u) - 2f(u,u,u) - C_1(u,u,u)\| \leq \frac{1}{8}\sum_{m=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{mj}u)}{8^{mj}} \tag{114}$$

for all $u \in U$. Now from (113) and (114), one can see that

$$\begin{aligned}
&\left\| f(u,u,u) + \frac{1}{6}A_1(u,u,u) - \frac{1}{6}C_1(u,u,u) \right\| \\
&\quad = \left\| \left\{ -\frac{f(2u,2u,2u)}{6} + \frac{8f(u,u,u)}{6} + \frac{A_1(u,u,u)}{6} \right\} + \left\{ \frac{f(2u,2u,2u)}{6} - \frac{2f(u,u,u)}{6} - \frac{C_1(u,u,u)}{6} \right\} \right\| \\
&\quad \leq \frac{1}{6}\left\{ \|f(2u,2u,2u) - 8f(u,u,u) - A_1(u,u,u)\| + \|f(2u,2u,2u) - 2f(u,u,u) - C_1(u,u,u)\| \right\} \\
&\quad \leq \frac{1}{6}\left\{ \frac{1}{2}\sum_{m=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{mj}u)}{2^{mj}} + \frac{1}{8}\sum_{m=\frac{1-j}{2}}^{\infty} \frac{\Phi(2^{mj}u)}{8^{mj}} \right\}
\end{aligned}$$

for all $u \in U$. Thus we obtain (114) by defining $A(u,u,u) = \frac{-1}{6}A_1(u,u,u)$ and $C(u,u,u) = \frac{1}{6}C_1(u,u,u)$, $\Phi(2^{mj}u)$, $A(u,u,u)$ and $C(u,u,u)$ are respectively defined in (58), (59) and (99) for all $u \in U$. □

The following corollary is the immediate consequence of Theorem 3.11, using Corollaries 3.2 and 3.7 concerning the stability of (25).

Corollary 3.12. *Let $F : U^3 \to V$ be a mapping and there exists real numbers ρ and s such that*

$$\|F(x,y,z,w,u,v))\| \leq \begin{cases} \rho, \\ \rho\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s + ||u||^s + ||v||^s\right\}, \\ \rho\ ||x||^s||y||^s||z||^s||w||^s||u||^s||v||^s, \\ \rho\left\{||x||^s||y||^s||z||^s||w||^s + ||u||^s||v||^s \right. \\ \qquad \left. + \left\{||x||^{6s} + ||y||^{6s} + ||z||^{6s} + ||w||^{6s} + ||u||^{6s} + ||v||^{6s}\right\}\right\}, \end{cases} \tag{115}$$

for all $x, y, z, w, u, v \in U$, then there exists a unique 3-variable additive mapping $A : U^3 \to V$ and a unique 3-variable cubic mapping $C : U^3 \to V$ such that

$$\|f(u,u,u) - A(u,u,u) - C(u,u,u)\| \leq \begin{cases} \frac{\rho_1}{6}\left(1 + \frac{1}{7}\right), \\ \frac{\rho_2}{6}\left(\frac{1}{|2-2^s|} + \frac{1}{|8-2^s|}\right)||u||^s, & s \neq 1,3; \\ \frac{\rho_3}{6}\left(\frac{1}{|2-2^{6s}|} + \frac{1}{|8-2^{6s}|}\right)||u||^{6s}, & 6s \neq 1,3; \\ \frac{\rho_4}{6}\left(\frac{1}{|2-2^{6s}|} + \frac{1}{|8-2^{6s}|}\right)||u||^{6s}, & 6s \neq 1,3; \end{cases} \tag{116}$$

where $\rho_1, \rho_2, \rho_3, \rho_4$ are defined in (85) for all $u \in U$.

4. Stability Results: Fixed Point Method

In this section, we apply a fixed point method for achieving stability of the 3-variable $k-$AC functional equation (25). Now, we present the following theorem due to B. Margolis and J.B. Diaz [24] for fixed point Theory.

Theorem 4.1. *[24] Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of (25).

Through out this section let U be a normed space and V be a Banach space.

Theorem 4.2. *Let $F : U^3 \to V$ be a mapping for which there exist a function $\phi : U^6 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\kappa_i^n} \phi(\kappa_i^n x, \kappa_i^n y, \kappa_i^n z, \kappa_i^n w, \kappa_i^n u, \kappa_i^n v,) = 0 \tag{117}$$

where

$$\kappa_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1, \end{cases}$$

such that the functional inequality

$$\|F(x, y, z, w, u, v)\| \leq \phi(x, y, z, w, u, v) \tag{118}$$

for all $x, y, z, w, u, v \in U$. If there exists $L = L(i) < 1$ such that the function $\Xi : U^6 \to [0, \infty)$ defined by

$$\Xi(u) = \ \Phi\left(\frac{u}{2}\right),$$

has the property

$$\Xi(u) = \frac{L}{\kappa_i}\ \Xi\left(\kappa_i u\right). \tag{119}$$

for all $u \in U$. Then there exists a unique 3-variable additive mapping $A : U^3 \to V$ satisfying the functional equation (25) and

$$\parallel f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u) \parallel \leq \frac{L^{1-i}}{1-L} \Xi(u) \tag{120}$$

for all $u \in U$. The mapping $\Phi(u)$ is defined in (58) for all $u \in U$.

Proof. Consider the set

$$\Omega = \{p/p : U^3 \to V,\ p(0, 0, 0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p_1, p_2) = \inf\{K \in (0, \infty) : \parallel p_1(u, u, u) - p_2(u, u, u) \parallel \leq K\Xi(u), u \in U\}.$$

It is easy to see that (Ω, d) is complete.

Define $T:\Omega^3 \to \Omega$ by

$$Tp(u,u,u) = \frac{1}{\kappa_i} p(\kappa_i u, \kappa_i u, \kappa_i u),$$

for all $u \in U$. Now $p_1, p_2 \in \Omega$,

$$\begin{aligned}
d(p_1,p_2) \leq K \Rightarrow & \parallel p_1(u,u,u) - p_2(u,u,u) \parallel \leq K\Xi(u), u \in U,\\
\Rightarrow & \left\| \frac{1}{\kappa_i} p_1(\kappa_i u, \kappa_i u, \kappa_i u) - \frac{1}{\kappa_i} p_2(\kappa_i u, \kappa_i u, \kappa_i u) \right\| \leq \frac{1}{\kappa_i} K\Xi(\kappa_i u), u \in U,\\
\Rightarrow & \left\| \frac{1}{\kappa_i} p_1(\kappa_i u, \kappa_i x, \kappa_i u) - \frac{1}{\kappa_i} p_2(\kappa_i u, \kappa_i u, \kappa_i u) \right\| \leq LK\Xi(u), u \in U,\\
\Rightarrow & \parallel Tp_1(u,u,u) - Tp_2(u,u,u) \parallel \leq LK\Xi(u), u \in U,\\
\Rightarrow & d(p_1,p_2) \leq LK.
\end{aligned}$$

This implies $d(Tp_1, Tp_2) \leq Ld(p_1, p_2)$, for all $p_1, p_2 \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

From (78), we have

$$\|g(2u,2u,2u) - 2g(u,u,u)\| \leq \Phi(u) \tag{121}$$

for all $u \in U$.

$$\left\| \frac{g(2u,2u,2u)}{2} - g(u,u,u) \right\| \leq \frac{\Phi(u)}{2} \tag{122}$$

for all $u \in U$. Using (119) for the case $i = 0$ it reduces to

$$\left\| \frac{g(2u,2u,2u)}{2} - g(u,u,u) \right\| \leq \ L\Xi(u)$$

for all $u \in U$,

$$\text{i.e.,}\quad d(g,Tg) \leq L \Rightarrow d(g,Tg) \leq L \leq L^1 < \infty.$$

Again replacing $u = \frac{u}{2}$ in (121), we get

$$\left\| g(u,u,u) - 2g\left(\frac{u}{2},\frac{u}{2},\frac{u}{2}\right) \right\| \leq \Phi\left(\frac{u}{2}\right) \tag{123}$$

Using (119) for the case $i = 1$ it reduces to

$$\left\| g(u,u,u) - 2g\left(\frac{u}{2},\frac{u}{2},\frac{u}{2}\right) \right\| \leq \ \Xi(u)$$

for all $u \in U$,

$$\text{i.e.,}\quad d(g,Tg) \leq 1 \Rightarrow d(g,Tg) \leq 1 \leq L^0 < \infty.$$

From the above two cases, we arrive

$$d(g,Tg) \leq L^{1-i}.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(u,u,u) = \lim_{n\to\infty} \frac{1}{\kappa_i^n} (f(\kappa_i^{(n+1)}u, \kappa_i^{(n+1)}u, \kappa_i^{(n+1)}u) - 8f(\kappa_i^n x, \kappa_i^n x, \kappa_i^n u)) \tag{124}$$

for all $u \in U$.

To prove $A : U^3 \to V$ is additive. Replacing (u, y, z, w, u, v) by $(\kappa_i^n x, \kappa_i^n y, \kappa_i^n z, \kappa_i^n w, \kappa_i^n u, \kappa_i^n v)$ in (118) and dividing by κ_i^n, it follows from (117) that

$$\begin{aligned}\|A(u,y,z,w,u,v)\| &= \lim_{n\to\infty} \frac{\|F(\kappa_i^n x, \kappa_i^n y, \kappa_i^n z, \kappa_i^n w, \kappa_i^n u, \kappa_i^n v)\|}{\kappa_i^n} \\ &\leq \lim_{n\to\infty} \frac{\phi(\kappa_i^n x, \kappa_i^n y, \kappa_i^n z, \kappa_i^n w, \kappa_i^n u, \kappa_i^n v)}{\kappa_i^n} = 0\end{aligned}$$

for all $x, y, z, w, u, v \in U$, i.e., A satisfies the functional equation (25).

According to the fixed point alternative, since A is the unique fixed point of T in the set $\Delta = \{A \in \Omega : d(f, A) < \infty\}$, A is the unique function such that

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)\| \leq K\Xi(u)$$

for all $u \in U$ and $K > 0$. Again using the fixed point alternative, we obtain

$$d(f, A) \leq \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f, A) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u) \| \leq \frac{L^{1-i}}{1-L}\Xi(u)$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.2 concerning the stability of (25).

Corollary 4.3. *Let $F : U^3 \to V$ be a mapping and there exists real numbers ρ and s such that*

$$\|F(x,y,z,w,u,v))\| \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s + ||u||^s + ||v||^s\right\}, & s \neq 1; \\ \rho\ ||x||^s ||y||^s ||z||^s ||w||^s ||u||^s ||v||^s, & 6s \neq 1; \\ \rho\left\{||x||^s ||y||^s ||z||^s ||w||^s ||u||^s ||v||^s \right. & \\ \qquad \left. + \left\{||x||^{6s} + ||y||^{6s} + ||z||^{6s} + ||w||^{6s} + ||u||^{6s} + ||v||^{6s}\right\}\right\}, & 6s \neq 1; \end{cases} \tag{125}$$

for all $x, y, z, w, u, v \in U$, then there exists a unique 3- variable additive function $A : U^3 \to V$ such that

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)\| \leq \begin{cases} |\rho_1|, \\ \dfrac{\rho_2 ||u||^s}{|2-2^s|}, \\ \dfrac{\rho_3 ||u||^{6s}}{|2-2^{6s}|} \\ \dfrac{\rho_4 ||u||^{6s}}{2-2^{6s}} \end{cases} \tag{126}$$

where $\rho_1, \rho_2, \rho_3, \rho_4$ are defined in (85) for all $u \in U$.

Proof. Setting

$$\phi(x,y,z,w,u,v) = \begin{cases} \rho, \\ \rho\left\{||x||^s+||y||^s+||z||^s+||w||^s+||u||^s+||v||^s\right\}, \\ \rho\ ||x||^s||y||^s||z||^s||w||^s||u||^s||v||^s, \\ \rho\left\{||x||^s||y||^s||z||^s||w||^s||u||^s||v||^s \right. \\ \qquad \left. +\left\{||x||^{6s}+||y||^{6s}+||z||^{6s}+||w||^{6s}+||u||^{6s}+||v||^{6s}\right\}\right\} \end{cases}$$

for all $x,y,z,w,u,v \in U$. Now

$$\frac{\phi(\kappa_i^n x,\kappa_i^n y,\kappa_i^n z,\kappa_i^n w,\kappa_i^n u,\kappa_i^n v)}{\kappa_i^n} = \begin{cases} \frac{\rho}{\kappa_i^n}, \\ \frac{\rho}{\kappa_i^n}\{||\kappa_i^n x||^s+||\kappa_i^n y||^s+||\kappa_i^n z||^s+||\kappa_i^n w||^s+||\kappa_i^n u||^s+||\kappa_i^n v||^s\}, \\ \frac{\rho}{\kappa_i^n}\quad ||\kappa_i^n x||^s\ ||\kappa_i^n y||^s\ ||\kappa_i^n z||^s\ ||\kappa_i^n w||^s\ ||\kappa_i^n u||^s\ ||\kappa_i^n v||^s, \\ \frac{\rho}{\kappa_i^n}\quad \Big\{||\kappa_i^n x||^s\ ||\kappa_i^n y||^s\ ||\kappa_i^n z||^s\ ||\kappa_i^n w||^s\ ||\kappa_i^n u||^s\ ||\kappa_i^n v||^s \\ \quad +\left\{||\kappa_i^n x||^{6s}+||\kappa_i^n y||^{6s}+||\kappa_i^n z||^{6s}+||\kappa_i^n w||^{6s}+||\kappa_i^n u||^{6s}+||\kappa_i^n v||^{6s}\right\}\Big\}, \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } \ n\to\infty, \\ \to 0 \text{ as } \ n\to\infty, \\ \to 0 \text{ as } \ n\to\infty, \\ \to 0 \text{ as } \ n\to\infty. \end{cases}$$

Thus, (117) is holds. But we have

$$\Xi(u) = \Phi\left(\frac{u}{2}\right)$$

has the property

$$\Xi(u) = L\cdot\frac{1}{\kappa_i}\ \Xi\left(\kappa_i u\right)$$

for all $u\in U$. Hence

$$\begin{aligned} \Xi(u) = &\ \Phi\left(\frac{u}{2}\right) \\ &= (4k^2-1)\phi\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right)+(-4k^2+2)\phi\left(u,\frac{u}{2},u,\frac{u}{2},u,\frac{u}{2}\right) \\ &\quad +2\phi\left(\frac{u}{2},u,\frac{u}{2},u,\frac{u}{2},u\right)+k^2\phi\left(u,u,u,u,u,u\right)+\phi\left(\frac{u}{2},\frac{3u}{2},\frac{u}{2},\frac{3u}{2},\frac{u}{2},\frac{3u}{2}\right) \\ &\quad +2\phi\left(\frac{(1+k)u}{2},\frac{u}{2},\frac{(1+k)u}{2},\frac{u}{2},\frac{(1+k)u}{2},\frac{u}{2}\right)+2\phi\left(\frac{(1-k)u}{2},\frac{u}{2},\frac{(1-k)u}{2},\frac{u}{2},\frac{(1-k)u}{2},\frac{u}{2}\right) \\ &\quad +\phi\left(\frac{(1+2k)u}{2},\frac{u}{2},\frac{(1+2k)u}{2},\frac{u}{2},\frac{(1+2k)u}{2},\frac{u}{2}\right)+\phi\left(\frac{(1-2k)u}{2},\frac{u}{2},\frac{(1-2k)u}{2},\frac{u}{2},\frac{(1-2k)u}{2},\frac{u}{2}\right) \\ &= \begin{cases} \rho(k^2+10), \\ \frac{\rho}{2^s}\left[12k^2-3k^2\cdot 2^{s+1}+3\left(3^s+2(1+k)^s+2(1-k)^s+(1+2k)^s+(1-2k)^s\right)+27\right]||u||^s, \\ \frac{\rho}{2^{6s}}\left[4k^2(1-2^{3s})+2^{3s+2}+k^2\cdot 2^{6s}+3^{3s}+2(1+k)^{3s}+2(1-k)^{3s}+(1+2k)^{3s}+(1-2k)^{3s}-1\right]||u||^{6s}, \\ \frac{\rho}{2^{6s}}\left(\left[4k^2(1-2^{3s})+2^{3s+2}+k^2\cdot 2^{6s}+3^{3s}+2(1+k)^{3s}+2(1-k)^{3s}+(1+2k)^{3s}+(1-2k)^{3s}-1\right]\right. \\ \quad \left.+\left[12k^2-3k^2\cdot 2^{6s+1}+3\left(3^{6s}+2(1+k)^{6s}+2(1-k)^{6s}+(1+2k)^{6s}+(1-2k)^{6s}\right)+27\right]\right)||u||^{6s}, \end{cases} \\ &= \begin{cases} \rho_1 \\ \frac{\rho_2||u||^s}{2^s}, \\ \frac{\rho_3||u||^{6s}}{2^{6s}}, \\ \frac{\rho_4||u||^{6s}}{2^{6s}}. \end{cases} \end{aligned}$$

Now,

$$\frac{1}{\kappa_i}\Xi(\kappa_i u) = \begin{cases} \frac{11\rho_1}{\kappa_i} \\ \frac{\rho_2||\kappa_i u||^s}{\kappa_i 2^s}, \\ \frac{\rho_3||\kappa_i u||^{6s}}{\kappa_i 2^{6s}}, \\ \frac{\rho_4||\kappa_i u||^{6s}}{2^{\kappa_i 6s}}, \end{cases} = \begin{cases} \kappa_i^{-1}\Xi(u), \\ \kappa_i^{s-1}\Xi(u), \\ \kappa_i^{6s-1}\Xi(u), \\ \kappa_i^{6s-1}\Xi(u). \end{cases}$$

Hence the inequality (119) holds for

(•) Either $L = 2^{-1}$ if $i = 0$ and $L = \frac{1}{2^{-1}}$ if $i = 1$.

(•) Either $L = 2^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$.

(•) Either $L = 2^{6s-1}$ for $6s < 1$ if $i = 0$ and $L = \frac{1}{2^{6s-1}}$ for $6s > 1$ if $i = 1$.

(•) Either $L = 2^{6s-1}$ for $6s < 1$ if $i = 0$ and $L = \frac{1}{2^{6s-1}}$ for $6s > 1$ if $i = 1$.

Now, from (120), we prove the following cases for condition (i).

Case:1 $L = 2^{-1}$ if $i = 0$

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)\| \le \frac{\left(2^{(-1)}\right)^{1-0}}{1 - 2^{(-1)}}\Xi(u) = 11\rho.$$

Case:2 $L = \frac{1}{2^{-1}}$ if $i = 1$

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)\| \le \frac{\left(\frac{1}{2^{(-1)}}\right)^{1-1}}{1 - \frac{1}{2^{(-1)}}}\Xi(u) = -11\rho.$$

Again, from (120), we prove the following cases for condition (ii).

Case:3 $L = 2^{s-1}$ for $s < 1$ if $i = 0$

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)\| \le \frac{\left(2^{(s-1)}\right)^{1-0}}{1 - 2^{(s-1)}}\Xi(u) = \frac{\rho_1||u||^s}{2 - 2^s}.$$

Case:4 $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)\| \le \frac{\left(\frac{1}{2^{(s-1)}}\right)^{1-1}}{1 - \frac{1}{2^{(s-1)}}}\Xi(u) = \frac{\rho_1||u||^s}{2^s - 2}.$$

Also, from (120), we prove the following cases for condition (iii).

Case:5 $L = 2^{6s-1}$ for $6s < 1$ if $i = 0$

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)\| \le \frac{\left(2^{(6s-1)}\right)^{1-0}}{1 - 2^{(6s-1)}}\Xi(u) = \frac{\rho_2||u||^{6s}}{2 - 2^{6s}}.$$

Case:6 $L = \frac{1}{2^{6s-1}}$ for $6s > 1$ if $i = 1$

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A(u, u, u)\| \le \frac{\left(\frac{1}{2^{(6s-1)}}\right)^{1-1}}{1 - \frac{1}{2^{(6s-1)}}}\Xi(u) = \frac{\rho_2||u||^{6s}}{2^{6s} - 2}.$$

Finally, to prove condition (iv) the proof is similar to that of condition (iii). Hence the proof is complete □

The proof of the following Theorem and Corollary is similar to that of Theorem 3.6 and Corollary 3.7. Hence, we omit the proofs.

Theorem 4.4. *Let $F : U^3 \to V$ be a mapping for which there exist a function $\phi : U^6 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\kappa_i^{3n}} \phi(\kappa_i^n x, \kappa_i^n y, \kappa_i^n z, \kappa_i^n w, \kappa_i^n u, \kappa_i^n v,) = 0 \tag{127}$$

where

$$\kappa_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1, \end{cases}$$

such that the functional inequality

$$\|F(x, y, z, w, u, v)\| \leq \phi(x, y, z, w, u, v) \tag{128}$$

for all $x, y, u, v, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Xi(u) = \ \Phi\left(\frac{u}{2}\right),$$

has the property

$$\Xi(u) = \frac{L}{\kappa_i^3}\ \Xi(\kappa_i x). \tag{129}$$

Then there exists a unique 3-variable cubic mapping $C : U^3 \to V$ satisfying the functional equation (25) and

$$\| \ f(2u, 2u, 2u) - 2f(u, u, u) - C(u, u, u) \ \| \leq \frac{L^{1-i}}{1 - L}\Xi(u) \tag{130}$$

for all $u \in U$. The mapping $\Phi(u)$ and $C(u, u, u)$ are defined in (58) and (99) respectively for all $u \in U$.

Corollary 4.5. *Let $F : U^3 \to V$ be a mapping and there exists real numbers ρ and s such that*

$$\|F(x, y, z, w, u, v)\| \leq \begin{cases} \rho, \\ \rho\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s + ||u||^s + ||v||^s\right\}, \\ \rho \ ||x||^s||y||^s||z||^s||w||^s||u||^s||v||^s, \\ \rho\left\{||x||^s||y||^s||z||^s||w||^s + ||u||^s||v||^s \right. \\ \qquad \left. + \left\{||x||^{6s} + ||y||^{6s} + ||z||^{6s} + ||w||^{6s} + ||u||^{6s} + ||v||^{6s}\right\}\right\}, \end{cases} \tag{131}$$

for all $x, y, u, v, z, w \in U$, then there exists a unique 3- variable cubic function $C : U^2 \to V$ such that

$$\|f(2u, 2u, 2u) - 2f(u, u, u) - C(u, u, u)\| \leq \begin{cases} \frac{\rho_1}{7}, \\ \frac{\rho_2||u||^s}{|8 - 2^s|}, & s \neq 3; \\ \frac{\rho_3||u||^{6s}}{|8 - 2^{6s}|}, & 6s \neq 3; \\ \frac{\rho_4||u||^{6s}}{|8 - 2^{6s}|}, & 6s \neq 3; \end{cases} \tag{132}$$

where $\rho_1, \rho_2, \rho_3, \rho_4$ are defined in (85) for all $u \ \in \ U$.

Now, we are ready to prove the main fixed point stability results.

Theorem 4.6. *Let $F : U^3 \to V$ be a mapping for which there exist a function $\phi : U^6 \to [0, \infty)$ with the conditions (117) and (127) where*

$$\kappa_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1, \end{cases}$$

such that the functional inequality

$$\|F(x, y, z, w, u, v)\| \leq \phi(x, y, z, w, u, v) \tag{133}$$

for all $u, y, z, w \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Xi(u) = \ \Phi\left(\frac{u}{2}\right),$$

has the properties (119) and (129) Then there exists a unique 3-variable additive mapping $A : U^3 \to V$ and a unique 3-variable cubic mapping $C : U^3 \to V$ satisfying the functional equation (25) and

$$\| \ f(u, u, u) - A(u, u, u) - C(u, u, u) \ \| \leq \frac{1}{3} \frac{L^{1-i}}{1 - L} \Xi(u) \tag{134}$$

for all $u \in U$. The mapping $\Phi(u), A(u, u, u)$ and $C(u, u, u)$ are defined in (58), (59) and (99) respectively for all $u \in U$.

Proof. By Theorems 4.2 and 4.4, there exists a unique 3-variable additive function $A_1 : U^3 \to V$ and a unique 3-variable cubic function $C_1 : U^3 \to V$ such that

$$\|f(2u, 2u, 2u) - 8f(u, u, u) - A_1(u, u, u)\| \leq \frac{L^{1-i}}{1 - L} \Xi(u) \tag{135}$$

and

$$\|f(2u, 2u, 2u) - 2f(u, u, u) - C_1(u, u, u)\| \leq \frac{L^{1-i}}{1 - L} \Xi(u) \tag{136}$$

for all $u \in U$. Now from (135) and (136), one can see that

$$\begin{aligned}
&\left\| f(u, u, u) + \frac{1}{6} A_1(u, u, u) - \frac{1}{6} C_1(u, u, u) \right\| \\
&= \left\| \left\{ -\frac{f(2u, 2u, 2u)}{6} + \frac{8f(u, u, u)}{6} + \frac{A_1(u, u, u)}{6} \right\} \left\{ \frac{f(2u, 2u, 2u)}{6} - \frac{2f(u, u, u)}{6} - \frac{C_1(u, u, u)}{6} \right\} \right\| \\
&\leq \frac{1}{6} \left\{ \|f(2u, 2u, 2u) - 8f(u, u, u) - A_1(u, u, u)\| + \|f(2u, 2u, 2u) - 2f(u, u, u) - C_1(u, u, u)\| \right\} \\
&\leq \frac{1}{6} \left\{ \frac{L^{1-i}}{1 - L} \Xi(u) + \frac{L^{1-i}}{1 - L} \Xi(u) \right\} \\
&\leq \frac{1}{3} \frac{L^{1-i}}{1 - L} \Xi(u)
\end{aligned}$$

for all $u \in U$. Thus we obtain (134) by defining $A(u, u, u) = \frac{-1}{6} A_1(u, u, u)$ and $C(u, u, u) = \frac{1}{6} C_1(u, u, u)$,$\Phi(u)$, $A(u, u, u)$ and $C(u, u, u)$ are respectively defined in (58), (59) and (99) for all $u \in U$. □

The following corollary is an immediate consequence of Theorem 4.6, using Corollaries 4.3 and 4.5 concerning the stability of (25).

Corollary 4.7. *Let $F : U^3 \to V$ be a mapping and there exists real numbers ρ and s such that*

$$\|F(x,y,z,w,u,v)\| \leq \begin{cases} \rho, \\ \rho\left\{||x||^s + ||y||^s + ||z||^s + ||w||^s + ||u||^s + ||v||^s\right\}, \\ \rho\ ||x||^s||y||^s||z||^s||w||^s||u||^s||v||^s, \\ \rho\left\{||x||^s||y||^s||z||^s||w||^s + ||u||^s||v||^s \right. \\ \qquad \left. + \left\{||x||^{6s} + ||y||^{6s} + ||z||^{6s} + ||w||^{6s} + ||u||^{6s} + ||v||^{6s}\right\}\right\}, \end{cases} \tag{137}$$

for all $x, y, z, w, u, v \in U$, then there exists a unique 3-variable additive mapping $A : U^3 \to V$ and a unique 3-variable cubic mapping $C : U^3 \to V$ such that

$$\|f(u,u,u) - A(u,u,u) - C(u,u,u)\| \leq \begin{cases} \rho_1\left(1 + \frac{1}{7}\right), \\ \frac{\rho_2}{3}\left(\frac{1}{|2-2^s|} + \frac{1}{|8-2^s|}\right)||u||^s, & s \neq 1,3; \\ \frac{\rho_3}{3}\left(\frac{1}{|2-2^{6s}|} + \frac{1}{|8-2^{6s}|}\right)||u||^{6s}, & 6s \neq 1,3; \\ \frac{\rho_4}{3}\left(\frac{1}{|2-2^{6s}|} + \frac{1}{|8-2^{6s}|}\right)||u||^{6s}, & 6s \neq 1,3; \end{cases} \tag{138}$$

where $\rho_1, \rho_2, \rho_3, \rho_4$ are defined in (85) for all $u \in U$.

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